Control Systems
05/06/2018

Exercise 1 Denoting $L(s) = G(s)P(s)$, one has

$$y(s) = W(s)v(s), \quad e(s) = W_e(s)v(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$ and $W_e(s) = \frac{1}{1+L(s)}$. As usual, we shall split the controller in two loops; namely, $G(s) = G_2(s)G_1(s)$ with $G_1(s)$ designed for satisfying steady-state specifications (i.e., (ii)) whereas the outer loop $G_2(s)$ is defined for transient and stability requirements (i.e., (iii) and (i)).

(ii) Set for the time being $G_2(s) = 1$. As $e_{ss}(t) = W_e(0)t + \frac{\partial W_e}{\partial s}(0)$, when $v(t) = t$, one needs $W_e(0) = 0$ and $\frac{\partial W_e}{\partial s}(0) = 0$. Accordingly, two integrating actions are needed. As the plant itself already possesses a pole at $s = 0$, we set the inner control loop $G_1(s) = \frac{1}{s}$.

(iii) By inspecting the Bode Plots of

$$L_1(s) = G_1(s)P(s) = \frac{1}{s^2(s-1)} \tag{1}$$

reported in Figure 1, one notices that the outer loop control action $G_2(s)$ needs to be chosen so to

1. increase the value of the phase at $\omega_t^* = 3$ rad/s as so that $m_\varphi^* = 180^o + \angle G_2(j\omega_t^*)| + \angle L_1(j\omega_t^*) \geq 30^o$ with $\angle L_1(j\omega_t^*) = -360^o + 71.57^o$ so implying $\angle G_2(j\omega_t^*)| \geq 138.43^o$.
2. decrease the magnitude at $\omega_t^* = 3$ rad/s so to guarantee $|G_2(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0$ with $|L_1(j\omega_t^*)|_{dB} \approx -29.08$

Accordingly, as no bound is apriori set over the gain of $G_2(s)$ (that is $G_2(0)$), we shall design the outer loop as composed of anticipating actions aimed at increasing the phase at $\omega_t^* = 3$ rad/s and a gain to make $\omega_t^* = 3$ rad/s the new cross-over frequency.
Thus, the structure we propose for $G_2(s)$ is the following one

$$G_2(s) = kG_a(s), \quad k > 0$$

with $G_a(s) = G_a^1(s)G_a^2(s)$. In particular, we introduce 3 anticipating actions of the form

$$G_a^1(s) = \Big(\frac{1 + \tau_a^1 s}{1 + \frac{\tau_a^1}{m_a^1} s}\Big)^2, \quad G_a^2(s) = \frac{1 + \tau_a^2 s}{1 + \frac{\tau_a^2}{m_a^2} s}$$

with

- $G_a^1(s)$ composed of two identical anticipating functions acting at $\omega_N^1 = 3$ rad/sec with $m_a^1 = 16$ (that is at $\tau_a^1 = 1$) so that $\angle G_a^1(j\omega_t^*) \approx 121^o$ and $|G_a^1(j\omega_t^*)|_{dB} \approx 20.08$.
- $G_a^2(s)$ being one anticipating function with $\omega_N^2 = 5$ rad/sec and $m_a^2 = 3$ (that is $\tau_a^2 = \frac{5}{3}$) so that $\angle G_a^2(j\omega_t^*) \approx 20^o$ and $|G_a^2(j\omega_t^*)|_{dB} \approx 8$.

In this way, as $k > 0$, $m_\varphi^* = 32.57^o$ whereas k needs to be chosen so that $|k|_{dB} + |G_a(j\omega_t^*)|_{dB} - 29.08 = 0$ with $|G_a(j\omega_t^*)|_{dB} \approx 28.08$ so requiring $k = 1.124$. The bode

