A13

$$\text{If} \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{matrix}]\, r \\ ]\, s \end{matrix}$$

where X_1, X_2 are random vectors, with density $p_X(x_1, x_2)$ (joint density of X_1 and X_2), then the so-called marginal densities $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$ of X_1 and, respectively, of X_2 are related to $p_X(x_1, x_2)$ as follows

$$\begin{cases} p_{X_1}(x_1) = \int_{\Omega_{X_2}} p_X(x_1, x_2)\, dx_2 \\ \\ p_{X_2}(x_2) = \int_{\Omega_{X_1}} p_X(x_1, x_2)\, dx_1 \end{cases} \qquad (A.2)$$

where Ω_{X_1} and Ω_{X_2} and Ω_X are the supports of p_{X_1}, p_{X_2} and respectively p_X and

$$\boxed{\Omega_X = \Omega_{X_1} \times \Omega_{X_2}}$$

A18

$$= \frac{1}{p_Y(y)} \int_{\Omega_X} p_{\binom{X}{Y}}(x,y)\,dx$$

$$= \frac{p_Y(y)}{p_Y(y)} = 1$$

N.B $p_Y(y) = \int_{\Omega_X} p_{\binom{X}{Y}}(x,y)\,dx$ by definition of marginal density

One can define with the family $p_{X|Y}(x,y)$ the expected value of $f(X)$ conditioned to Y:

$$E[f(X)|Y] = \int_{\Omega_{X|Y}} f(x)\,p_{X|Y}(x,y)\,dx$$

the covariance of X conditioned to Y

$$\Psi_{X|Y} = E\left[(X - E[X|Y])(X - E[X|Y])^T | Y\right]$$

The invertibility of the function $g^{(k)}(\theta)$ is a necessary condition for the property of identifiability of θ in the case that

$$p_{z^{(k)}|\theta}(s^{(k)}, \theta) = p_{v^{(k)}}(s^{(k)} - g^{(k)}(\theta))$$

$$\left(\text{or } p_{z^{(k)}}(s^{(k)}, \theta) = p_{v^{(k)}}(s^{(k)} - g^{(k)}(\theta))\right)$$

Condition for "local" invertibility of $g^{(k)}(\theta)$ are:

$$\boxed{\operatorname{rank}\left\{ \frac{\partial g^{(k)}}{\partial \theta} \bigg|_{\theta = \bar{\theta}} \right\} = p}\ , \quad \bar{\theta} \in D_\theta \ (\text{or } \bar{\theta} \in R_\theta)$$

(by the implicit function theorem).

A necessary condition for this is

$$\boxed{kq \geq p}$$

If $g^{(k)}(\theta) = C^{(k)}\theta$ (linearity), Assume also

$$D_\eta = f(D_\theta) \qquad (20)$$

in the sense that $\forall \eta \in D_\eta \; \exists \, \theta \in D_\theta$ such that $\bar{\eta} = f(\bar{\theta})$.

If $\hat{\theta}|_k$ is a maximum likelihood estimate of θ, then

$$\hat{\eta}|_k = f(\hat{\theta}|_k)$$

is a maximum likelihood estimate of η. Conversely, if $\hat{\eta}|_k$ is a maximum likelihood estimate of η then all the estimates

$$\hat{\theta}|_k \in \hat{\Theta} := \{ \hat{\theta} \in D_\theta : \hat{\eta}|_k = f(\hat{\theta}) \}$$

are maximum likelihood estimates.

Proof. Define the cost functions

$$J(\theta) := \ln p_{z^{(k)}}(\breve{z}^{(k)}, \theta)$$

where $p_{z^{(k)}}(z^{(k)}, \theta) = p_{v^{(k)}}(z^{(k)} - g^{(k)}(\theta))$

and

with

$$\boxed{\begin{aligned}\Psi_{\theta|z^{(k)}} &= A^{-1} = \Psi_\theta - \Psi_{\theta z^{(k)}} \Psi_{z^{(k)}}^{-1} \Psi_{z^{(k)}\theta} \\ E[\theta|z^{(k)}] &= \bar{\theta} + \Psi_{\theta z^{(k)}} \Psi_{z^{(k)}}^{-1} (z^{(k)} - \bar{z}^{(k)})\end{aligned}} \quad (25)$$

and this proves also that the MMSE estimate $\tilde{\theta}|_k = E[\theta|z^{(k)}]$ is affine with respect to $z^{(k)}$. $\triangleleft$

When given $p_{\nu^{(k)}}$ and p_θ gaussian, also $p\begin{pmatrix}\theta \\ z^{(k)}\end{pmatrix}$ is gaussian? We want to discuss this issue here.

