STABILITY OF NONAUTONOMOUS SYSTEMS

<u>EX.</u> $\dot{x} = (6t\sin t - 2t)x, \; x(t_0) = x_0.$

the solution is

$$x(t, t_0, x_0) = x_0 \, e^{\int_{t_0}^{t} (6\tau \sin\tau - 2\tau)\, d\tau}$$

$$= x_0 \, e^{6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0 \cos t_0 + t_0^2}$$

We have for some $c(t_0) > 0$

$$\|x(t, t_0, x_0)\| \leq \|x_0\| \, c(t_0) \quad \forall t \geq t_0 .$$

Therefore if $\|x_0\| < \delta \triangleq \dfrac{\epsilon}{c(t_0)}$ for some fixed $\epsilon > 0$:

$$\|x(t, t_0, x_0)\| \leq \epsilon \quad \forall t \geq t_0$$

which implies stability of the equilibrium $x = 0$. Note that δ cannot be independent of t_0.

Suppose $t_0 = 2n\pi$, $n = 0, 1, 2, \dots$

and $t = t_0 + \pi$:

$$x(t, t_0, x_0) = x(t_0+\pi, t_0, x_0)$$
$$= x_0 \cdot e^{(4n+1)(6-\pi)\pi}$$

Clearly if $x_0 \neq 0$

$$\frac{x(t, t_0, x_0)}{x(t_0)} \to \infty \quad \text{as } n \to +\infty$$

This proves that δ must depend on t_0!

EX. $\dot{x} = -\dfrac{x}{1+t}$, $x(t_0) = x_0$

We have the solution

$$x(t, t_0, x_0) = x_0 \, e^{\int_{t_0}^{t} -\frac{1}{1+\tau} d\tau}$$

$$= x_0 \frac{1+t_0}{1+t}$$

It follows

$$\|x(t, t_0, x_0)\| \leq \|x_0\| \quad \forall t \geq t_0$$

which implies stability of the equilibrium $x=0$.

It is also clear

$x(t,t_0,x_0) \to 0$ as $t \to \infty$

and $x=0$ is asymptotically stable. Note that the convergence is not uniform with respect to t_0. Indeed, $x(t,t_0,x_0) \to 0$ as $t \to \infty$ means that $\forall \varepsilon > 0$ there exists $T = T(\varepsilon, t_0) > 0$ such that $\|x(t,t_0,x_0)\| < \varepsilon$ for $t > t_0 + T$. Since

$$x(t,t_0,x_0) = \frac{1+t_0}{1+t} x_0$$

T cannot be independent of t_0!

2.1. NOTIONS OF STABILITY

of $x=0$, assumed to be an equilibrium point of $\dot{x}=f(x,t)$, $x_0=x(t_0)$, $t_0\geq 0$.

UNIFORM STABILITY (US) There exists $\alpha \in \mathcal{K}$ and $c>0$, independent of t_0, such that

$$\|x(t,t_0,x_0)\| \leq \alpha(\|x_0\|) \quad \forall t\geq t_0\geq 0, \ \forall \|x_0\|<c$$

UNIFORM ASYMPTOTIC STABILITY (UAS)

There exists $\beta \in \mathcal{KL}$ and $c>0$, independent of t_0, such that

$$\|x(t,t_0,x_0)\| \leq \beta(\|x_0\|, t-t_0)$$

$$\forall t\geq t_0\geq 0, \forall \|x_0\|<c$$

GLOBAL UNIFORM ASYMPTOTIC STABILITY (GUAS)

The (UAS) property holds $\forall x_0 \in \mathbb{R}^n$.

EXPONENTIAL STABILITY (ES)

The (UAS) property holds with $\beta(r,s) \triangleq k r e^{-\gamma s}$, $k, \gamma > 0$

✓ CONNECTION WITH LYAPUNOV STABILITY

The equilibrium $x=0$ of $\dot{x}=f(t,x)$ is

- uniformly stable if and only if $\forall\, \varepsilon > 0$ there exist $\delta \triangleq \delta(\varepsilon)$, independent of t_0, such that

$$\|x_0\| < \delta \Rightarrow \|x(t,t_0,x_0)\| < \varepsilon \quad \forall\, t \geq t_0 \geq 0$$

- uniformly asymptotically stable if and only if it is uniformly stable and there exists $c > 0$, independent of t_0, such that

$$x(t, t_0, x_0) \to 0 \quad \text{as} \quad t \to \infty$$

uniformly in t_0, $\forall \; \|x_0\| < c$.

