Control Systems
16/09/2019

Exercise 1 Denoting $L(s) = G(s)P(s)$ and

$$P(s) = (1 + P_1(s))P_2(s) = \frac{s-2}{s(s+2)}$$

in the Laplace domain the input-output evolutions are described by

$$y(s) = W(s)v(s) + W_d(s)d(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$ and $W_d(s) = \frac{P_2(s)}{1+L(s)}$.

First, let us note that the invariant spectrum with respect to controllability is provided by $\mathcal{I}_C = \{-1\} \subset \mathcal{C}^-$ so that the system is still stabilizable.

At this point, let us set $G(s) = G_2(s)G_1(s)$ where $G_1(s)$ and $G_2(s)$ are designed so to fulfil, respectively, the steady-state and transient specifications.

(i) By the structure of the system, one has that an integrator is already acting before the entering point of the disturbance so that $y_{ss}(t) = 0$ under constant disturbances $d(t)$. Setting $G_1(s) = \kappa_1$ and for the time-being $G_2(s) = 1$, one gets that $|e_{ss}(t)| \leq M = 0.1$ if

$$|\frac{W_e(s)}{s}|_{s=0} \leq 0.1$$

with $W_e(s) = \frac{1}{1+L(s)}$ which is satisfied setting $|\kappa_1| \geq 10$. Moreover, by investigating the

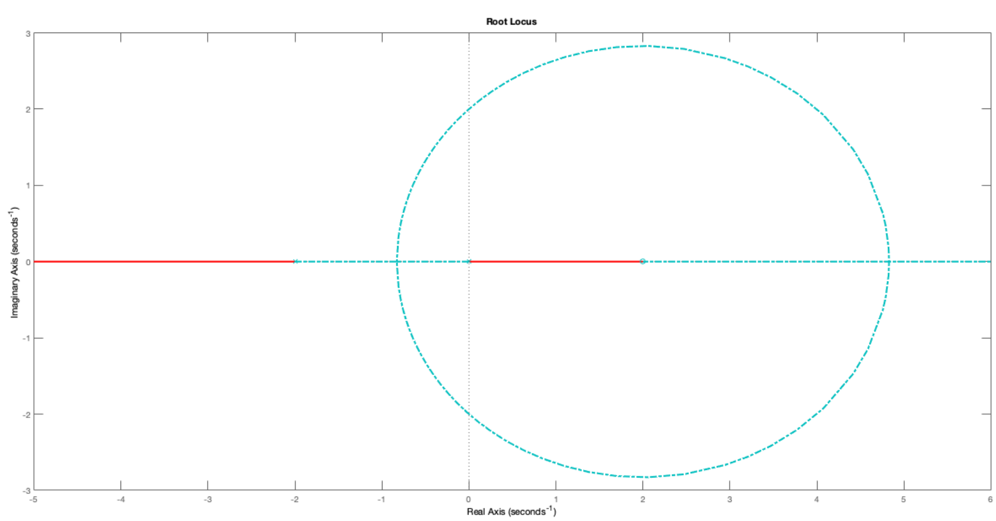

Figure 1: Root locus of $P(s) = \frac{s-2}{s(s+2)}$

root locus associated to $P(s)$ (Figure 1), one immediately verifies that a negative gain is necessary for asymptotically stabilizing the closed-loop system. As a consequence, we set $\kappa_1 = -10$ and $G_2(s) = \kappa_2 \bar{G}_2(s)$with $\kappa_2 > 1$ and denote