We need basic results on gaussian variables. Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be gaussian gaussian and uncorrelated. Consider any linear combination

$$aX + bY, \quad a, b \in \mathbb{R}.$$

It follows again from (A.1) that

$$p_{\binom{P}{Q}}(p,q) = p_{\binom{X}{Y}}(f^{-1}(p,q)) \left| \det \frac{\partial f^{-1}}{\partial(p,q)}(p,q) \right|$$

But X, Y, since jointly gaussian and uncorrelated are also independent:

$$p_{\binom{X}{Y}}(x,y) = p_X(x)\, p_Y(y)$$

Therefore, since $f^{-1}(p,q) = (p-q, q)$

$$p_{\binom{P}{Q}}(p,q) = p_X(p-q)\, p_Y(q) \frac{1}{|a|}$$

If $\bar{X} := E[X]$ and $\bar{Y} := E[Y]$, then

$$p_P(p) = \int_{\Omega_Q} p_{\binom{P}{Q}}(p,q)\, dq$$

$$= \int_{\mathbb{R}} p_X(p-q)\, p_Y(q)\, dq$$

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which proves that $X+Y$ is gaussian with mean $\overline{X}+\overline{Y}$ and variance $\sigma_X^2+\sigma_Y^2$. Therefore, $aX+bY$ is gaussian for all $a,b\in\mathbb{R}$ with mean $a\overline{X}+b\overline{Y}$ and variance $a^2\sigma_X^2+b^2\sigma_Y^2$. We conclude

2. $X,Y\in\mathbb{R}$, jointly gaussian and uncorrelated $\Rightarrow aX+bY$ gaussian for all $a,b\in\mathbb{R}$

KALMAN FILTER

Consider

$$\boxed{\begin{aligned} x(j+1) &= A(j)\,x(j) + B(j)\,m(j) \\ &\qquad\qquad j = 0, \dots, k, \\ z(j) &= C(j)\,x(j) + v(j) \\ &\qquad\qquad j = 1, \dots, k, \end{aligned}} \tag{27}$$

where $x(j) \in \mathbb{R}^n$, $m(j) \in \mathbb{R}^p$, $z(j) \in \mathbb{R}^q$, the sequences $\{u(j)\}$ and $\{v(j)\}$ are white, mutually uncorrelated, $E[v(j)] = 0$, $E[m(j)] = 0$ $\forall j$, known $\Psi_{v(j)}$ and $\Psi_{m(j)}$, $\forall j$; $x(0)$ has mean $\bar{x}(0)$ and covariance $\Psi_{x(0)}$ (both known), uncorrelated with each $\{v(j)\}$ and $\{m(j)\}$. Moreover, x_0, $\{v(j)\}$ and $\{m(j)\}$ are jointly gaussian. We refer to these assumptions as standard kalman assumptions.

with

$$C^{(k)} = \begin{pmatrix} c(1) \\ c(2) \\ \vdots \\ c(k) \end{pmatrix}$$

To guarantee identifiability of θ we need that k is such that:

$$\operatorname{rank} C^{(k)} = p = 2 \qquad (72)$$

or in other words that there exist $i < j \leq k$, such that

$$\det \begin{pmatrix} \sin\omega i & \cos\omega i \\ \sin\omega j & \cos\omega j \end{pmatrix} \neq 0$$

i.e. $0 \neq \sin\omega i \cos\omega j - \cos\omega i \sin\omega j =$

$$= \sin(\omega(i-j))$$

$\Leftrightarrow \omega(i-j) \neq h\pi$, integer h

$\Leftrightarrow$ $\boxed{\omega \text{ is not a multiple of } \pi}$