Control Systems
8/1/2019(A)

Exercise 1 In Laplace domain the disturbance and input-to-output responses are given by

$$y(s) = L(s)e(s) + d_2(s) + \frac{P_1(s)}{1+L_1(s)} d_1(s)$$

with

$$L(s) = G_1(s)P_2(s)P(s),$$
$$L_1(s) = G_1(s)P_1(s),$$
$$P(s) = \frac{L_1(s)}{1+L_1(s)}.$$

In order to meet requirements (ii) and (iii) set

$$G_1(s) = \frac{1}{s},\ G_2 = \frac{1}{s}\bar{G}(s)$$

with one-dimensional $\bar{G}(s)$ (recall that G_1 is required to be one dimensional and G_2 two-dimensional). Therefore

$$P(s) = \frac{2.1s+0.1}{s^2+1.1s0.1},\ L(s) = \frac{\bar{G}(s)}{s}\frac{1}{2.1s+0.1}\frac{2.1s+0.1}{s^2+1.1s0.1} = \frac{\bar{G}(s)}{s}\frac{10}{s(s+1)(1+10s)}.$$

From the Bode plot of $L(s) = P_2(s)P(s)$ (Fig. 1) we see that we have to increase the phase (to maximize the phase margin) using an anticipative+proportional action $\bar{G}(s) = KR_a(s) = K\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}$.

In order to maximize the phase margin, we choose $m_a = 16$ with $\omega_N = 4$ rad/sec (maximum phase value) at $\omega_t^* = 0.0001$ rad/sec (where the Bode plot of the phase of $L(s)$ is higher: actually, any $\omega_t^* \leq 0.0001$ is good as well). We obtain $\tau_a = 4/0.0001 = 4000$. Therefore, the anticipative action is $R_a(s) = \frac{1+4000s}{1+\frac{4000}{16}s}$. For colocating ω_t^* at 0.0001 rad/sec we see from the Bode plots of $L(s)\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}$ (Fig. 2) that we need a proportional attenuation $K =\approx -93dB = 2.23 * 10^{-5}$.

The controller $G_1(s)$ is given finally by

$$G_1(s) = \frac{2.23 * 10^{-5}}{s}\frac{1+4000s}{1+\frac{4000}{16}s}.$$

The Bode plot of $G_1(s)L(s)$ is drawn in Fig. 3 and shows that we have a crossover frequency $\omega_t^* = 10^{-3}$ rad/sec with a phase margin $m_\phi^* \approx 150°$. The Nyquist plot shows that the closed-loop system is asymptotically stable (we have 0 counterclockwise tours around the point $-1+0j$).

Exercise 2 (a) The root locus of $P(s) = \frac{s+3}{s(s-3)(s+10)^2}$ is drawn in Fig. 1.