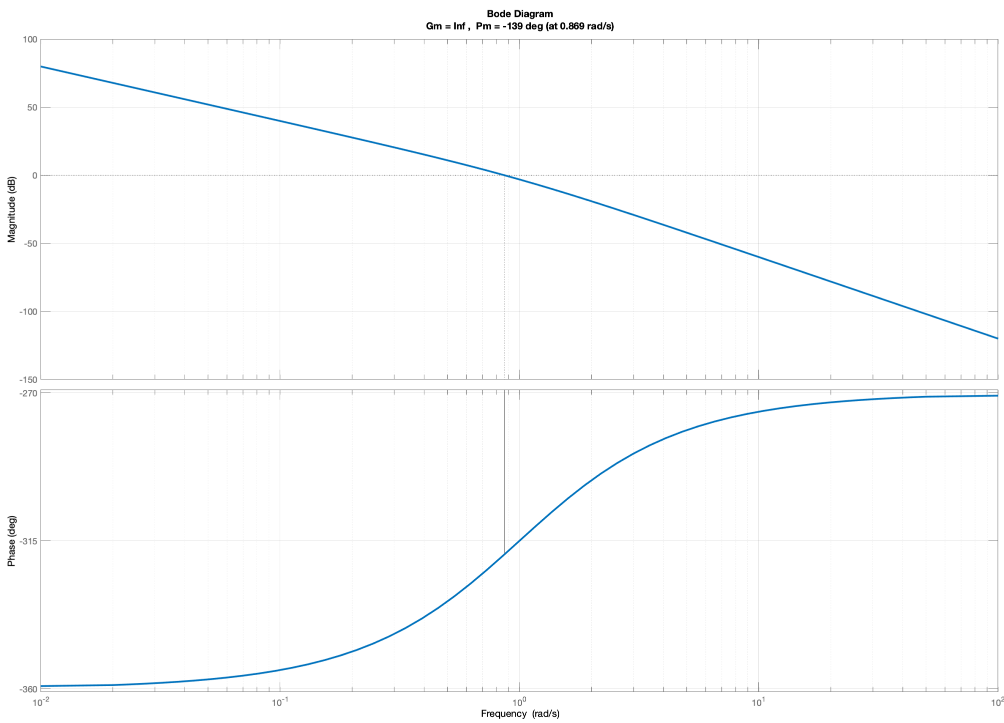

Figure 1: Bode plots of (1)

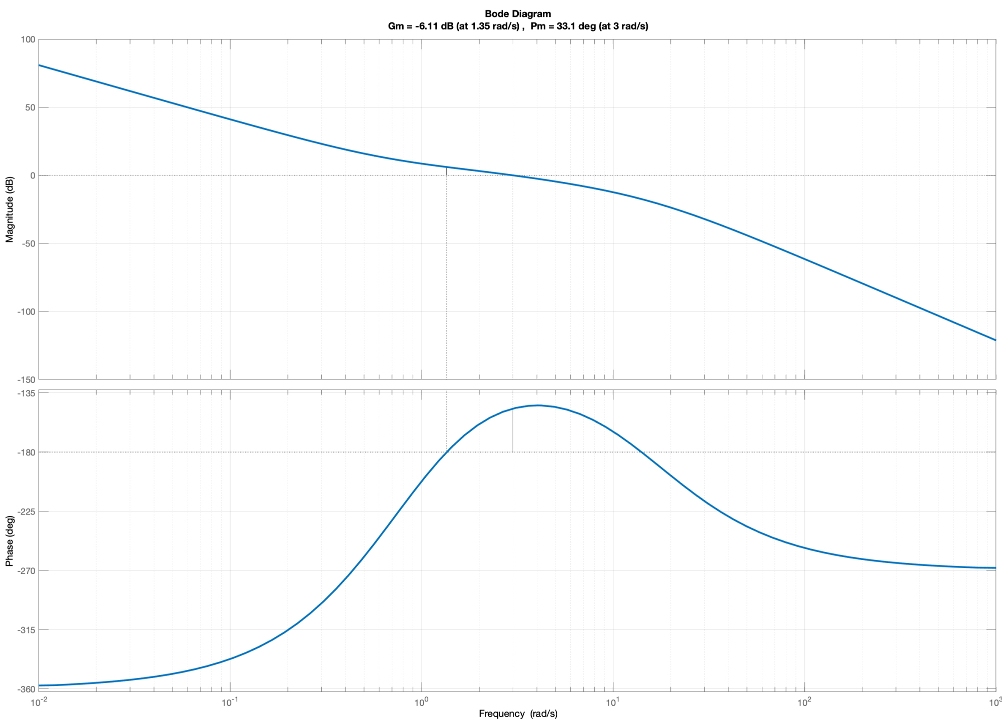

Figure 2: Bode plots of (2)

