V can be chosen independent of

2.6. INVARIANCE THEOREMS

For autonomous systems, if for some Lyapunov function $V(x)$ we have

$$\dot{V}(x) \leq -W(x) \leq 0$$

it is known from La Salle's invariance theorem that the trajectory of the system approaches the largest invariant set in $E \triangleq \{x \in \mathbb{R}^n : W(x) = 0\}$.

LASALLE'S THEOREM

Let Ω be a compact set with the property that every solution of $\dot{x} = f(x)$ which starts in Ω remains for all future time in Ω. Let $V: \Omega \to \mathbb{R}$ be a continuosly differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let $E \triangleq \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ and M the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

EX.
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -k_1 \sin(x_1) - k_2 x_2$$
with $k_1, k_2 > 0$.
Consider the Lyapunov function
$$V(x) = k_1 \int_0^{x_1} \sin(s)\,ds + \frac{1}{2} x_2^2$$
$$= k_1 (1 - \cos(x_1)) + \frac{1}{2} x_2^2$$
which is positive definite on
$\{x \in \mathbb{R}^2 : -\pi < x_1 < \pi\}$

We have

$$\dot{V}(x) = -kx_2^2 \leq 0$$

But $E = \{ x \in \mathbb{R}^2 : \dot{V}(x) = 0 \}$

$$= \{ x \in \mathbb{R}^2 : x_2 = 0 \}$$

Then by Lasalle's theorem, $x(t, x_0)$ approaches the largest invariant set $E \subset M$: this invariant set is necessarily $x = 0$ $\Rightarrow$

$$x(t) \to 0 \text{ as } t \to +\infty$$

For nonautonomous systems

$$\dot{x} = f(t, x)$$

it is not even clear how to define the set E. However, if

$$\dot{V}(t, x) \leq -W(x) \leq 0$$

then $E = \{ x \in \mathbb{R}^n : W(x) = 0 \}$.

We expect that $x(t, x_0)$ approaches E.

We give first the so called <u>BARBALAT's LEMMA</u>.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\,d\tau$ exists and it is finite. Then

$$\phi(t) \to 0 \quad \text{as} \quad t \to \infty$$

<u>Short proof</u>. Assume this is false. There exists a $k_1 > 0$ such that $\forall\, T > 0$ we find $T_1 \geq T$ with $|\phi(T_1)| \geq k_1$. Since $\phi(t)$ is uniformly continuous there is k_2 such that

$$|\phi(t+\tau) - \phi(t)| < \frac{k_1}{2}$$

for all $t \geq 0$ and $0 \leq \tau \leq k_2$. Hence

$$|\phi(t)| = |\phi(t) - \phi(T_1) + \phi(T_1)| \geq$$

$$\geq |\phi(T_1)| - |\phi(t) - \phi(T_1)| > k_1 - \frac{1}{2}k_1 = \frac{1}{2}k_1$$

$$\forall\, t \in [T_1, T_1 + k_2].$$

therefore

$$\left| \int_{T_1}^{T_1+k_2} \phi(t)\,dt \right| = \int_{T_1}^{T_1+k_2} |\phi(t)|\,dt$$

$$> \frac{1}{2} k_1 k_2$$

$\left\{ \begin{array}{l} \text{since } \phi(t) \text{ has the} \\ \text{same sign for } T_1 \leq t \leq T_1 + k_2 \end{array} \right.$

Thus $\int_0^t \phi(\tau)\,d\tau$ cannot converge to a finite limit as $t \to \infty$ $\Rightarrow$ contradiction!

<u>THEOREM</u> 2.6.1. Let $D_r = \{ x \in \mathbb{R}^n \mid \|x\| < r \}$ and suppose $f(t,x)$ is locally Lipschitz in x, uniformly on t, on $[0,\infty) \times D_r$.

Let $V : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

and

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x) \leq 0$$

$\forall\, t \geq 0$, $\forall\, x \in D_2$, where $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, 2)$ and $W(x)$ is continuous on D_2. Then all solutions of $\dot{x} = f(t,x)$ with $\|x_0\| < \alpha_2^{-1}(\alpha(2))$ are bounded and satisfy

$$W(x(t)) \to 0 \text{ as } t \to \infty$$

<u>Proof</u> For any $\|x_0\| < \alpha_2^{-1}(\alpha_1(2))$ we can choose $\rho < 2$ such that $\|x_0\| < \alpha_2^{-1}(\alpha_1(\rho))$. As in the proof of theorem 2.3

$$\underset{\substack{\| \\ x_0}}{\|x(t_0)\|} < \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow$$

$$x(t) \in \Omega_{t,\rho} \quad \forall t \geq t_0$$

since $\dot{V}(t,x) \leq -W(x) \leq 0$. Hence $\|x(t)\| \leq \rho \quad \forall t \geq t_0$.