$$\bar{P}(s) = -P(s) = -10\frac{s-2}{s(s+2)} \tag{1}$$

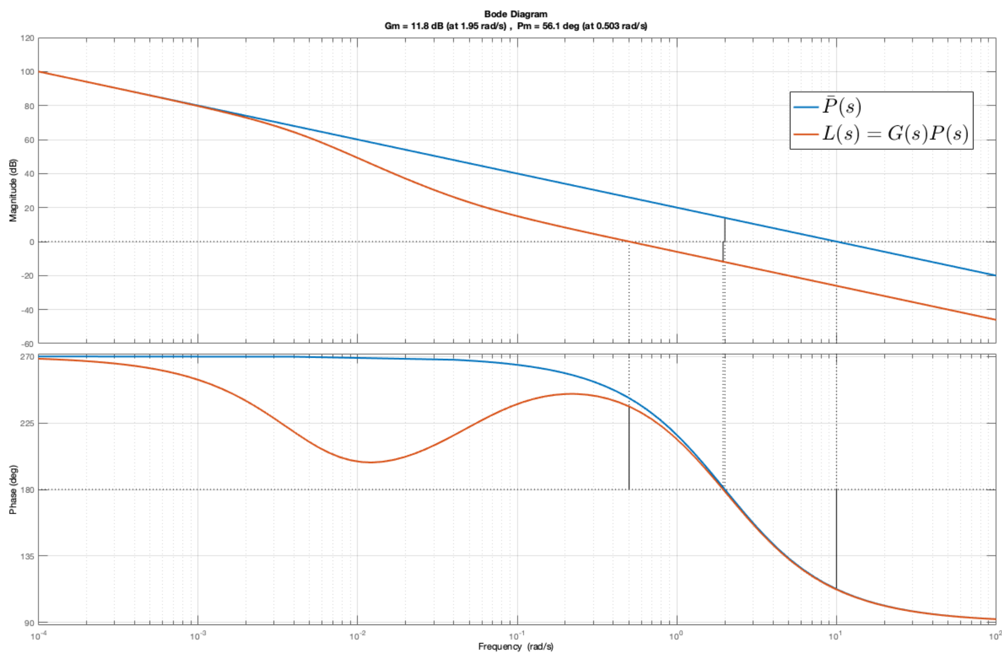

Figure 2: Bode plots of (1) and (2)

(ii) By inspecting (Figure 2) the Bode plots of (1), one has that at $\omega = 0.5$ rad/s

$$|P(0.5j)|_{dB} \approx 26.02dB \quad \text{and} \quad \angle\bar{P}(0.5j) + 180^\circ \approx 61.93^\circ$$

with hence decreasing values as $\omega > 0.5$ rad/s. Accordingly, as $\kappa_2 \geq 1$ for fulfilling specification (i), we set $\bar{G}_2(s)$ so to decrease the cross-over frequency at $\omega_t^\star \approx 0.5 + \varepsilon$ rad/s (with $\varepsilon > 0$ small) while ensuring that

$$\angle\bar{P}(0.5j) + \angle\bar{G}_2(0.5j) + 180^\circ = \angle\bar{G}_2(0.5j) + 61.93^\circ \geq 45^\circ \quad \Longrightarrow \quad \angle\bar{G}_2(0.5j) \geq -16.93^\circ.$$

To this end, we set $\kappa_2 = 1$ and thus

$$\bar{G}_2(s) = \frac{1+\frac{\tau}{m_1}s}{1+\tau s}\frac{1+\frac{\tau}{m_2}s}{1+\tau s}$$

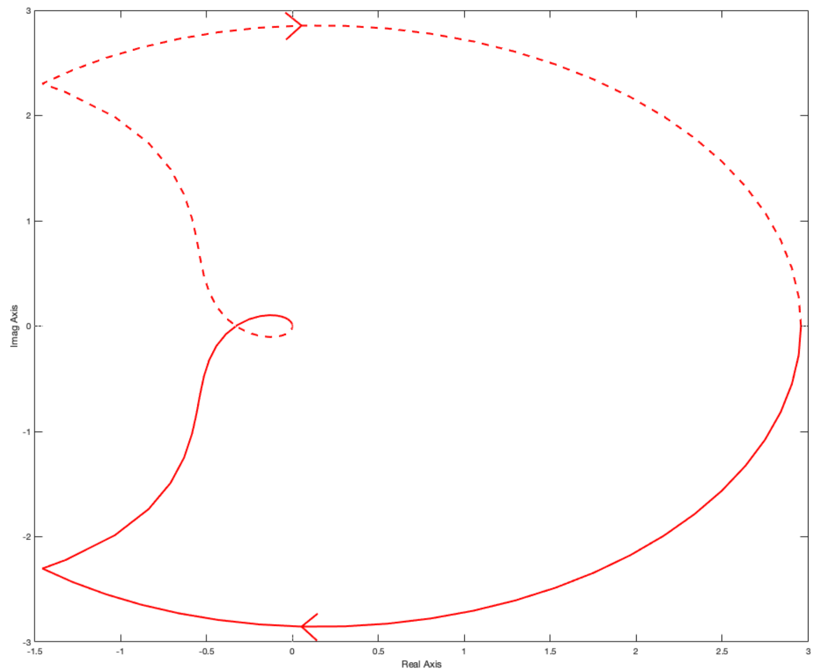

Figure 3: Nyquist plot of (2)

with $m_1 = 10$ and $m_2 = 2$ and $\tau = \frac{\omega_N}{\omega_t^\star}$ with $\omega_N = 100$ so to get

$$\angle \bar{G}_2(0.5j) \approx -5.83°, \quad |\bar{G}_2(0.5j)|_{dB} \approx -26dB.$$

