I. MATHEMATICAL TOOLS

I.1 $\mathcal{L}_p$ spaces, norms.

- $|x|$ absolute value of $x \in \mathbb{R}$
- $\|x\|$ euclidean norm of $x \in \mathbb{R}^n$
- $\|A\|$ induced norm of $A \in \mathbb{R}^{m \times n}$ (from $\|\cdot\|$)

$$\triangleq \sup_{\|x\|=1} \|Ax\| = \sup_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$$

- for measurable functions $u : [0, \infty) \longrightarrow \mathbb{R}$ s.t. $\int_0^\infty \|u(\tau)\|^p d\tau < +\infty$

$$\underbrace{\|u\|_p}_{\mathcal{L}_p\text{-norm}} \triangleq \left(\int_0^\infty \|u(\tau)\|^p d\tau \right)^{1/p}$$

$\boxed{\text{for any } p \in [1, +\infty)}$

$\rightarrow$ $\boxed{\mathcal{L}_p\text{-spaces (spaces with } \mathcal{L}_p\text{-norm)}}$

- Truncated functions (at $s>0$):

$$u_s(t) \triangleq \begin{cases} u(t) & 0 \leq t \leq s \\ 0 & t > s \end{cases}$$

where $u : [0,\infty) \to \mathbb{R}$ is a measurable function.

Extended $\mathcal{L}_p$-spaces $\triangleq \mathcal{L}_{pe}$:

$$\mathcal{L}_{pe} = \{ u \mid \forall s < \infty , u_s \in \mathcal{L}_p \}$$

- if $p = \infty \longrightarrow \mathcal{L}_\infty , \mathcal{L}_{\infty e}$:

$$\| u \|_\infty = \sup_{t \geq 0} \| u(t) \|$$

$\boxed{\text{EX.1}}$ e^t is not in $\mathcal{L}_\infty$ but it is in $\mathcal{L}_{\infty e}$

<u>FACT #1</u> $u \in \mathcal{L}_1$ may be not bounded (i.e. in $\mathcal{L}_\infty$)

<u>FACT #2</u> $u \in \mathcal{L}_\infty$ may be not in $\mathcal{L}_1$

<u>FACT #3</u> if $u \in \mathcal{L}_1 \cap \mathcal{L}_\infty \Rightarrow$ $u \in \mathcal{L}_p \ \forall p \in [1, +\infty]$

<u>FACT #4</u> if $u \in \mathcal{L}_p \not\Rightarrow u(t) \to 0$ as $t \to +\infty$

$\downarrow$

<u>BARBALAT'S LEMMA</u>

- u uniformly continuous
- $\lim_{t\to+\infty} \int_0^t u(\tau)d\tau$ exists and it is finite

$\Rightarrow \boxed{u(t) \to 0 \text{ as } t \to +\infty}$

<u>COROLLARY #1</u>

- $u, \dot{u} \in \mathcal{L}_\infty$, $u \in \mathcal{L}_p$ for $p \in [1, \infty)$

$\Rightarrow u(t) \to 0$ as $t \to +\infty$

(HINT. take $|u|^p$ and apply B's Lemma)

I.2 POSITIVE DEFINITENESS

✓ $A \in \mathbb{R}^{n \times n}$ is

- *positive definite* if $x^T A x > 0$ $\forall x \in \mathbb{R}^n \setminus \{0\}$ $\Rightarrow \boxed{A > 0}$
- *positive semidefinite* if $x^T A x \geq 0$ $\forall x \in \mathbb{R}^n$ $\Rightarrow \boxed{A \geq 0}$
- *negative definite* if $-A$ is positive definite $\Rightarrow \boxed{A < 0}$
- *negative semidefinite* if $-A$ is positive semidefinite $\Rightarrow \boxed{A \leq 0}$

