Notes on Linear Control Systems: Module IV

Stefano Battilotti

***Abstract*—Stability and attractivity of equilibrium points. Lyapunov stability. Lyapunov functions. Lyapunov criteria for stability. Eigenvalues criterion for stability of linear systems. The Routh table. I/S and I/O stability. System performances versus eigenvalues and pole placement in the complex plane.**

I. Lyapunov stability

Let us consider a nonlinear differential system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \tag{1}$$

with initial state $\mathbf{x}_0 = x_0$ and assume $f(0) = 0$. The asymptotic behaviour of its solutions, i.e. when t tends to infinity, can be analyzed with respect to *constant* solutions x_e.

Definition *1.1: An equilibrium point x_e of (1) is any point $x_e \in \mathbb{R}^n$ such that $f(x_e) = 0$, i.e. a constant solution of (1).*

The system (1) has at least the equilibrium point $x_e := 0$. The equilibrium points may either appear as a continuum of points or isolated points.

Exercize *1.1:* The system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) := \mathbf{x}(t) - \mathbf{x}^2(t) \tag{2}$$

has two equilibrium points which correspond to the roots of $f(x) = 0$. This roots are $x_{e,1} = 0$ and $x_{e,2} = 1$, which are isolated points. On the hand, the equilibrium points of

$$\dot{\mathbf{x}}(t) = 0 \tag{3}$$

are all the points $x \in \mathbb{R}$, which is a continuum of points. $\lhd$

A *open neighbourhood* $U(x_e)$ of a point $x_e \in \mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n : \|x - x_e\| < \rho\}$ with $\rho > 0$.

Definition *1.2: An equilibrium point x_e of (1) is locally attractive if there exists a open neighbourhood $U(x_e)$ of x_e such that*

$$\lim_{t \to +\infty} \|\mathbf{x}(t, x_0) - x_e\| = 0 \tag{4}$$

for all $x_0 \in U(x_e)$. The point x_e is globally attractive if (4) holds for all $x_0 \in \mathbb{R}^n$.

Definition *1.3: An equilibrium point x_e of (1) is (Lyapunov) stable if*

$$\begin{aligned} &\forall \varepsilon > 0\ \exists \delta > 0 : \|x_0 - x_e\| < \delta \\ &\Rightarrow \|\mathbf{x}(t, x_0) - x_e\| < \varepsilon\ \forall t \geqslant 0 \end{aligned} \tag{5}$$

An equilibrium point x_e of (1) is

- *locally asymptotically (Lyapunov) stable if it is locally attractive and (Lyapunov) stable,*
- *globally asymptotically (Lyapunov) stable if it is globally attractive and (Lyapunov) stable.*
- *(Lyapunov) unstable if it is not (Lyapunov) stable.*

Notice that while local asymptotic stability implies stability, the converse is false. Property (5) means nothing but that $\mathbf{x}(t, x_0)$ remains for all times close to x_e if x_0 is close to x_e.

Proposition *1.1: The following properties are equivalent:*
(i) x_e is (Lyapunov) stable
(ii) there exist an increasing continuous function $\kappa : [0, +\infty) \to [0, +\infty)$ such that $\kappa(0) = 0$ and

$$\forall \varepsilon > 0 : \|x_0 - x_e\| < \kappa(\varepsilon) \Rightarrow \|\mathbf{x}(t, x_0) - x_e\| < \varepsilon\ \forall t \geqslant 0 \tag{6}$$

A longstanding criterion for studying the local and global stability of equilibrium points is the *Lyapunov criterion.* A function $V : \mathbb{R}^n \to \mathbb{R}$ is said

- positive definite (resp. semidefinite) at $x = x_e$ if $V(x) > 0$ for all $x \neq x_e$ in a neighbourhood of x_e (resp. $V(x) \geqslant 0$ for all x in a neighbourhood of x_e) and $V(x_e) = 0$.
- positive definite (resp. semidefinite) at $x = x_e$ on $\mathbb{R}^n$ if $V(x) > 0$ for all $x \neq x_e$ (resp. $V(x) \geqslant 0$ for all x) and $V(x_e) = 0$.
- negative definite (resp. semidefinite) at $x = x_e$ if $-V(x)$ is positive definite (resp. semidefinite) at $x = x_e$.
- negative definite (resp. semidefinite) at $x = x_e$ on $\mathbb{R}^n$ if $-V(x)$ is positive definite (resp. semidefinite) at $x = x_e$ on $\mathbb{R}^n$.