Accordingly, the overall controller is given by

$$G(s) = -10\frac{2000s^2 + 120s + 1}{40000s^2 + 400s + 1}.$$

(i) The Nyquist plot of the open loop system

$$L(s) = G(s)P(s) = -10\frac{2000s^2 + 120s + 1}{40000s^2 + 400s + 1}\frac{s-2}{s(s+2)} \tag{2}$$

is reported in Figure 3. The number of counter-clockwise encirclements of $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0 as the number the open loop poles of L(s) with positive real part. Thus, the system is asymptotically stable in closed loop.

Exercise 2 It is a matter of computation to verify that the system (A, B, C) is not controllable. As a matter of fact, the invariant spectrum with respect to controllability is given by $\mathcal{I}_C = \{-1\} \subset \mathbb{C}^-$ so that the system is stabilizable under feedback.

At this point, we compute the transfer function associated to (A, B, C) as given by

$$P(s) = \frac{1}{s+2}$$

so getting

$$y(s) = W(s)v(s) + W_d(s)d(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$, $W_d(s) = \frac{1}{1+L(s)}$ and $L(s) = G(s)P(s)$.

(i) For fulfilling specification (i) it is necessary to embed a copy of the signals to reject in the open loop transfer function $L(s)$ so guaranteeing that the corresponding steady-state responses are zero. As a consequence, we set

$$G(s) = \frac{1}{s(s^2+1)}G_r(s).$$

(ii) As the dimension of the feedback is lower bounded by specification (i) we set $G_r(s) = as^2 + bs + c$ so to increase the relative degree (i.e., the pole-zero excess) of the corresponding open loop transfer function $L(s)$ to $r = 2$ so getting

$$L(s) = \frac{as^2+bs+c}{s(s^2+1)(s+2)}.$$

Also, under a suitable choice of $a, b, c \in \mathbb{R}$ the center of the asymptotes of $L(s)$ (denoted by $s_0 \in \mathbb{R}$) can be constrained to be $s_0 < -0.3$. By computing the pole polynomial associated to the input-output transfer function $W(s) = \frac{L(s)}{1+L(s)}$ one gets

$$\mathbf{p}(s; a, b, c) = s^4 + 2s^3 + (1+a)s^2 + (2+b)s + c$$

so getting that the poles of the closed-loop system can be all assigned at a proper $-p < -0.3$ that is the following set admits a solution

$$\begin{aligned} &\mathbf{p}(s; a, b, c) = (s+p)^4 \\ &p > 0.3. \end{aligned}$$

In particular, one gets

$$p^4 = c \quad 4p^3 = 2+b, \quad 6p^2 = 1+a, \quad 4p = 2$$

and thus the solution

$$p = \frac{1}{2}, a = \frac{1}{2}, \quad b = -\frac{3}{2}, \quad c = \frac{1}{6}.$$

