Notes on Linear Control Systems: Module I

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***Abstract*—Nonlinear differential models. Linearization of nonlinear models. State and output solutions of a linear differential model: the matrix exponential. Natural modes and modal decomposition. Structural properties of natural modes: observability from the outputs and excitability with impulsive inputs.**

I. Introduction: the mathematical model

A *system* is the agglomerate of a certain number of interconnected elements, described by some *variables* which may vary during time. Some of these variables influence other ones. For this reason it is convenient to distinguish

- input (or independent) variables
- output (or dependent) variables

The input variables can be considered as *causes* and the output variables as *effects*. A system in which we specify the set of input and output variables is said *oriented*. For example the level of a fluid in a tank is an output variable and the incoming flow rate, regulated by a valve, is an input variable. By varying the incoming flow rate (cause) we vary the level of the tank (effect). On the other hand, if one or more customers are connected to the tank, the level of the tank may vary also as a consequence of the outside requests of fluid. The outside request of fluid is an input variable (cause) but it has a different nature from the incoming flow rate since the first may have an unpredictable variation during time. On this account, the input variables can be distinguished into

- controllable inputs (if it is possible to control or influence it)
- uncontrollable inputs (if it is not possible to control or influence it)

The uncontrollable inputs are also called *disturbances*. The requests of fluid in the tank from customers are a typical example of disturbance. The controllable inputs are usually transmitted to the system through *actuators* (for example the electric or pneumatic valve in the tank).

There are some outputs which we want to assume a desired behavior. To continue with the example of the tank, it may be requested that the level of the fluid in a tank remains constant in time. We say that these variables are *controlled* outputs.

On the other hand, if the task is to cause the controlled variables assume desired behaviors, it is important that some information (outputs) of the system be available in such a way to conceive a strategy (control) for achieving this task. In order that the level of the fluid in a tank remains constant in time, it can be useful to have information about the level itself. If the level of the tank is at some time below/beyond the desired constant value, it is necessary then to increase/decrease the incoming flow rate by regulating the opening of the valve. For this reasons, besides controlled outputs, we need to distinguish *measured* outputs. There may be some controlled outputs which are not measured: for example, we may want to control the velocity of a moving point by measuring its position. The measured outputs are usually obtained from the system through *sensors* (for example electric or digital devices which measure the level of the fluid in the tank). In summary, we distinguish the outputs into

- controlled outputs
- measured outputs

In general the set of the outputs (effects) consists of those variables which completely describe or characterize the behavior of the system as a consequence of the behavior of the inputs (causes). This set of variables is also known as *state* of the system.

A system can be represented as an agglomerate of blocks while the relations between its variables can be represented as the interconnections among these blocks.

A *mathematical model* is the description of a system through equations which determine the outputs as a function of the inputs. A good mathematical model is the compromise between the accurate mathematical description of the principles which characterize the system and the necessary approximations for which the model is "tractable" from the point of view of the available analysis techniques and computational methods.

A model may be of several types and different types of models can describe the same system. A *static* model can be adopted if the variation of the inputs is sufficiently slow with respect to the reaction times of the system or a sufficiently long time is elapsed so that the output variables have reached a steady-state condition. For static models the outputs are usually some functions of the inputs. The static models give no information on the behavior of the system during the period before reaching some steady-state conditions. In these cases it is better to adopt *dynamical* models, consisting of a certain number differential equations which establish the relations between the input and output variables. The system is assumed to be in some initial condition, in the sense that the variables have some initial values which do not change if no input (or no variation of the input) is applied to the system. When the inputs change, the output variables behaves according to the input change but also to the initial conditions of the system. If the (differential) equations describing the model are nonlinear (resp. linear), the model is said to be *nonlinear* (resp. linear).

Once a mathematical model is available, it is possible to study the mathematical properties of the model and the possibility of achieving a desired behavior of the controlled outputs by suitable input strategies or behaviors. The analysis

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and design methodologies for such strategies is the aim of these notes. We will consider only models for which the measured outputs coincide with the controlled outputs and we will study design methodologies which are applicable only to models with one input and one output.