✓ $A \in \mathbb{R}^{n \times n}$ *symmetric*: $A = A^T$

orthogonal: $A = A^{-T}$

For a symmetric $A \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix T such that

$$A = T^T \Delta T$$

with $\Delta = \{\lambda_1, \ldots, \lambda_n\}$ and $\lambda_1 \ldots \lambda_n$ are the real eigenvalues of A.

square root of $A \geq 0$ symmetric

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$$A^{1/2} \triangleq T^T D^{1/2} T$$

with

$$\begin{cases} A^{1/2} A^{1/2} = A \\ (A^{1/2})^T = A^{1/2} \end{cases}$$

Fact #4 $\quad A, B \geq 0 \Rightarrow A+B \geq 0$

$$\not\Rightarrow A \cdot B \geq 0$$

for symmetric $A \geq 0$

$$\lambda_{min}(A) \|x\|^2 \leq x^T A x \leq \lambda_{max}(A)$$

minimum eigenvalue $\quad$ maximum eigenvalue

Moreover,

$$\|A\| = \lambda_{max}(A)$$

$$\|A^{-1}\| = \frac{1}{\lambda_{min}(A)} \quad (\text{if } A > 0)$$

2. $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions

- A continuous function $\alpha : [0, a)$
$\to [0, +\infty)$ is of class $\mathcal{K}$ if it is
strictly increasing and $\alpha(0) = 0$. It
is of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$
as $r \to \infty$ (we say $\alpha \in \mathcal{K}$, $\alpha \in \mathcal{K}_\infty$).

- A continuous function $\beta : [0, a) \times [0, \infty)$
is of class $\mathcal{KL}$ if for each fixed s
the mapping $\beta(r, s)$ is of class $\mathcal{K}$
with respect to r and for each fixed
r is decreasing with respect to s
and $\beta(r, s) \to 0$ as $s \to \infty$ (we say
$\beta \in \mathcal{KL}$).

ex. $\alpha(r) = \tan^{-1} r$ $\quad (\in \mathcal{K})$

$\alpha(r) = r^c, \; c > 0$ $\quad (\in \mathcal{K}_\infty)$

$\beta(r, s) = r^c e^{-s}, \; c > 0$ $\quad (\in \mathcal{KL})$

Some properties

Let $\alpha_1, \alpha_2 \in \mathcal{K}$ on $[0, a)$,

$\alpha_3, \alpha_4 \in \mathcal{K}_\infty$,

$\beta \in \mathcal{KL}$:

- $\alpha_1^{-1} \in \mathcal{K}$ on $[0, \alpha_1(a))$
- $\alpha_3^{-1} \in \mathcal{K}_\infty$ (also $\alpha_4^{-1} \in \mathcal{K}_\infty$)
- $\alpha_1 \circ \alpha_2 \in \mathcal{K}$
- $\alpha_3 \circ \alpha_4 \in \mathcal{K}_\infty$
- $\sigma(r, s) \triangleq \alpha_1(\beta(\alpha_2(r), s)) \in \mathcal{KL}$

EXISTENCE OF SOLUTION

$$\dot{x} = f(t,x) \ , \ x(t_0) = x_0 \qquad (*)$$

- piecewise continuous in t
- $\|f(t,x) - f(t,y)\| \leq L \|x-y\|$
  $\forall \ x, y \in B_r \triangleq \{x \in \mathbb{R}^n \ | \ \|x - x_0\| < r\}$
  $\forall \ t \in [t_0, t_1]$

There exists some $\delta > 0$ such that $x(t, t_0, x_0)$ is the unique solution of $(*)$ over $[t_0, t_0 + \delta]$.

EQUILIBRIUM POINTS

$$x_e \in \mathbb{R}^n : \quad f(t, x_e) = 0 \quad \forall t \geq 0$$

Translation of x_e in $0 \Rightarrow$

$$z \triangleq x - x_e \Rightarrow$$

$$\dot{z} = \underbrace{f(t, z + x_e)}_{\tilde{f}(t,z)} \Rightarrow \tilde{f}(t, 0) = 0$$