Accordingly, the overall feedback is given by

$$G(s) = \frac{1}{16}\frac{8s^2 - 24s + 1}{s(s^2+1)}$$

assigning for poles in $p = -\frac{1}{2}$. Accordingly, the root locus of

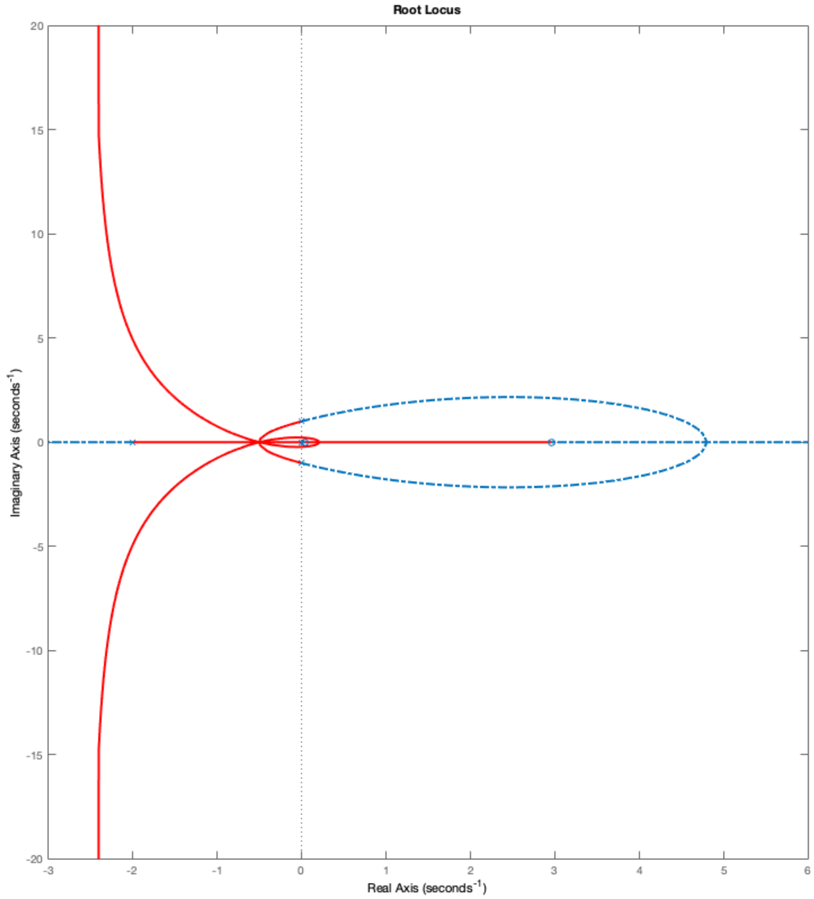

Figure 4: Root Locus of $KL(s)$ with $L(s)$ in (3).

$$KL(s) = G(s)P(s) = \frac{K}{16}\frac{(s-2.9577)(s-0.0423)}{s(s^2+1)(s+2)}. \tag{3}$$

possesses relative degree $r = 2$ and center of asymptotes $s_0 \approx 2.5$. By construction of $G(s)$, it possesses one singularity of order $\mu = 4$ at $(s_1^\star, K_1^\star) = (-\frac{1}{2}, 1)$. In addition, considering the pole-polynomial of the closed-loop transfer function $\tilde{W}(s) = \frac{KL(s)}{1+KL(s)}$ provided by

$$\tilde{p}(s,K) = s^4 + 2\,s^3 + \left(\frac{K}{2}+1\right)\,s^2 + \left(2 - \frac{3\,K}{2}\right)\,s + \frac{K}{16}$$

one gets that other two singularities of order $\mu = 2$ arise corresponding to $(s_2^\star, K_2^\star) = (0.208, 2.10)$ and $(s_3^\star, K_3^\star) = (4.79, -179.04)$. The point in which the locus crosses the imaginary axis correspond to $K \in \mathbb{R}$ making the Routh table non-regular that is

r^4	1 $\quad 1+\frac{K}{2}$ $\quad \frac{K}{16}$
r^3	2 $\quad 2-\frac{3}{2}K$
r^2	$5K$ $\quad \frac{K}{4}$
r^1	$19-15K$
r^0	$\frac{K}{4}$

so getting $K \in \{0, \frac{19}{15}\}$. Also, it is immediate to verify that the closed-loop system is asymptotically stable as $K \in (0, \frac{19}{15})$. The locus is reported in Figure 4.

Exercise 3 The eigenvalues of the systems are given by $\lambda_1 = -3$ and $\lambda_2 = 2$ with

$$u_1 = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

being the corresponding eigenvectors. Thus, the system possesses two aperiodical modes describing the corresponding free evolution

$$x(t) = e^{At}x_0 = c_1 e^{-3t}u_1 + c_2 e^{2t}u_2$$

with $c_1, c_2 \in \mathbb{R}$ provided by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = U^{-1}x_0, \quad U = \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix}.$$

By noticing that $x(t_f) \in \text{span}\{u_2\}$ one concludes that necessarily $x_0 \in \text{span}\{u_2\}$ so that $c_1 = 0$ and thus $x_0 = \begin{pmatrix} 0 & x_0^2 \end{pmatrix}^\top$ and $x_0^2 = e^{-4}$ as the solution to

$$x(t_f) = Ue^{\Lambda t}U^{-1}x_0, \quad e^{\Lambda t} = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$

for $t = t_f = 2$ and $x(t_f) = (0\ 1)^\top$. As $t \to \infty$, the corresponding solutions from $x_0 = e^{-4}u_2$ diverge that is $\|x(t)\| \to \infty$.

On the other side, solutions converge to the origin if and only if $x_0 \in \text{span}\{u_1\}$ so to guarantee $c_2 = 0$.