Control Systems
02/02/2018(A)

Exercise 1 Denoting $L(s) = G(s)P(s)$, one has

$$y(s) = W(s)v(s) + W_d(s)d(s)$$

with input-output and disturbance-output transfer functions respectively provided by

$$W(s) = \frac{L(s)}{1 + L(s)}, \quad W_d(s) = \frac{P(s)}{1 + L(s)}.$$

In particular, we shall define $G(s) = G_2(s)G_1(s)$ so that $G_1(s)$ is designed so to fulfil specification (ii) whereas $G_2(s)$ is designed for specifications (iii) and (i).

(ii) Assuming for the time-being $G_2(s) = 1$, as the input-disturbance is a ramp (i.e., $d(t) = t$) an integrator is needed right before the entering point of the disturbance. This requires

$$G_1(s) = \frac{k_1}{s}$$

with $k_1 \in \mathbb{R}$ in such a way to guarantee

$$\left|\frac{W_d(s)}{s}\right|_{s=0} = \left|\frac{1}{s^2 + s + 1}\right|_{s=0} \leq 0.1$$

so getting $k_1 \geq 10$. Accordingly, we can set $k_1 = 10$ by constraining $|G_2(0)| > 1$ so to preserve the required specification. As a consequence of the choice of $G_1(s)$ it is then clear that the dimension of $G_2(s)$ must be at most 1 so to fulfil the constraint on the dimension of $G(s) = G_2(s)G_1(s)$.

For understanding how to design $G_2(s)$, let us draw the Bode plots of

$$L_1(s) = G_1(s)P(s) = \frac{10}{s(s+1)} \tag{1}$$

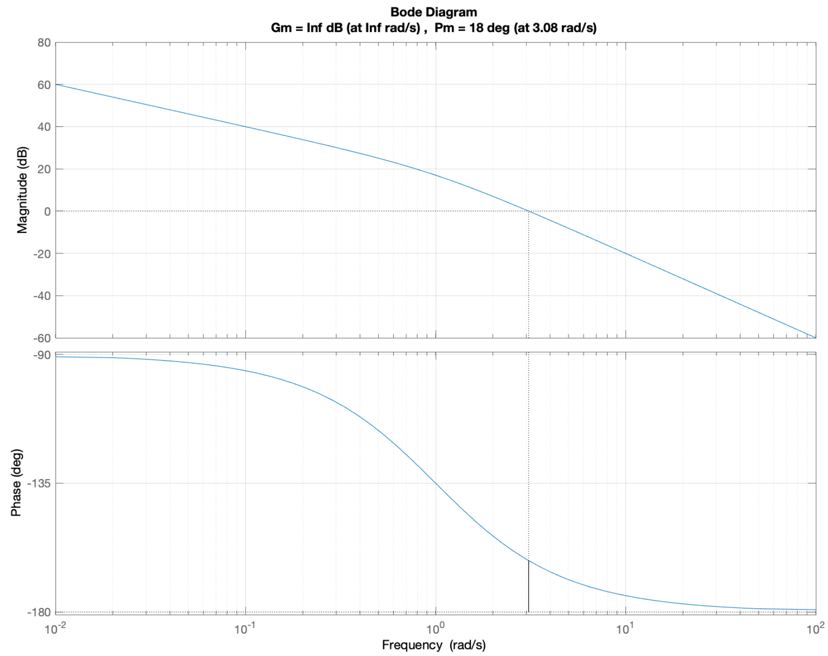
reported in Figure 1. It is evident that the cross-over frequency of the actual system $L_1(s)$ is given by $\omega_t = 3.08\ rad/s$ with corresponding phase margin $m_\varphi = 18^o$.