Proposition *1.2: Let x_e be an equilibrium point. If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that:*

- *V is positive definite at $x = x_e$, $\frac{\partial V}{\partial x}(x) f(x)$ is negative semidefinite at $x = x_e$ then x_e is stable.*
- *V is positive definite at $x = x_e$, $\frac{\partial V}{\partial x}(x) f(x)$ is negative definite at $x = x_e$ then x_e is locally asymptotically stable.*
- *V is positive definite at $x = x_e$ on $\mathbb{R}^n$, $\frac{\partial V}{\partial x}(x) f(x)$ is negative definite at $x = x_e$ on $\mathbb{R}^n$ and $\lim_{\|x\| \to +\infty} V(x) = +\infty$ then x_e is globally asymptotically stable.*

If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, positive definite at $x = x_e$, such that

- *$\frac{\partial V}{\partial x}(x) f(x)$ is negative semidefinite at x_e then x_e is (Lyapunov) stable.*
- *$\frac{\partial V}{\partial x}(x) f(x)$ is negative definite at $x = x_e$ then then x_e is locally asymptotically stable.*
- *$\frac{\partial V}{\partial x}(x) f(x)$ is negative definite at $x = x_e$ on $\mathbb{R}^n$ and $\lim_{\|x\| \to +\infty} V(x) = +\infty$ then x_e is globally asymptotically stable.*

The functions $V(x)$ satisfying the above conditions are called *Lyapunov functions.*

S. Battilotti is with Department of Computer, Control, and Management Engineering "Antonio Ruberti", Sapienza University of Rome, Via Ariosto 25, Italy.

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Exercize *1.2:* For the system (2) consider the equilibrium point $x_{e,1} = 1$ and the (candidate) Lyapunov function $V(x) = (x-1)^2$, which is positive definite at $x = 1$ on $\mathbb{R}$. We have

$$\frac{\partial V}{\partial x}(x)f(x) = 2(x-1)(x-x^2) = -2x(x-1)^2 := a(x) \quad (7)$$

Notice that $a(x)$ is negative definite at $x = 1$. It follows that $x_{e,1} = 1$ is locally asymptotically stable. However, $x_{e,1} = 1$ cannot be globally asymptotically stable, since any initial condition $\mathbf{x}_0 < 0$ in (2) generates a solution $\mathbf{x}(t, x_0)$ such that $\mathbf{x}(t, x_0) \to -\infty$ as $t \to \infty$.

If we consider the equilibrium point $x_{e,2} = 0$, we find out that it is not (Lyapunov) stable. Indeed, as alredy mentioned, any initial condition $\mathbf{x}_0 < 0$ in (2) generates a solution $\mathbf{x}(t)(x_0)$ such that $\mathbf{x}(t)(x_0) \to -\infty$ as $t \to \infty$.

Exercize *1.3:* Consider the simple pendulum with no friction ($k = 0$)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) = \begin{pmatrix} \mathbf{x}_{2,t} \\ -\frac{g}{l}\sin(\mathbf{x}_{1,t}) \end{pmatrix} \quad (8)$$

and a Lyapunov function

$$V(x) = \frac{1}{2}x_2^2 + \frac{g}{l}(1-\cos(x_1)). \quad (9)$$

$V(x)$ is positive definite at $x = 0$ (over the domain $-2\pi < x_1 < 2\pi$) and

$$\frac{\partial V}{\partial x}(x)f(x) = 0. \quad (10)$$

which is negative semidefinite at $x = 0$. We conclude that $x_e = 0$ is Lyapunov stable. If we consider non-zero friction ($k \neq 0$) we take

$$V(x) = \frac{1}{2}x^\top Px + \frac{g}{l}(1-\cos(x_1)) \quad (11)$$

where

$$P = \begin{pmatrix} \frac{k^2}{2m^2} & \frac{k}{2m} \\ \frac{k}{2m} & 1 \end{pmatrix} \quad (12)$$