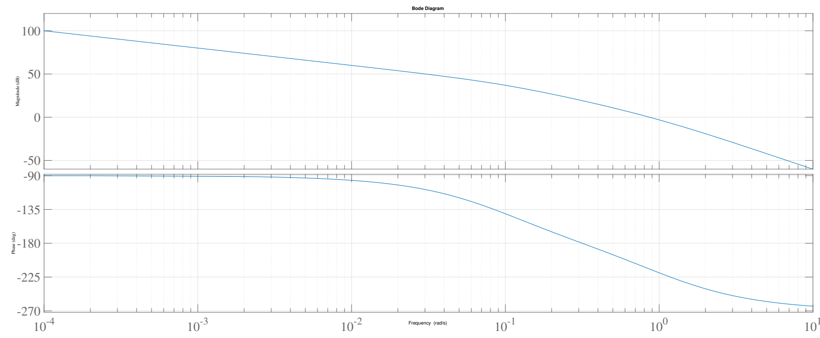

Figure 1: Bode plots of $L(s) = P_2(s)P(s)$

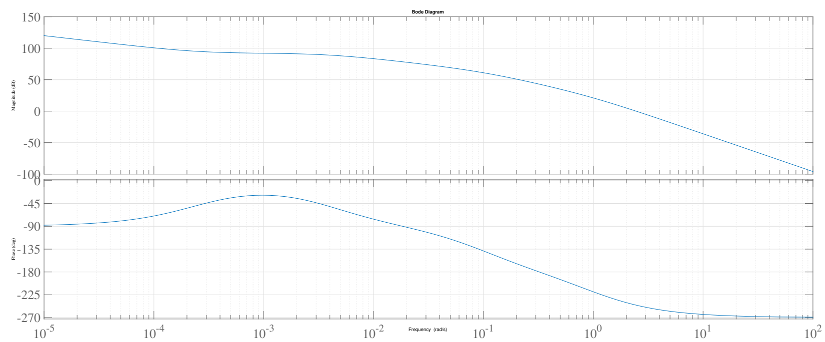

Figure 2: Bode plots of $L(s)R_a(s)$

The zero-pole excess is $n - m = 3$ and the asymptote center is at $s_0 = \frac{3-20+3}{3} = -\frac{14}{3} \approx 4.67$. The Routh table applied to $NUM(1 + KP(s)) = s^4 + 17s^3 + 40s^2 + (K - 300)s + 3K$ has the first column given by

$$\begin{array}{r} 1 \\ 17 \\ 980 - K \\ \dfrac{-K^2 + 413K - 294000}{980 - K} \\ 51K \end{array}$$

The number of sign variation in this column confirms the presence of 2 closed-loop poles with positive real part for $K > 0$ and 1 closed-loop pole with positive real part for $K < 0$. Moreover, neither locus crosses the imaginary axis. Therefore, there is no K such that the closed-loop system $W(s) = \frac{KP(s)}{1+KP(s)}$ is asymptotically stable (point (b)).

(c) It is required to find one dimensional $G(s)$ such that the closed-loop system $W(s) = \frac{G(s)P(s)}{1+G(s)P(s)}$ is asymptotically stable with poles having real part ≤ -2 and steady state error

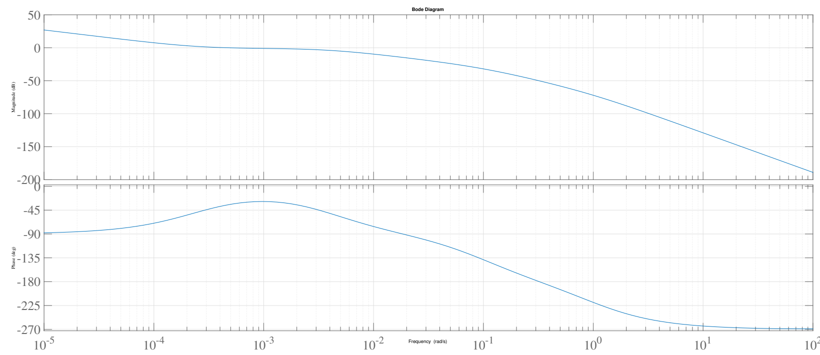

Figure 3: Bode plots of $G_1(s)L(s)$

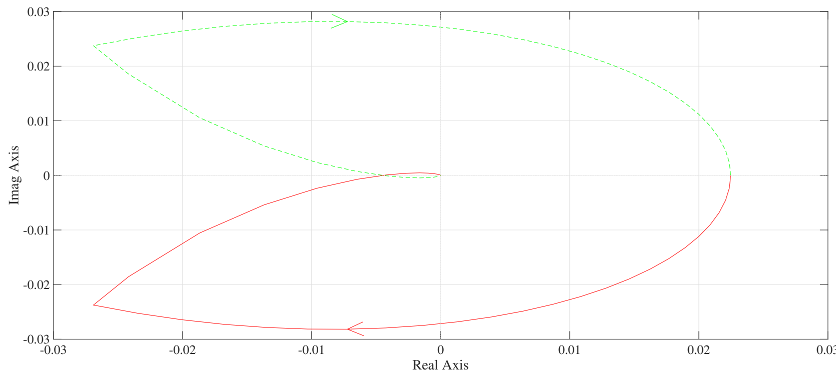

Figure 4: Nyquist plot of $G(s)P(s)$

to unit ramp input $|e_1| \leq 0.1$. Since the asymptote center is < -2 and the zeroes of $P(s)$ have real part < -2, we have only to decrease the zero-pole excess from 3 to 2 (keeping the asymptote center < -2) and then increase the gain to move the poles. Let

$$G(s) = \frac{G_1(s)}{1 + Ts}$$

with $G_1(s) = K(1 + \bar{z}s)$ and $z > 0$. Choose K in such a way that, whatever $\bar{z} > 0$ is,

$$|e_1| = \left|\frac{1}{s}W_e(s)\right|_{s=0} = \left|\frac{1}{s}\frac{1}{1+G(s)P(s)}\right|_{s=0} = \left|\frac{1}{s}\frac{1}{1+G_1(s)P(s)}\right|_{s=0}$$
$$= \left|\frac{(s-3)(s+10)^2}{K(1+\bar{z}s)(s+3)+(s-3)(s+10)^2 s}\right|_{s=0} = \left|\frac{100}{K}\right| \leq 0.1$$

which gives $|K| \geq 1000$. Next, noticing that

$$P(s)G_1(s) = K\bar{z}\frac{(s+\frac{1}{\bar{z}})(s+3)}{(s-3)(s+10)^2} = \bar{K}\frac{(s+\frac{1}{\bar{z}})(s+3)}{(s-3)(s+10)^2}$$