2.2. ~~AN~~ USEFUL LEMMA

Consider

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

where $\alpha \in \mathcal{K}$, locally Lipschitz defined on $[0, a)$. For all $0 \leq y_0 < a$,

$$y(t, t_0, y_0) = \sigma(y_0, t - t_0) \quad \forall\, t \geq t_0$$

with $\sigma \in \mathcal{KL}$ defined on $[0, a) \times [0, \infty)$.

EX. $\dot{y} = -ky$, $k > 0$:

$$y(t, t_0, y_0) = y_0 e^{-k(t-t_0)}$$
$$\triangleq \sigma(y_0, t-t_0) \in \mathcal{KL}$$

$\dot{y} = -ky^2$, $k > 0$:

$$y(t, t_0, y_0) = \frac{y_0}{k(t-t_0)y_0 + 1}$$

$$\triangleq \sigma(y_0, t-t_0)$$

2.3 Sufficient conditions for UAS

THEOREM 2.3. Let $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and

$$D_r \triangleq \{ x \in \mathbb{R}^n \mid \|x\| < r \}, \; r > 0.$$

Let $V : [0, \infty) \times D_r \to \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -\alpha_3(\|x\|)$$

$\forall t \geq 0$, $\forall x \in D_r$ with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$ defined on $[0, r)$. Then $x = 0$ is UAS.

Proof. We have

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -\alpha_3(\|x\|)$$

Let $\rho < r$ define

$$\Omega_{t,\rho} \triangleq \{ x \in D_r \mid V(t,x) \leq \alpha_1(\rho) \}$$

This set contains $\{ \|x\| \leq \alpha_2^{-1}(\alpha_1(\rho)) \}$: indeed

$$\|x\| \leq \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow \alpha_2(\|x\|) \leq \alpha_1(\rho)$$

$$\Rightarrow V(t,x) \leq \alpha_1(\rho) \Rightarrow x \in \Omega_{t,\rho}$$

On the other hand $\Omega_{t,e}$ is a subset of $\{\|x\| \leq e\}$. Indeed,

$$V(t,x) \leq \alpha_1(e) \Rightarrow \alpha_1(\|x\|) \leq \alpha_1(e)$$

$$\Rightarrow \|x\| \leq e$$

Therefore $\forall t \geq 0$

$$\{x \in \mathbb{R}^n \mid \|x\| \leq \alpha_2^{-1}(\alpha_1(e))\} \subset \Omega_{t,e}$$
$$\subset \{x \in \mathbb{R}^n \mid \|x\| \leq e\} \subset D_r$$

since $e < r$. For any $t_0 \geq 0$ and

$x_0 \in \Omega_{t_0,e}$

$$x(t,t_0,x_0) \in \Omega_{t,e} \quad \forall t \geq t_0 .$$

Indeed, $\dot{V}(t,x) < 0$ on $D_r - \{0\}$ hence $V(t,x(t))$ is decreasing and $\forall t \geq t_0$

$$V(t,x(t)) \leq V(t_0,x_0) \leq \alpha_1(\rho)$$

$$\Rightarrow x(t,t_0,x_0) \in \Omega_{t,e} \quad \forall t \geq t_0 .$$

In what follows, we assume

$$\|x_0\| \leq \alpha_2^{-1}(\alpha_1(e)) \quad (\Rightarrow x_0 \in \Omega_{t,e})$$

We have

$$\dot{V} \le -\alpha_3(\|x\|) \le$$
$$\le -\alpha_3(\alpha_2^{-1}(V)) \triangleq -\alpha(V)$$

where $\alpha \in \mathcal{K}$ defined on $[0, \alpha_1(e))$ (since α_2 is defined on $[0, r)$ and α_2^{-1} on $[0, \alpha_2(r))$), with $\alpha_1(e) \le \alpha_2(e) < \alpha_2(r)$). Let $y(t)$ satisfy

$$\dot{y} = -\alpha(y) \quad \text{with}$$
$$y(t_0) \triangleq V(t_0, x_0) \quad \left(\begin{array}{l}\text{we can assume}\\ \alpha \text{ locally Lipschitz}\end{array}\right)$$

Clearly,

$$V(t, x(t)) \le y(t) \qquad \forall t \ge t_0$$

By lemma 2.2 there exists $\sigma \in \mathcal{KL}$ defined on $[0, \alpha_1(e)) \times [0, \infty)$ such that

$$V(t, x(t)) \le \sigma(V(t_0, x_0), t - t_0)$$
$$\forall\ V(t_0, x_0) \in [0, \alpha_1(e))$$