For maximizing the phase margin of $L(s) = G_2(s)L_1(s)$, only one anticipating action can be introduced, that is

$$G_2(s) = k_2 G_a(s), \quad G_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}$$

and $|k_2| > 1$. Accordingly, for increasing the phase margin as much as possible the anticipating function cannot be set to be any; indeed, one has to set ω_t^* and all other functions in such a way that

$$|k_2|_{dB} + |G_a(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0$$
$$180^o + \angle G_a(j\omega_t^*) + \angle L_1(j\omega_t^*)$$



(iii)

Figure 1: Bode plots of (1)

so providing, as $|k_2| > 1$ implies $|k_2|_{dB} > 0$,

$$|k_2|_{dB} = -|G_a(j\omega_t^*)|_{dB} - |L_1(j\omega_t^*)|_{dB} \geq 0 \implies |G_a(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} \leq 0.$$

Thus, for maximizing the phase margin m_φ^* we set the anticipating function labeled by $m_a = 16$ acting at $\omega_n = 4$ rad/sec so getting $|G_a(j\omega_t^*)|_{dB} \approx -12$ and $\angle G_a(j\omega_t^*) = 62^o$. According to the above constraint, we set hence $\omega_t^* = 6.5$ rad/sec so ensuring $|G_a(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} \approx 0$ and requiring $k_2 = 1$. Accordingly, the phase margin of

$$L(s) = k_2 G_a(s) L_1(s) = \frac{0.6154s + 1}{0.0385s + 1} \frac{10}{s(s+1)} \tag{2}$$

will be equal to $m_\varphi^* = 71.2^o$. The corresponding Bode plots are in Figure 2 and emphasize on the fact that the actual cross-over frequency is not $\omega_t^* = 6.5$ rad/s due to approximation errors. However, as the specification does not require a specific value for the cross-over frequency, it is not necessary to include a further action through the gain k_2 (that would be admissible in this case) to move it to $\omega_t^* = 6.5$ rad/s.

(i) The Nyquist plot of (2) is reported in Figure 3. The number of counter-clockwise encirclements of $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0 that is also coincident with the open loop poles with positive real part. Thus, the system is asymptotically stable in closed loop.

Exercise 2 a) Denoting by n and m the number of poles and zeros of the transfer function, the relative degree of $P(s)$ is given by $r = n - m = 1$. Accordingly, the root locus possesses

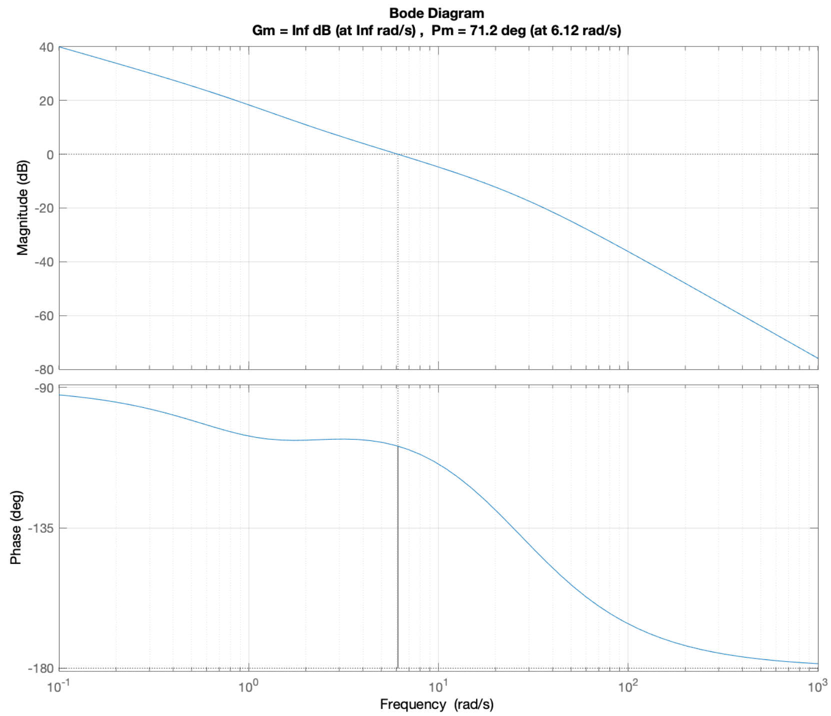


Figure 2: Bode plots of (2)

an horizontal asymptote centered at

$$s_0 = \frac{1-6-4+4}{1} = -5$$

that can be discarded. Introducing $k \in \mathbb{R}$ and defining $p(s,k) = (s-1)(s+4)(s+6) + k(s+2)^2$ as the polynomial defining the closed-loop poles under $G(s) = k$, one gets that singularities are the solutions to

$$p(s,k) = s^3 + (k+9)s^2 + (4k+14)s + 4k - 24 = 0$$
$$\frac{\partial p(s,k)}{\partial s} = 3s^2 + 2(k+9)s + (4k+14) = 0$$
$$\frac{\partial^2 p(s,k)}{\partial s^2} = 6s + 2(k+9) = 0.$$