Since $V(t, x(t))$ is monotonically nonincreasing and bounded from below by zero, it converges as $t \to \infty$. Now

$$\int_{t_0}^{t} W(x(\tau))d\tau \leq -\int_{t_0}^{t} \dot{V}(\tau, x(\tau))d\tau$$

$$= V(t_0, x(t_0)) - V(t, x(t)) \leq V(t_0, x(t_0))$$

therefore $\int_{t_0}^{t} W(x(\tau))d\tau$ exists and it is finite. Since $\|x(t)\| < \rho$ $\forall\, t \geq t_0$ and $f(t,x)$ is locally Lipschitz in x, uniformly in t, we conclude that $x(t)$ is uniformly continuous in t on $[t_0, \infty)$ (indeed, $\dot{x}(t)$ is bounded on $[t_0, \infty)$). Hence by Barbalat's lemma

$$W(x(t)) \to 0 \text{ as } t \to \infty$$

If all assumptions of theorem 2.6.1 hold globally and $\alpha_1 \in K_\infty$ then the statement of the theorem is true for $\forall\, x(t_0) = x_0 \in \mathbb{R}^n$

The limit $W(x(t)) \to 0$ implies that $x(t)$ approaches

$$E = \{ x \in D \mid W(x) = 0 \}$$

as $t \to \infty$. In general, E is not an invariant set and $x(t)$ approaches a set contained in E which is not invariant. Only for autonomous systems $x(t)$ approaches an invariant set contained in E. As a generalization of LaSalle's theorem we have for nonautonomous system the following result.

<u>Theorem</u> 2.6.2. Let $x=0$ be an equilibrium point for $\dot{x}=f(t,x)$ and $D_r=\{x\in\mathbb{R}^n \mid \|x\|<r\}$. Let $V:[0,\infty)\times D_r\to\mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|x\|)\le V(t,x)\le \alpha_2(\|x\|)$$

$$\dot{V}(t,x)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\le 0$$

$$\int_t^{t+\delta}\dot{V}(\tau,\psi(\tau,t,x))\,d\tau\le-\lambda V(t,x)$$

for $0<\lambda<1$

$\forall t\ge 0$, $\forall x\in D_r$,

for some $\delta>0$, $\alpha_1,\alpha_2\in\mathcal{K}$ defined on $[0,r)$ and $\psi(\tau,t,x)$ is the solution of the system that starts from x at time t. Then $x=0$ is UAS. If all the assumptions hold globally and $\alpha_1\in\mathcal{K}_\infty$, then $x=0$ is GUAS. If

$$\alpha_i(r)=k_i r^c,\ k_i>0,\ c>0,\ i=1,2,$$

then $x=0$ is GES

Proof. As in previous results, we can prove

$$\|x(t_0)\| \le \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow x(t) \in \Omega_{t,\rho} \quad \forall t \ge t_0$$

where $\rho < r$ (since $\dot{V}(t,x) \le 0$). For $t \ge t_0$

$$V(t+\delta, x(t+\delta)) = V(t, x(t)) + \int_t^{t+\delta} \dot{V}(\tau, \psi(\tau, t, x(t))) \, d\tau \qquad (1)$$

$$\le V(t, x(t)) - \lambda V(t, x(t)) = (1-\lambda) V(t, x(t))$$

Since $\dot{V}(t,x) \le 0$

(2) $V(\tau, x(\tau)) \le V(t, x(t)) \quad \forall \tau \in [t, t+\delta]$.

For any $t \ge t_0$, N be the smallest positive integer such that $t \le t_0 + N\delta$. Divide the interval $[t_0, t_0 + (N-1)\delta]$ into $(N-1)$ equal subintervals of length δ each. Hence

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from 2

$$V(t, x(t)) \leq V(t_0 + (N-1)\delta, x(t_0 + (N-1)\delta))$$

$$\leq (1-\lambda) V(t_0 + (N-2), x(t_0 + (N-2)\delta))$$

from 1

$$\vdots$$

$$\leq (1-\lambda)^{N-1} V(t_0, x(t_0))$$

$$\leq \frac{1}{1-\lambda} (1-\lambda)^{(t-t_0)/\delta} V(t_0, x(t_0))$$

$$= \frac{1}{1-\lambda} e^{-b(t-t_0)} V(t_0, x(t_0))$$

where $b \triangleq \frac{1}{\delta} \ln \frac{1}{1-\lambda}$

If $\quad \sigma(r, s) = \frac{r}{1-\lambda} e^{-bs} \Rightarrow$

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0)$$

$$\forall\, V(t_0, x(t_0)) \in [0, \alpha_1(\rho)]$$

from which we get $\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0)$ for any $x(t_0) \in \Omega_{t_0, \rho}$, where

$$\beta(r,s) = \alpha_1^{-1}(\sigma(\alpha_2(r),s))$$

<u>Example</u> $\dot{x} = A(t)x$

with continuous $A(t)$, $t \geq 0$.