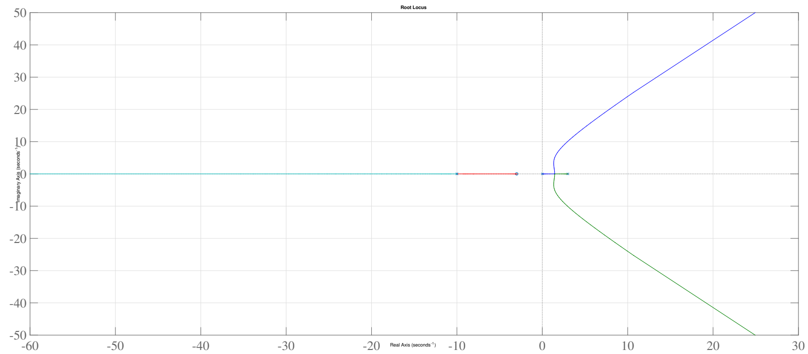

Figure 5: Positive root locus of $P(s)$

with $\bar{K} = K\bar{z}$, choose $\bar{z} > 0$ such that the new asymptote center s_0' remains < -2

$$s_0' = \frac{-14 + \frac{1}{\bar{z}}}{2} < -2 \Rightarrow \bar{z} > 0.1$$

Let's try the values $\bar{z} = 1/4$ and (tentatively large) $K = 10^3$. The Routh table applied to $NUM(1 + G_1(s)P(s)) = s^4 + 9s^3 + 212s^2 + 462s + 1140$ has the first column given by

$$\begin{array}{r} 1 \\ 3 \\ 241/3 \\ 1 \\ 570 \end{array}$$

which implies stability of the closed-loop $\frac{G_1(s)P(s)}{1+G_1(s)P(s)}$. However, $G_1(s)$ is not implemntable as such and we have to add the pole $\frac{1}{1+Ts}$ for obtaining the implementable controller

$$G(s) = \frac{G_1(s)}{1 + Ts}$$

Choose tentatively (small) $T = 10^{-4}$ and check through the Routh table, applied to $NUM(1+G(s)P(s))$, not to have sign variations in the first column.

(d) We seek a controller

$$G(s) = K\frac{(s+10)^2}{s+3}\frac{s+z}{s+p}\frac{1}{1+Ts}$$

where we are canceling as many stable poles and zeroes of $P(s)$ as possible. The closed-loop transfer function is

$$W(s) = \frac{L(s)}{1 + L(s)},\ L(s) = K\frac{s+z}{(s-3)(s+p)(1+Ts)} = \bar{K}\frac{s+z}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

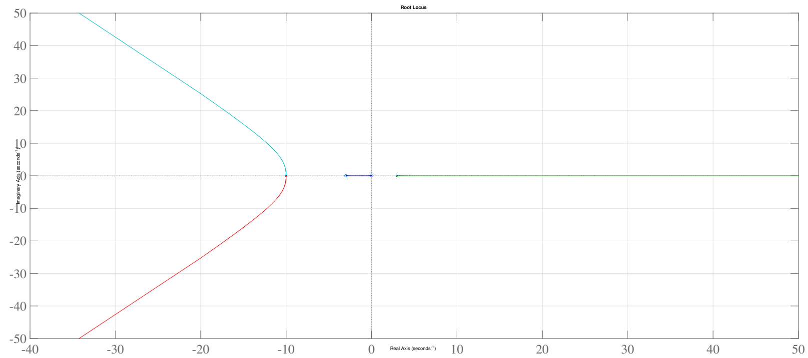

Figure 6: Negative root locus of $P(s)$

for $\bar{K} = \frac{K}{T}$ and for some $p_1, p_2, p_3, p_4 > 0$ such that

$$(s+p_1)(s+p_2)(s+p_3)(s+p_4) = s(s+\bar{T})(s+p)(s-3) + \bar{K}(s+z)$$

where $\bar{T}\frac{1}{T}$. In particular, we obtain by comparison from above

$$\begin{aligned} \bar{T} + p - 3 &= p_1 + p_2 + p_3 + p_4 \\ \bar{T}p - 3p - 3T &= p_3(p_1+p_2) + p_1p_2 + p_4(p_1+p_2+p_3) \\ K - 3\bar{T}p &= p_4(p_3(p_1+p_2) + p_1p_2) + p_1p_2p_3 \\ Kz &= p_1p_2p_3p_4 \end{aligned} \tag{1}$$