By solving the equations above, it turns out that the negative locus possesses one singularity with multiplicity $\mu = 2$ in correspondence of $(s^*, k^*) \approx (-4.74, -0.713)$. What is left to do is now is to quantify the number of intersection of the root locus with the imaginary axis. Those intersecting points correspond to values of $k \in \mathbb{R}$ for which the Routh table of $p(s,k) = s^3 + (k+9)s^2 + (4k+14)s + 4k - 24$ is not regular. Thus, by developing computations one gets

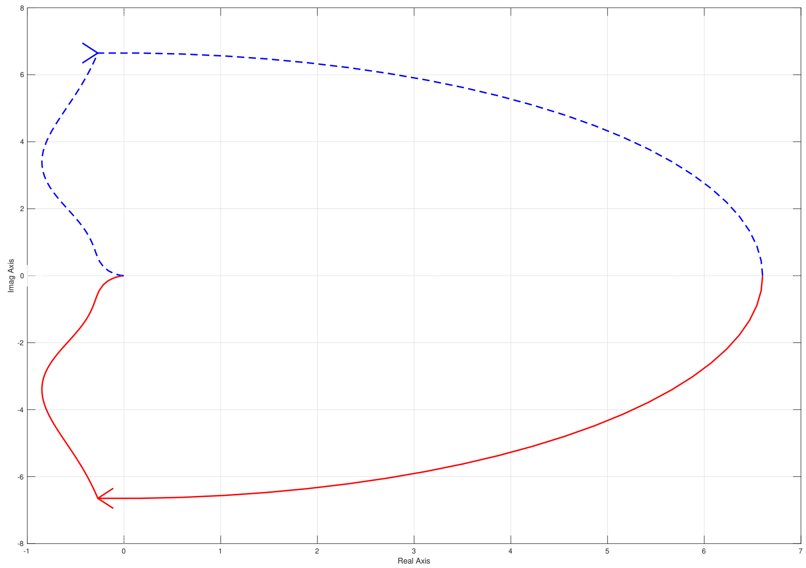


Figure 3: Nyquist plot of (2)

$$
\begin{array}{c|cc}
r^3 & 1 & 4k+14 \\
r^2 & k+9 & 4k-24 \\
r^1 & \frac{2k^2+23k+75}{k+9} & \\
r^0 & k-6. &
\end{array}
$$

The Routh table is not regular for $k = 6$ so implying that the positive locus intersects the imaginary axis in correspondance of $k = 6$ corresponding to the closed-loop pole $s = 0$. The root locus is reported in Figures 4 and 5.

b) From the root locus of $P(s)$, it is evident that a static feedback $G(s) = k$ is not enough for assigning all poles with real part smaller than -3 as it does not exhibit three branches on the left hand side of the vertical line cantered at -3. Thus, a dynamical feedback is needed. First of all, such a feedback must increase the relative degree to $r' = 2$. To this end, as no limitation is made on the dimension of $G(s)$ we propose a feedback of the form

$$G(s) = k\frac{(s+4)(s+6)}{(s+2)^2(s+p)}$$

with $k, p \in \mathbb{R}$ so getting

$$L(s) = G(s)P(s) = k\frac{1}{(s-1)(s+p)}.$$

Note that the controller above generates unobservability of the mode associated to the eigenvalue -2 and uncontrollability of the ones associated to the eigenvalues -4 and -6. However, those *cancellations* do not affect asymptotic stabilizability of the closed loop. At this point, $k, p \in \mathbb{R}$ need to be chosen so that the closed-loop system possesses al