It can be seen that $V(x)$ is positive definite at $x = 0$. Moreover,

$$\frac{\partial V}{\partial x}(x)f(x) = -\frac{1}{2}\frac{g}{l}\frac{k}{m}x_1\sin(x_1) - \frac{1}{2}\frac{k}{m}x_2^2 \quad (13)$$

which is negative definite at $x = 0$. We conclude that $x_e = 0$ is locally asymptotically stable. $\lhd$

Next, consider as a particular case of (1) the linear system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \quad (14)$$

The equilibrium points x_e of (14) are the solutions of $Ax_e = 0$, i.e. $x_e \in \mathrm{Ker}A$ which is a vector subspace of $\mathbb{R}^n$. Therefore, for a linear system there cannot be any isolated equilibrium point other than $x_e = 0$.

Since the solution $\mathbf{x}(t, x_0)$ of (14) is a linear combination of modes, the following facts hold true:

- If all the modes of (14) are convergent then $\lim_{t\to+\infty}\|\mathbf{x}(t, x_0)\| = 0$ for all $x_0 \in \mathbb{R}^n$ and, therefore, the equilibrium point $x_e := 0$ of (14) is globally attractive.
- if the equilibrium point $x_e := 0$ of (14) is globally attractive then $\lim_{t\to+\infty}\|\mathbf{x}(t, x_0)\| = 0$ and all the modes of (14) are convergent.
- if all the modes of (14) are convergent, $x_e := 0$ is the only equilibrium point and A is nonsingular.

By summing up,

Proposition *1.3: The equilibrium point $x_e := 0$ of (14) is globally attractive if and only if all the modes of (1) are convergent.*

Since a mode associated to a certain eigenvalue λ of A is convergent if and only if $\mathrm{Re}(\lambda) < 0$ we obtain the following important conclusion.

Proposition *1.4: The equilibrium point $x_e := 0$ of (14) is globally attractive if and only if $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$.*

This, together with Proposition 1.3, gives:

Proposition *1.5: The equilibrium point $x_e := 0$ of (14) is globally asymptotically stable if and only if $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$.*

This follows from the fact that the solution $\mathbf{x}(t, x_0) = e^{At}x_0$ is linear with respect to x_0.

Since only the equilibrium point $x_e := 0$ of (14) can be globally asymptotically stable, in this case we simply say that (14) itself is (globally) asymptotically stable.

For an asymptotically stable linear system (14) typical Lyapunov functions are quadratic: $V(x) = x^\top Px$, where P is a $(n \times n)$ positive definite matrix. A positive definite matrix P is a $(n \times n)$ is a matrix such that $x^\top Px \geqslant 0$ for all $x \in \mathbb{R}^n$. Notice that that if P is positive definite then $V(x) = x^\top Px$ is positive definite at $x = 0$ on $\mathbb{R}^n$. Moreover, $\lim_{\|x\|\to+\infty} V(x) = +\infty$. We have

$$\frac{\partial V}{\partial x}(x)Ax = 2x^\top PAx = x^\top(PA + A^\top P)x \quad (15)$$

If

$$PA + A^\top P = -Q \quad (16)$$

with Q a $(n \times n)$ positive definite matrix then

$$\frac{\partial V}{\partial x}(x)Ax = -x^\top Qx := a(x) \quad (17)$$

where $a(x)$ is negative definite at $x = 0$. We have the following important conclusion.

Proposition *1.6: The equilibrium point $x_e := 0$ of (14) is globally asymptotically stable if and only if for each $(n \times n)$ positive definite matrix Q there exist a $(n \times n)$ positive definite matrix P such that*

$$PA + A^\top P = -Q. \quad (18)$$

This is the Lyapunov criterion for asymptotic stability of linear systems. In this case, $V(x) = x^\top Px$ is a Lypunov function for (14).