Since the output response in Laplace domain to a unit step input is

$$Y(s) = W(s)\frac{1}{s} = \bar{K}\frac{s+z}{s(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

we obtain in time

$$y(t) = \bar{K}[R_1 e_+^{-p_1 t} + R_2 e_+^{-p_2 t} + R_3 e_+^{-p_3 t} + R_4 e_+^{-p_4 t}\frac{z}{p_1p_2p_3p_4}\delta_{-1}(t)$$

with residuals

$$\begin{aligned} R_1 &= -\frac{z-p_1}{(p_2-p_1)(p_3-p_1)(p_4-p_1)p_1}, R_2 = -\frac{z-p_2}{(p_1-p_2)(p_3-p_2)(p_4-p_2)p_2} \\ R_3 &= -\frac{z-p_3}{(p_1-p_3)(p_2-p_3)(p_4-p_3)p_3}, R_4 = -\frac{z-p_4}{(p_1-p_4)(p_2-p_4)(p_3-p_4)p_4} \end{aligned}$$

The steady state output response is

$$y_{ss}(t) = \bar{K}\frac{z}{p_1p_2p_3p_4}$$

The transient output respons is

$$|y(t) - y_{ss}(t)| = |\bar{K}[R_1 e_+^{-p_1 t} + R_2 e_+^{-p_2 t} R_3 e_+^{-p_3 t} + R_4 e_+^{-p_4 t}]| \leq [|R_1| + |R_2| + |R_3| + |R_4|]e^{-\min_i p_i t}$$

We require that

$$|y(t) - y_{ss}(t)| \leq \frac{5}{100}|y_{ss}(t)|, \forall t \geq T_a = 20^{-2}.$$

We obtain the condition

$$e^{T_a \min_i p_i} \geq 25\frac{p_1p_2p_3p_4}{z}[|R_1| + |R_2| + |R_3| + |R_4|]$$
$$\Rightarrow T_a \min_i p_i \geq \ln(25\frac{p_1p_2p_3p_4}{z}[|R_1| + |R_2| + |R_3| + |R_4|]) \tag{2}$$

For instance, if $p_1 = z$ and $p_3 = p_4 + 1$, $p_2 = p_3 + 1$, $p_3 = p_2 + 1$, and setting $T_a = 20^{-2}$ in (2), we get

$$25 \cdot 10^{-4} p_4 \geq \ln 25 + \ln(3p_4 + 2) \tag{3}$$

from which $p_4 = 10^4$.

Exercise 3. The closed-loop I/O transfer function is

$$W(s) = \frac{K_d P(s)}{1 + K_d K_r P(s)} = \frac{K_d}{s + 1 + K_d K_r}$$

The steady state forced response to the input $v(t) = 1 - t = v_1(t) - v_2(t)$ with $v_1(t) = 1$ and $v_2(t) = t$

$$y_{ss}(t) = y_{ss,1}(t) - y_{ss,2}(t) = W(0) - (W(0)t + \frac{dW}{ds}|_{s=0}) = -W(0)t + (W(0) - \frac{dW}{ds}|_{s=0})$$

We must require that $y_{ss}(t) = 2t + 1$ which implies

$$-W(0) = 2,\ W(0) - \frac{dW}{ds}|_{s=0} = 1$$

i.e.

$$\frac{K_d}{1 + K_d K_r} = -2,\ \frac{K_d}{(1 + K_d K_r)^2} = 3$$

Moreover, for the existence of steady state regime we must require that the closed-loop is asymptotically stable, i.e. the closed-loop poles are in $\mathbb{C}^-$:

$$1 + K_d K_r > 0$$

From the first condition we obtain $K_d = 4/3$, $K_r = -5/4$ which however do not satisfy the second condition since $1 + K_d K_r = 1 - 20/12 < 0$. We conclude that there are no values of K_r and K_d for which we have the desired steady state output response.