II. From nonlinear to linear models

Consider the following nonlinear model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}(t),\mathbf{u}(t))\\ \mathbf{y}(t) &= h(\mathbf{x}(t),\mathbf{u}(t))\end{aligned} \tag{1}$$

where $t \geqslant 0$, $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the input vector and $\mathbf{y}(t)$ the output vector. We say that a solution of (1) is a triple $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)$ such that for $t \geqslant 0$

$$\begin{aligned}\dot{\mathbf{x}}(t)^* &= f(\mathbf{x}^*(t),\mathbf{u}^*(t))\\ \mathbf{y}(t)^* &= h(\mathbf{x}^*(t),\mathbf{u}^*(t))\end{aligned} \tag{2}$$

Let $\mathcal{X}\times\mathcal{U}\times\mathcal{Y}$ be the set of solutions $(\mathbf{x}(t),\mathbf{u}(t),\mathbf{y}(t))$ of (1). It is possible to study the solutions $(\mathbf{x}(t),\mathbf{u}(t),\mathbf{y}(t))$ of (1) with good approximation around some given $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)\in\mathcal{X}\times\mathcal{U}\times\mathcal{Y}$ through the *linearization* of (1) around $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)$. The analysis and control of the linearization of (1) is a powerful tool for studying and controlling (1) whenever its solutions $(\mathbf{x}(t),\mathbf{u}(t),\mathbf{y}(t))$ remain close to $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)$. The linearized model is obtained by linearizing the functions f and h around $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)$. The triple $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)$ is selected according to physical intuition or practical issues. For example, in the mathematical model of a spacecraft which is designed to track a certain orbit around the earth with constant angular velocity, for a local analysis of its motion along the orbit it is reasonable to linearize the model around a constant value of the angular velocity and a control input which forces the spacecraft to move exactly with constant angular velocity.

We can define a linearization of (1) around a given $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)\in\mathcal{X}\times\mathcal{U}\times\mathcal{Y}$ as follows. By considering the Taylor expansions of f and h around $(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{y}(t)^*)$ we get

$$\begin{aligned}f(\mathbf{x}(t),\mathbf{u}(t)) \;&=\; f(\mathbf{x}^*(t),\mathbf{u}^*(t))\\ &\quad+\frac{\partial f}{\partial x}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))}(\mathbf{x}(t)-\mathbf{x}^*(t))\\ &\quad+\frac{\partial f}{\partial u}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))}(\mathbf{u}(t)-\mathbf{u}^*(t))\\ &\quad+\mathbf{R}(\mathbf{x}(t)-\mathbf{x}^*(t),\mathbf{u}(t)-\mathbf{u}^*(t),t)\\ h(\mathbf{x}(t),\mathbf{u}(t)) \;&=\; h(\mathbf{x}^*(t),\mathbf{u}^*(t))\\ &\quad+\frac{\partial h}{\partial x}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))}(\mathbf{x}(t)-\mathbf{x}^*(t))\\ &\quad+\frac{\partial h}{\partial u}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))}(\mathbf{u}(t)-\mathbf{u}^*(t))\\ &\quad+\mathbf{S}(\mathbf{x}(t)-\mathbf{x}^*(t),\mathbf{u}(t)-\mathbf{u}^*(t),t)\end{aligned}$$

where $\mathbf{R}(\cdot,\cdot,t)$ and $\mathbf{S}(\cdot,\cdot,t)$ are higher order terms, i.e. such that for each $t \geqslant 0$

$$\lim_{\mathbf{x},\mathbf{u}\to 0}\frac{\|\mathbf{R}(\mathbf{x},\mathbf{u},t)\|}{\sqrt{\|\mathbf{x}\|^2+\|\mathbf{u}\|^2}}=0,\;\lim_{\mathbf{x},\mathbf{u}\to 0}\frac{\|\mathbf{S}(\mathbf{x},\mathbf{u},t)\|}{\sqrt{\|\mathbf{x}\|^2+\|\mathbf{u}\|^2}}=0$$