Exercize *1.4:* Consider the linearization of (8) around the equilibrium point $x_e = (0, 0)^\top$:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) = \begin{pmatrix} \mathbf{x}_{2,t} \\ -\frac{g}{l}\mathbf{x}_{1,t} - \frac{k}{m}\mathbf{x}_{2,t} \end{pmatrix} \quad (19)$$

and assume $\frac{k^2}{m^2} > \frac{4g}{l}$. The system (19) is (globally) asymptotically stable since the eigenvalues of A are all negative real. A (candidate) Lyapunov function for (19) is

$$V(x) = x^\top Px, \; P := \begin{pmatrix} \frac{5k}{4m}\frac{g}{l} & \frac{g}{l} \\ \frac{g}{l} & \frac{k}{4m} \end{pmatrix} \quad (20)$$

Indeed

$$PA + A^\top P = -Q = -2\begin{pmatrix} \frac{g^2}{l^2} & 0 \\ 0 & \frac{k^2}{4m^2} - \frac{g}{l} \end{pmatrix} . \lhd \tag{21}$$

In conclusion, we can state also a criterion of stability for an equilibrium point x_e of (1) from the the stability of its linearization around x_e.

Proposition *1.7: Consider the nonlinear system (1) and let x_e be an equilibrium point. If the linearization of (1) around x_e is globally asymptotically stable, then x_e is locally asymptotically stable for (1).*

Exercize *1.5:* Since (19) is (globally) asymptotically stable, then the equilibrium point $x_e = (0,0)^\top$ of the simple pendulum (8) is locally asymptotically stable (but not globally asymptotically stable since we have another equilibrium point $x_e = (\pi, 0)^\top$). $\lhd$

Proposition *1.8: Consider the nonlinear system (1) and let x_e be an equilibrium point. If the linearization of (1) around x_e is unstable for the presence of eigenvalues with positive real part, then x_e is unstable for (1).*

Exercize *1.6:* Since the linearization of (8) around the equilibrium point $x_e = (\pi, 0)^\top$ has one positive eigenvalue, then the equilibrium point $x_e = (\pi, 0)^\top$ of the simple pendulum (8) is unstable. $\lhd$

II. Stability tests: the Routh criterion

As we mentioned in the previous section, asymptotic stability of a linear system (14) can be assessed by the eigenvalues of A. However, it is not always possible to calculate exactly the eigenvalues of A since they are roots of a n-degree polynomial

$$\mathbf{p}(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{22}$$

On the other hand, for stability it is sufficient to check if all the eigenvalues of A have negative real parts, rather than determine exactly their values. We will say that $\mathbf{p}(\lambda)$ is *Hurwitz* if all its roots have negative real parts. A first easy *necessary condition* for assuring if a polynomial is Hurwitz is the following.

Proposition *2.1: If $\mathbf{p}(\lambda)$ is Hurwitz then $a_n a_i > 0$ for all $i = 0, 1, \ldots, n-1$, i.e. all the coefficients a_i are nonzero and have the same sign.*

Proof. Assume that $\mathbf{p}(\lambda)$ is Hurwitz. The polynomial $\mathbf{p}(\lambda)$ can be factorized as

$$K\prod_{i=1}^{r}(1+\lambda\tau_i)^{\mu_i}\prod_{i=1}^{s}(1+2\zeta_i\frac{\lambda}{\omega_{n,i}} + \frac{\lambda^2}{\omega_{n,i}^2})^{\nu_i}$$

where $\tau_i := -\frac{1}{\lambda_i} > 0$ for real roots $\lambda_i < 0$ and $\omega_{n,i} := \sqrt{\alpha_i^2 + \omega_i^2}$ and $\zeta_i := -\frac{\alpha_i}{\omega_{n,i}} > 0$ for pairs of complex conjugate roots $\mu_i = \alpha_i \pm j\omega_i$ with $\alpha_i < 0$. The coefficients of each term $1 + \lambda\tau_i$ and $1 + 2\zeta_i\frac{\lambda}{\omega_{n,i}} + \frac{\lambda^2}{\omega_{n,i}^2}$ have the same sign. Therefore, the coefficients of the products of these terms must have the same sign. This prove the proposition. $\square$

The converse of proposition (2.1) does not hold, unless $n = 1$ or $n = 2$.

Proposition *2.2: If either $n = 1$ or $n = 2$, $\mathbf{p}(\lambda)$ is Hurwitz if and only if $a_n a_i > 0$ for all $i = 0, 1, \ldots, n-1$.*

Sufficient and necessary conditions for the roots of a polynomial $\mathbf{p}(\lambda)$ being all with negative real part can be obtained from the so-call Routh criterion. This criterion is based on the construction of a table (the *Routh table*) as follows.