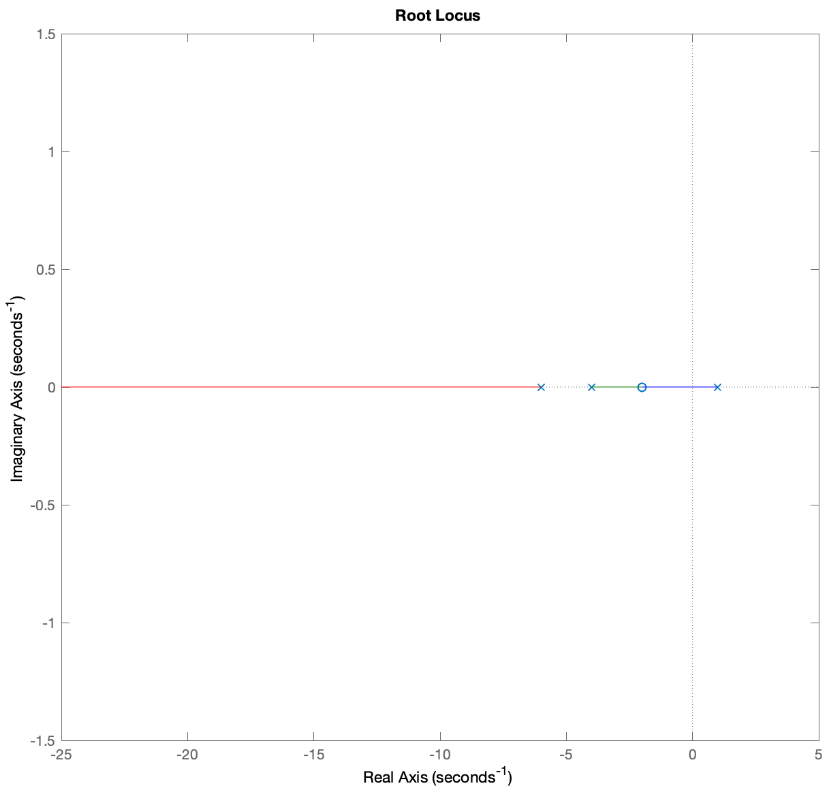


Figure 4: Positive root locus of $P(s) = \frac{(s+2)^2}{(s-1)(s+4)(s+6)}$

poles with real part smaller than -3. By denoting $p_L(s,k,p) = (s-1)(s+p)+k$ the polynomial of the closed-loop poles, it is enough to invoke the Routh criterion and set k, p so that the shifted polynomial $p_L^*(s,k,p) = p_L(s-3,k,p) = (s-4)(s+p-3)+k = s^2+(p-7)s+k-4(p-3)$ is Hurwitz. Thus, one gets that the specification is satisfied for all $p, k \in \mathbb{R}$ satisfying

$$p > 7, \quad k > 4(p-3).$$

Exercise 3 The system

$$\begin{aligned}\dot{x} &= Ax + Bd \\ y &= Cx\end{aligned}$$

possesses two aperiodic modes associated to the eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.

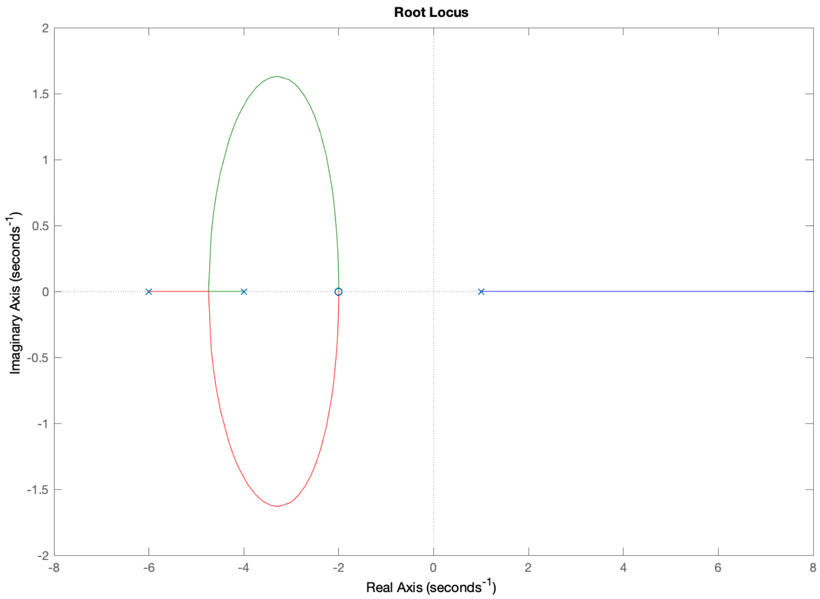


Figure 5: Negative root locus of $P(s) = \frac{(s+2)^2}{(s-1)(s+4)(s+6)}$

(i) The output response of the system for $d = 0$ gets the form

$$\begin{aligned} y(t) &= Cx(t) \\ x(t) &= c_1 e^{-3t} z_1 + c_2 e^{2t} z_2 \end{aligned}$$

with $z_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $z_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ being the eigenvectors associated to $\lambda_1 = -3$ and $\lambda_2 = 2$ and