Set

$$\begin{aligned}\mathbf{z}(t) &:= \mathbf{x}(t)-\mathbf{x}^*(t),\\ \mathbf{v}^*(t) &:= \mathbf{u}(t)-\mathbf{u}^*(t),\\ \mathbf{w}(t) &:= \mathbf{y}(t)-h(\mathbf{x}^*(t),\mathbf{u}^*(t)).\end{aligned} \tag{3}$$

and

$$\begin{aligned}\mathbf{A}(t) &:= \frac{\partial f}{\partial x}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))},\\ \mathbf{B}(t) &:= \frac{\partial f}{\partial u}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))}\\ \mathbf{C}(t) &:= \frac{\partial h}{\partial x}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))},\\ \mathbf{D}(t) &:= \frac{\partial h}{\partial u}|_{(x,u)=(\mathbf{x}^*(t),\mathbf{u}^*(t))}\end{aligned} \tag{4}$$

On account of the above definitions and approximating f and h by taking only constant and first order terms in their Taylor expansions, from (1) we obtain

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{A}(t)\mathbf{z}(t)+\mathbf{B}(t)\mathbf{v}(t)\\ \mathbf{w}(t) &= \mathbf{C}(t)\mathbf{z}(t)+\mathbf{D}(t)\mathbf{v}(t)\end{aligned} \tag{5}$$

The linearized model (5) has state $\mathbf{z}(t)$, i.e. the displacement of the state $\mathbf{x}(t)$ from $\mathbf{x}^*(t)$, input $\mathbf{v}(t)$, i.e. the displacement of the input $\mathbf{u}(t)$ from $\mathbf{u}^*(t)$ and output $\mathbf{y}(t)$, i.e. the displacement of $\mathbf{y}(t)$ from $h(\mathbf{x}^*(t),\mathbf{u}^*(t))$. It is important to notice that, on account of fact that we are approximating f and h up to the first order terms in the Taylor expansion around $(\mathbf{x}^*(t),\mathbf{u}^*(t))$, $(\mathbf{z}(t),\mathbf{v}(t))$ is a valid approximation of $(\mathbf{x}(t)-\mathbf{x}^*(t),\mathbf{u}(t)-\mathbf{u}^*(t))$ to the extent $\|\mathbf{x}(t)-\mathbf{x}^*(t)\|+\|\mathbf{u}(t)-\mathbf{u}^*(t)\|$ remains sufficiently small.

A. A robotic platform

The model of a robotic platform is

$$\begin{aligned}\dot{\mathbf{x}}_1(t) &= \mathbf{v}(t)\cos\mathbf{x}_3(t)\\ \dot{\mathbf{x}}_2(t) &= \mathbf{v}(t)\sin\mathbf{x}_3(t)\\ \dot{\mathbf{x}}_3(t) &= \omega(t)\end{aligned} \tag{6}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the coordinates of the contact point P of the wheel on the ground, $\mathbf{x}_3$ is the angle formed by the wheel with the $\mathbf{x}_1$-axis, ω is the angular velocity of the robot around the vertical axis (z-axis) and $\mathbf{v}$ is the linear velocity of P (equal to the product of the wheel radius with its angular velocity). The input variables are $\mathbf{v}$ and ω and we set $\mathbf{u} = \begin{pmatrix}\mathbf{v} & \omega\end{pmatrix}^\top$. A reasonable choice of the solution $(\mathbf{x}^*(t),\mathbf{u}^*(t))$ is motivated by figuring out the robotic platform starting form the origin of the plane $(\mathbf{x}_1,\mathbf{x}_2)$, moving forward with constant velocity $\mathbf{v}^*$ of the wheels and rotating around its z-axis with constant velocity ω^*. Correspondingly, we select

$$\mathbf{u}^*(t) := \begin{pmatrix}v^* & \omega^*\end{pmatrix}^\top. \tag{7}$$

From (6) we get also $\mathbf{x}^*(t)$ which corresponds to apply an input $\mathbf{u}(t)=\mathbf{u}^*(t)$:

$$\mathbf{x}^*(t) = \begin{pmatrix}\frac{v^*}{\omega^*}\sin(\omega^* t)\\ -\frac{v^*}{\omega^*}\cos(\omega^* t)\\ \omega^* t\end{pmatrix} \tag{8}$$

The matrices of the linearized model around $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ are

$$\mathbf{A}(t) := \begin{pmatrix} 0 & 0 & -v_e \sin(\omega^* t) \\ 0 & 0 & v^* \cos(\omega^* t) \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{B}(t) := \begin{pmatrix} \cos(\omega^* t) & 0 \\ \sin(\omega^* t) & 0 \\ 0 & 1 \end{pmatrix}$$

The choice of the solution $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ around which we linearize is important for obtaining tractable linear models in the sense of being amenable to being controlled. By linearizing around

$$\begin{pmatrix} \mathbf{x}^*(t) \\ \mathbf{u}^*(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

which is another solution of (6) (i.e. null linear and angular velocities, platform still at the origin), we would have obtained

$$\mathbf{A}(t) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{B}(t) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

This linearized model has the drawback that we are not able to control the direction x_2 in the plane.