Any solution starting inside $\Omega_{t_0, c}$ satisfies

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t)))$$
$$\leq \alpha_1^{-1}(\sigma(V(t_0, x_0), t - t_0))$$
$$\leq \alpha_1^{-1}(\sigma(\alpha_2(\|x_0\|), t - t_0))$$
$$\triangleq \beta(\|x_0\|, t - t_0)$$

But $\beta \in \mathcal{KL}$ since

$$\beta(r, s) \triangleq \alpha_1^{-1}(\sigma(\alpha_2(r), s))$$

and it follows UAS. $\triangleleft$

Notice the condition

$$\underbrace{\alpha_1(\|x\|) \le V(t,x)}_{\substack{V \text{ is said to be} \\ \text{POSITIVE DEFINITE}}} \overbrace{\le \alpha_2(\|x\|)}^{\substack{V \text{ is said to be} \\ \text{DECRESCENT}}}$$

V is said to be POSITIVE DEFINITE → if $\alpha_1 \in \mathcal{K}_\infty$ V is said to be RADIALLY UNBOUNDED

Functions V satisfying the conditions of Theorem 2.3 are LYAPUNOV functions.

<u>COROLLARY 2.3.1</u> Assume all the assumptions of theorem 4.1 are satisfied globally (i.e. $\forall x \in \mathbb{R}^n$) and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Then $x=0$ is GUAS.

<u>Proof</u> Since $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ also $\alpha_2^{-1}(\alpha_1(\rho)) \in \mathcal{K}_\infty$. Hence $\alpha_2^{-1}(\alpha_1(\rho)) \to \infty$ as $\rho \to \infty \Rightarrow \forall x_0 \in \mathbb{R}^n$.

$\forall\ x_0 \in \mathbb{R}^n$ we can select ρ large enough so that $\|x_0\| \le \alpha_2^{-1}(\alpha_1(\rho))$.
From this point, proceed as in the proof of theorem 2.3. $\triangleleft$

COROLLORY 2.3.2 Assume all the assumptions of theorem 4.1 are satisfied with

$$\alpha_i(r) = k_i r^c, \quad k_i, c, > 0.$$

Then $x=0$ is ES. Moreover, if the assumption hold globally, then $x=0$ is GES.

Proof. The function α in the proof of theorem 2.3 gives

$$\alpha(r) \triangleq \alpha_3(\alpha_2^{-1}(r))$$

$$= k_3\left[\left(\frac{r}{k_2}\right)^{1/c}\right]^c = \frac{k_3}{k_2} r$$

and $\sigma \in \mathcal{KL}$ is (coming from $\dot{y} = -\alpha(y) = -(k_3/k_2)\, y$)

$$\alpha(r,s) = r e^{-(k_3/k_2)s}$$ (21)

and
$$\beta(r,s) = \left[\frac{k_2 r^c e^{-(k_3/k_2)s}}{k_1}\right]^{1/c}$$
$$= \left(\frac{k_2}{k_1}\right)^{1/c} \cdot r e^{-(k_3/k_2 c)s}.$$

It follows $x=0$ is ES. If all the assumptions hold globally (i.e. $\forall x$)

$$\Rightarrow \|x(t)\| \le \left(\frac{k_2}{k_1}\right)^{1/c} \|x(t_0)\| e^{-(k_3/k_2 c)(t-t_0)}$$

$\forall\, x(t_0) \in \mathbb{R}^n$, since $\alpha_1(r) = k_1 r^c \in \mathcal{K}_\infty$.

EX. 2.3.1. Consider

$$\dot{x} = A(t)x$$

and assume $A(t)$ be piecewise continuous $\forall t \geq 0$. Suppose also the existence of a piecewise continuously differentiable, symmetric bounded, positive definite matrix $P(t)$ (i.e.

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall t \geq 0)$$

such that $\forall\, t \geq 0$

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

with continuous symmetric and positive definite $Q(t)$ (i.e.

$$Q(t) \geq c_3 I > 0 \quad \forall t \geq 0 \quad).$$

Consider the (candidate) Lyapunov function

$$V(t, x) = x^T P(t) x$$

which is positive definite and decrescent:

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|_2^2$$

(but also radially unbounded)

We have

$$\dot{V}(t,x) = x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x$$

$$= x^T [\dot{P}(t) + P(t) A(t) + A^T(t) P(t)] x$$

$$= - x^T Q(t) x \leq -c_3 \|x\|^2$$

Therefore, $\dot{V}$ is negative definite. All assumptions of theorem 2.3 are met with $\alpha_i = c_i r^2$. Therefore, $x=0$ is GES $\triangle$