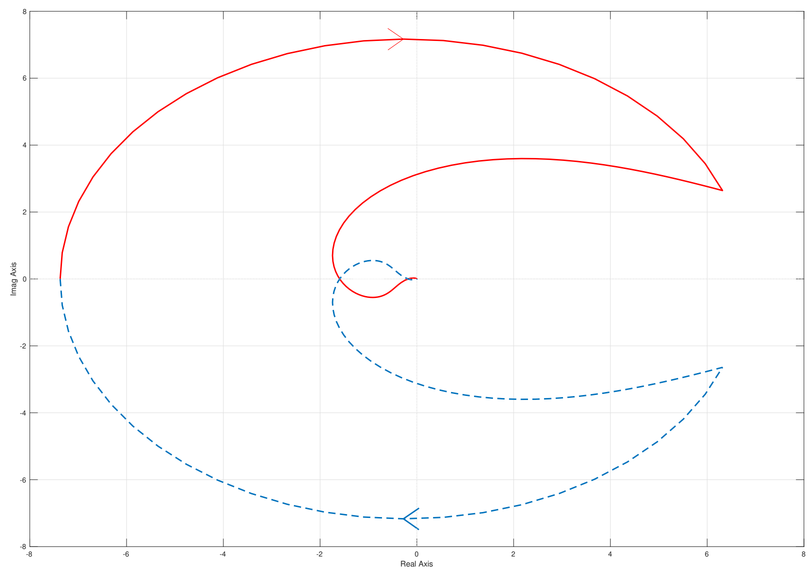

Figure 3: Nyquist plot of (2)

plots of

$$L(s) = G_2(s)G_1(s)P(s) = 1.124\Big(\frac{1+s}{1+\frac{1}{16s}}\Big)^2 \frac{1+\frac{5}{3}s}{1+\frac{5}{9}s} \frac{1}{s^2(s-1)} \tag{2}$$

are reported in Figure 2.

(i) The Nyquist plot of (2) is reported in Figure 3. The number of counter-clockwise encirclements of $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 1 that is also coincident with the open loop poles with positive real part. Thus, the system is asymptotically stable in closed loop.

Exercise 2 a) Denoting by n and m the number of poles and zeros of the transfer function, the relative degree of $P(s)$ is given by $r = n - m = 1$. Accordingly, the root locus possesses two asymptotes centered at

$$s_0 = \frac{1+5}{2} = 3$$

that can be discarded. Introducing $k \in \mathbb{R}$ and defining $p(s,k) = (s^2+1)(s-1)+k(s+5)$ as the polynomial defining the closed-loop poles under $G(s) = k$, one gets that singularities are the solutions to

$$p(s,k) = s^3 - s^2 + (k+1)s + 5k - 1 = 0$$
$$\frac{\partial p(s,k)}{\partial s} = 3s^2 - 2s + k + 1 = 0$$

By solving the equations above, it turns out that the negative locus possesses one sin-