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \end{pmatrix}^{-1} x_0$$

with x_0 being the initial condition. Accordingly, one gets

$$y(t) = c_1 e^{-3t} + c_2 e^{2t}$$

so that the $y(t) \to 0$ as $t \to \infty$ for all initial conditions annihiliating the divergent a-periodical mode. Because a-periodical modes evolve along the corresponding eigenvectors, it is enough to set $x_0 \in \text{span}\{z_1\}$ so getting, as a result, $c_2 = 0$.

(ii) For ensuring $y(t) = 0 \ \forall t \geq 0$ and d it is enough to chose B so that its image belongs to the unobservable subspace $\mathcal{I}$ of the system. Indeed, one gets

$$\mathcal{I} = \ker\{\begin{pmatrix} C \\ CA \end{pmatrix}\} = \{0\}$$

and thus the requirement is satisfied only for the trivial case of $B = 0$.

One other way of solving the specification would have been to set $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ in such a way to make the input-output transfer function identically zero; namely,

$$y(s) = P(s)d(s) = 0, \forall d \iff P(s) = C(sI - A)^{-1}B = 0.$$

(iii) For $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the transfer function of the system is given by

$$P(s) = C(sI - A)^{-1}B = \frac{1}{s+3}.$$

Accordingly, by rewriting $u(t) = a_1 \sin_+(t-1) + a_2 \cos_+(t-1)$ with $a_1 = \cos 1 = 0.54$ and $a_2 = \sin 1 = 0.84$, one computes the forced response of the system as

$$\begin{aligned} y(t) =& \mathcal{L}^{-1}(P(s)u(s))[t] = a_1 \mathcal{L}^{-1}(P(s)e^{-s}u_1(s))[t] + a_2 \mathcal{L}^{-1}(P(s)e^{-s}u_2(s))[t] \\ =& a_1 \mathcal{L}^{-1}(P(s)u_1(s))[t-1] + a_2 \mathcal{L}^{-1}(P(s)u_2(s))[t-1] \end{aligned}$$

with $u_1(s) = \mathcal{L}(\sin_+(t))[s]$ and $u_2(s) = \mathcal{L}(\cos_+(t))[s]$. Thus, it is enough to compute $y_i(s) = P(s)u_i(s)$ by neglecting proportional terms and time-delays affecting the input. For, one has

$$y_1(s) = \frac{1}{(s+3)(s^2+1)} = \frac{R_{11}}{s+3} + \frac{A_1 s + B_1}{s^2+1}$$

with

$$R_{11} = \frac{1}{10}, \quad A_1 = -\frac{1}{10}, \quad B_1 = \frac{3}{10}$$

and consequently

$$y_1(t) = \mathcal{L}^{-1}(y_1(s))[t] = R_{11} e_+^{-3t} A_1 (\cos t)_+ + B_1 (\sin t)_+.$$

Similarly, one has

$$y_2(s) = \frac{s}{(s+3)(s^2+1)} = \frac{R_{21}}{s+3} + \frac{A_2 s + B_2}{s^2+1}$$

with

$$R_{21} = -\frac{3}{10}, \quad A_2 = \frac{3}{10}, \quad B_2 = \frac{1}{10}$$

so getting

$$y_2(t) = \mathcal{L}^{-1}(y_2(s))[t] = R_{21} e_+^{-3t} A_2 (\cos t)_+ + B_2 (\sin t)_+.$$

Accordingly, the overall response is

$$y(t) = (a_1 R_{11} + a_2 R_{21}) e_+^{-3(t-1)} + (a_1 A_1 + a_2 A_2)(\sin(t-1))_+ + (a_1 B_1 + a_2 B_2)(\cos(t-1))_+.$$