Suppose there is a continuously differentiable symmetric $P(t)$ satisfying

$$0 \leq c_1 I \leq P(t) \leq c_2 I \quad \forall t \geq 0$$

as well as

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + C^T(t)C(t)$$

with continuous $C(t)$. If

$$V(t,x) = x^T P(t) x$$

we have

$$\dot{V}(t,x) = -x^T C^T(t) C(t) x \leq 0 .$$

The solution of $\dot{x} = A(t)x$ is

$$\psi(\tau,t,x) = \phi(\tau,t)x$$

so that

$$\int_t^{t+\delta} \dot{V}(\tau, \varphi(\tau, t, x))\, d\tau =$$

$$- x^T \int_t^{t+\delta} \Psi^T(\tau, t)\, C^T(\tau)\, C(\tau)\, \Psi(\tau, t)\, d\tau \cdot x$$

$$= - x^T W(t, t+\delta)\, x$$

where

$$W(t, t+\delta) = \int_t^{t+\delta} \Phi^T(\tau, t)\, C^T(\tau)\, C(\tau)\, \Phi(\tau, t)\, d\tau$$

If for some $k < c_2$:

$$\boxed{W(t, t+\delta) \geq k I, \quad \forall t \geq 0} \qquad (3)$$

then

$$\int_t^{t+\delta} \dot{V}(\tau, \varphi(\tau, t, x))\, d\tau \leq -k \|x\|^2$$

$$\leq - \frac{k}{c_2} V(t, x)$$

All assumptions of the theorem 2.6.2 are satisfied with

$$\alpha_i(\tau) = c_i \tau^2, \; i = 1, 2, \quad \lambda = \frac{k}{c_2} < 1$$

and we conclude that $x=0$ is GES.

The matrix $W(t, t+\delta)$ is the observability grammian of the pair $(A(t), C(t))$. Condition 3 comes from the uniform complete observability assumption on $(A(t), C(t))$:

UNIFORM COMPLETE OBSERVABILITY (UCO)

of
$$\begin{cases} \dot{x} = A(t)x \\ y = C(t)x \end{cases}$$

There exist strictly positive k_1, k_2 and $\delta > 0$ such that $\forall\ t_0 \geq 0$

$$\boxed{k_1 I \leq W(t_0, t_0+\delta) \leq k_2 I} \qquad 4$$

The observability is uniform since it holds uniformly on t_0 and complete since it holds $\forall\ x(t_0)$

If the system is observable on $[t_0, t_0+\delta]$, i.e. (4) is satisfied on $[t_0, t_0+\delta]$, then $x(t_0)$ can be reconstructed from $y(\cdot)$

$$x(t_0) = W^{-1}(t_0, t_0+\delta)\int_{t_0}^{t_0+\delta} \Phi^T(\tau, t_0) C^T(\tau) y(\tau)\, d\tau$$

INVARIANCE OF UCO UNDER OUTPUT INJECTION

$$\text{UCO of } \begin{cases} \dot{x} = A(t)x \\ y = C(t)x \end{cases}$$

is preserved under output injection:

i.e.
$$\begin{cases} \dot{x} = \left(A(t) + K(t)C(t)\right)x \\ y = C(t)x \end{cases}$$

is UCO if $\forall \delta > 0$ there exists $k \geq 0$ such that $\forall t_0 \geq 0$

$$\int_{t_0}^{t_0+\delta} \|K(\tau)\|^2\, d\tau \leq k.$$

In particular, if

$$\beta_1 I \leq W(t_0, t_0+\sigma) \leq \beta_2 I$$

then

$$\beta_1' I \leq W_k(t_0, t_0+\sigma) \leq \beta_2' I$$

where W_k is the observability grammian of $(A(t)+k(t)C(t), C(t))$

and

$$\beta_1' \triangleq \frac{\beta_1}{(1+\sqrt{k\beta_2})^2}$$

$$\beta_2' \triangleq \beta_2 e^{k\beta_2}$$