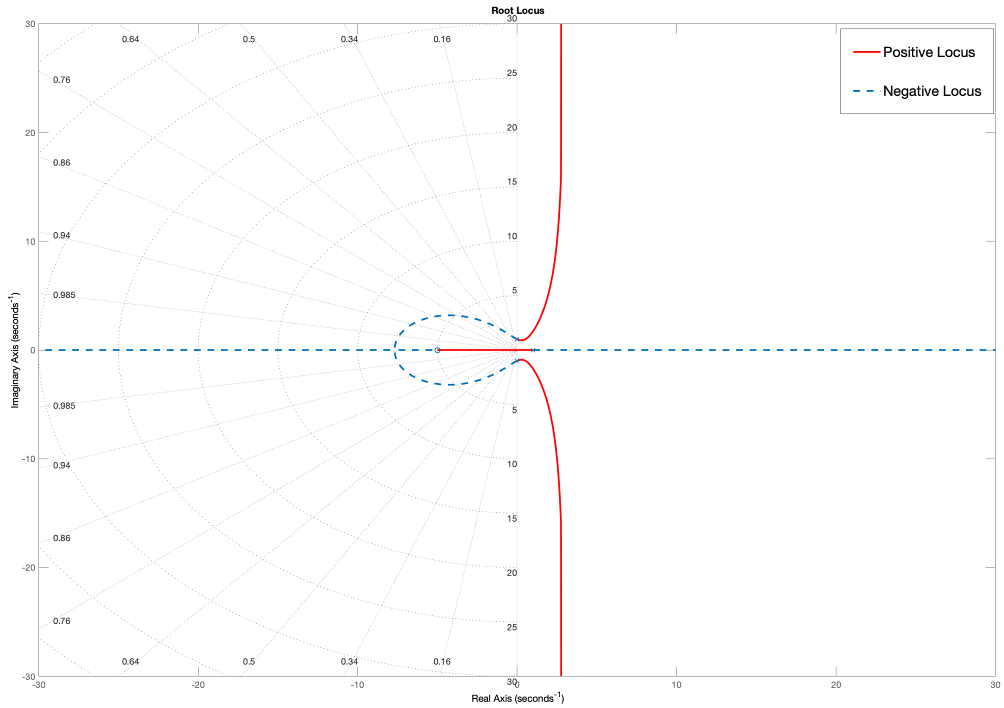

Figure 4: Root Locus of $P(s) = \frac{s+5}{(s-1)(s^2+1)}$.

gularity with multiplicity $\mu = 2$ in correspondence of $(s^*, k^*) \approx (-7.7, -194.27)$. What is left to do is now is to quantify the number of intersection of the root locus with the imaginary axis. Those intersecting points correspond to values of $k \in \mathbb{R}$ for which the Routh table of $p(s,k) = s^3 + (k+9)s^2 + (4k+14)s + 4k - 24$ is not regular. Thus, by developing computations one gets

$$\begin{array}{c|cc} r^3 & 1 & k+1 \\ r^2 & -1 & 5k-1 \\ r^1 & -6k & \\ r^0 & 5k-1. & \end{array}$$

The Routh table is not regular for $k = \frac{1}{5}$ and $k = 0$ so implying that the positive locus intersects the imaginary axis in correspondence of $k = \frac{1}{5}$ corresponding to the closed-loop pole $s = 0$ and at $k = 0$ corresponding to the open loop poles $s = \pm j$. The root locus is reported in Figure 4.

b) From the above root locus and the Routh table it is evident that there exists no controller $G(s) = k$ asymptotically stabilizing the feedback system.

c) For ensuring zero-steady state error to constant inputs, the controller $G(s)$ must possess a pole at $s = 0$. Thus, we set $p_1 = 0$ and, for the sake of notations, we shall denote hereinafter $p = p_2$. Thus, by denoting

$$L(s) = k\frac{(1+z_1 s)(1+z_2 s)(s+5)}{s(s^2+1)(s-1)(s+p)} = \hat{k}\frac{(s+\hat{z}_1)(s+\hat{z}_2)(s+5)}{s(s^2+1)(s-1)(s+p)}$$

one gets that a necessary condition for assigning the poles with real part smaller or equal than 3 is that the new center of the asymptotes satisfies

$$s_0' = \frac{3-p+\hat{z}_1+\hat{z}_2}{2} < -3.$$

Accordingly, $p, \hat{z}_1$ and $\hat{z}_2$ can be fixed as $p = 25, \hat{z}_1 = 3$ and $\hat{z}_2 = 4$ so getting $s_0' = -6$ and implying $z_1 = \frac{1}{3}$, $z_2 = \frac{1}{4}$ and $k = 12\hat{k}$. At this point, one can set $\hat{k} \in \mathbb{R}$ (or equivalently $k \in \mathbb{R}$) by invoking the extended Routh criterion. Namely, one sets $\hat{k}$ so to make the shifted closed-loop polynomial

$$\begin{aligned} p_L^*(s,\hat{k}) = p_L(s-3,\hat{k}) =&(s-3)(s^2-6s+10)(s-4)(s+22) + \hat{k}s(s+1)(s+2) \\ =&s^5 + 9s^4 + (\hat{k}-222)s^3 + (3\hat{k}+1266)s^2 + (2\hat{k}-3004)s + 2640 \end{aligned}$$