B. The simple pendulum

Consider a simple pendulum with a massless, inextensible and always taut rod. Assume also that motion occurs only in two dimensions, i.e. the bob at the end of the rod does not trace an ellipse but an arc. Let θ be the angular position of the rod (positive counterclockwise). The differential equation which represents the motion of a simple pendulum is

$$\ddot{\theta}(t) + \frac{g}{l} \sin \theta(t) + \frac{k}{m} \dot{\theta}(t) = 0 \tag{11}$$

where g is the gravity constant, m is the mass of the pendulum (concentrated in the bob) and k is the constant friction coefficient of the mean. By taking θ and $\dot{\theta}$ as state variables $\mathbf{x}_1$ and, respectively, $\mathbf{x}_2$ we obtain from (11)

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= -\frac{g}{l} \sin \mathbf{x}_1(t) - \frac{k}{m} \mathbf{x}_2(t) \end{aligned} \tag{12}$$

The state space model is nonlinear for the presence of $\sin \mathbf{x}_1$ and we have

$$f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{pmatrix}.$$

We look for constant solutions $\mathbf{x}^*(t) = x^*$ of (12), which are easily obtained from the equations:

$$\begin{aligned} x_2^* &= 0 \\ -\frac{g}{l} \sin x_1^* - \frac{k}{m} x_2^* &= 0 \end{aligned} \tag{13}$$

Therefore, we have two constant solutions which are $x^* = (0, 0)^\top$ and $x^* = (\pi, 0)^\top$. We have

$$\frac{\partial f}{\partial x}(x) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} cos x_1 & -\frac{k}{m} \end{pmatrix} \tag{14}$$

and the linearized model around $\mathbf{x}^*(t) = (0, 0)^\top$ is

$$\begin{aligned} \dot{\mathbf{z}}_1(t) &= \mathbf{z}_2(t) \\ \dot{\mathbf{z}}_2(t) &= -\frac{g}{l} \mathbf{z}_1(t) - \frac{k}{m} \mathbf{z}_2(t) \end{aligned} \tag{15}$$

where $\mathbf{z}_1(t) := \mathbf{x}_1(t)$ and $\mathbf{z}_2(t) := \mathbf{x}_2(t)$. On the other hand, the linearized model around $\mathbf{x}^*(t) = (\pi, 0)^\top$ is

$$\begin{aligned} \dot{\mathbf{z}}_1(t) &= \mathbf{z}_2(t) \\ \dot{\mathbf{z}}_2(t) &= \frac{g}{l} \mathbf{z}_1(t) - \frac{k}{m} \mathbf{z}_2(t) \end{aligned} \tag{16}$$

where $\mathbf{z}_1(t) := \mathbf{x}_1(t) - \pi$ and $\mathbf{z}_2(t) := \mathbf{x}_2(t)$.

C. The tank example

Consider a cylindrical tank designed to contain a certain quantity of fluid (with density ϱ) and to serve a local users' network. The horizontal section of the tank is S and the level of the fluid in the tank is h. The tank has an input fluid rate q_i, regulated by an input valve, and an output fluid rate q_o, determined uniquely by the users' network and unknown. The level h of the tank is also measured by suitable sensor devices. Since

$$\varrho S \dot{\mathbf{h}}(t) = \mathbf{q}_i(t) - \mathbf{q}_o(t) \tag{17}$$

the mathematical model of the tank is easily obtained as

$$\dot{\mathbf{x}}(t) = \frac{1}{\varrho S}(\mathbf{u}(t) - \mathbf{d}(t)), \mathbf{y}(t) = \mathbf{x}(t), \tag{18}$$

where the state is $\mathbf{x} = h$, the control input is $\mathbf{u} = q_i$, the regulated (and measured) output is $\mathbf{y} = h$ and $\mathbf{d} = q_o$ is a disturbance acting on the tank system.