Hurwitz. By computing the Routh table

r^5	1	$\hat{k}-222$	$2\hat{k}-3004$
r^4	3	$\hat{k}+422$	880
r^3	$\hat{k}-544$	$3\hat{k}-4946$	
r^2	$\frac{\hat{k}^2-131\hat{k}-214730}{\hat{k}-544}$	880	
r^1	$\frac{(\hat{k}^3-2073\hat{k}^2+320392\hat{k}+267210300)}{(\hat{k}-544)^2}$		
r^0	880		

one gets the specification satisfied for $\hat{k} > 1815.4$.

Exercise 3 **(i)** For computing the forced response, one needs to rewrite the input

$$u(t) = \begin{cases} 1 - e^{t-1} \text{ as } t \in [0,1) \\ t-1 \text{ as } t \in [1,2) \\ 1 \text{ as } t \geq 2. \end{cases}$$

as the linear combination of elementary signals. Accordingly, one gets

$$u(t) = u_1(t) - e^{-1}u_2(t) - u_1(t-1) + u_2(t-1) + u_3(t-1) - u_3(t-2)$$

with

$$u_1(t) = 1_+, \quad u_2(t) = e^t_+, \quad u_3(t) = t_+.$$

Accordingly, as the system is time-invariant and linear, the output response can be computed as

$$y(t) = y_1(t) - e^{-1}y_2(t) - y_1(t-1) + y_2(t-1) + y_3(t-1) - y_3(t-2) \tag{3}$$

with

$$y_i(t) = \mathcal{L}^{-1}(P(s)u_i(s))[t], \quad u_i(s) = \mathrm{L}(u_i(t))[s], \quad i = 1, 2, 3.$$

In particular, one has

$$\begin{aligned}
y_1(t) &= K\mathcal{L}^{-1}(\frac{1}{s(1+s)})[t] = K\mathcal{L}^{-1}(\frac{1}{s})[t] - K\mathcal{L}^{-1}(\frac{1}{1+s})[t] = K(1_+ - e_+^{-t})\\
y_2(t) &= K\mathcal{L}^{-1}(\frac{1}{(s-1)(s+1)})[t] = \frac{K}{2}\mathcal{L}^{-1}(\frac{1}{s-1})[t] - \frac{K}{2}\mathcal{L}^{-1}(\frac{1}{s+1})[t] = \frac{K}{2}(e_+^{t} - e_+^{-t})\\
y_3(t) &= K\mathcal{L}^{-1}(\frac{1}{s^2(s+1)})[t] = K\mathcal{L}^{-1}(\frac{1}{s+1})[t] - K\mathcal{L}^{-1}(\frac{1}{s})[t] + \mathcal{L}^{-1}(\frac{1}{s^2})[t]\\
&\quad K(e_+^{-t} - 1_+ + t_+).
\end{aligned}$$

By substituting, after suitable time-shift, the above equalities in (3) one gets the result.

(ii) The system has a well-define steady-state response as it is asymptotically stable (all poles, that we assume also as eigenvalues, are with negative real part). The steady-state response can be computed starting from (3) by neglecting all terms whose effect is vanishing in time. Accordingly, one gets

$$y_{ss}(t) = K.$$

(iii) The settling time is defined as the time instant $T_s > 0$ for which the output response remains within 5% of its steady-state values for all $t \geq T_s$. Accordingly, by defining the transient response as $y_{\mathrm{tran}}(t) = y(t) - y_{ss}(t)$ one gets that K needs to be chosen so that, for $T_s \leq 10^{-3}$ and for all $t \geq T_s$

$$|y_{\mathrm{tran}}(t)| \leq 0.05|y_{ss}(t)|.$$

By rewriting $y_{\mathrm{tran}}(t) = K\bar{y}_{\mathrm{tran}}(t)$ and $y_{ss}(t) = K\bar{y}_{ss}(t)$ one gets that the above equality is independent upon K so that it is not possible to set the gain to decrease at will the settling time.