Step (I): construction of the n-th and $(n-1)$-th rows. Let $r^{(n)}$ be the row of the coefficients a_j's in (22) which correspond to powers $\lambda^n, \lambda^{n-2}, \ldots$ and $r^{(n-1)}$ be the row of the coefficients a_j's in (22) which correspond to powers $\lambda^{n-1}, \lambda^{n-3}, \ldots$

$$r^{(n)} = \begin{pmatrix} a_n & a_{n-2} & a_{n-4} & \cdots \end{pmatrix}$$
$$r^{(n-1)} = \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \end{pmatrix}$$

Note that $r^{(n)}$ may have one element more than $r^{(n-1)}$ (n is even) or it may have the same number of elements (n is odd). Set $k \to n-2$.

Step (II): construction of the k-th row.

If $(k \geqslant 0) \& (r_1^{(k+1)} > 0)$ **then**

$$\gamma := \frac{r_1^{(k+2)}}{r_1^{(k+1)}};$$

$$\textbf{if } k \text{ is odd } \textbf{then } \mu := \frac{k+1}{2}$$

$$\textbf{else } \mu := \frac{k}{2} + 1;$$

else goto (III). (23)

If $r^{(k+1)}$ has one element less than $r^{(k+2)}$ **then** complete $r^{(k+1)}$ with one zero so that $r^{(k+2)}$ and $r_{j+1}^{(k+1)}$ have the same number of elements.

For $j = 1, \ldots, \mu$ **repeat** $r_j^{(k)} = r_{j+1}^{(k+2)} - \gamma r_{j+1}^{(k+1)}$;

Set $k \to k-1$ and **goto** (II).

Step (III): **end**.

Note that at each step $r^{(k+2)}$ we may have one element more than $r^{(k+1)}$ (k is even) or it may have the same number of elements (k is odd). Each element $r_j^{(k)}$ of a row $r^{(k)}$ can be calculated alternatively as

$$r_j^{(k)} := -\frac{1}{r_1^{(k+1)}}\det\begin{pmatrix} r_1^{(k+2)} & r_{j+1}^{(k+2)} \\ r_1^{(k+1)} & r_{j+1}^{(k+1)} \end{pmatrix}$$

Notice also that the algorithm stops at step k if $r_1^{(k+1)} = 0$. It is also possible to simplify a row $r^{(k)}$ by replacing it with $\alpha r^{(k)}$ where α is any positive number.

The Routh table is said to be *regular* if the numbers $r_1^{(j)}$, $j = 0, \ldots, n$ are all nonzero. We will say that there is a *permanency* between $r_1^{(k)}$ and $r_1^{(k+1)}$ if $r_1^{(k)} r_1^{(k+1)} > 0$. Otherwise we will say that there is a *variation*. We will denote by $N_V(\mathbf{p})$ and $N_P(\mathbf{p})$ the number of variations and, respectively, of permanencies in the Routh table generated by the polynomial $\mathbf{p}(\lambda)$. Clearly, for a polynomial $\mathbf{p}(\lambda)$ with a regular Routh table we have $N_V(\mathbf{p}) + N_P(\mathbf{p}) = n$. Also, denote by $N_-(\mathbf{p})$ the number of roots of $\mathbf{p}(\lambda)$ with negative real part, by $N_+(\mathbf{p})$ the number of roots of $\mathbf{p}(\lambda)$ with positive real part and by $N_0(\mathbf{p})$ the number of roots of $\mathbf{p}(\lambda)$ with null real part.

Proposition *2.3: If a polynomial $\mathbf{p}(\lambda)$ generates a regular Routh table then $N_V(\mathbf{p}) = N_+(\mathbf{p})$, $N_P(\mathbf{p}) = N_-(\mathbf{p})$ and $N_0(\mathbf{p}) = 0$.*

A consequence of the above proposition is that a sufficient condition for $\mathbf{p}(\lambda)$ being Hurwitz is that $\mathbf{p}(\lambda)$ generates a regular Routh table and $N_V(\mathbf{p}) = 0$. This sufficient condition is actually also necessary.

Theorem *2.1: A polynomial $\mathbf{p}(\lambda)$ is Hurwitz if and only if $\mathbf{p}(\lambda)$ generates a regular Routh table and $N_V(\mathbf{p}) = 0$.*

Note that theorem 2.3 implies that if $\mathbf{p}(\lambda)$ has at least one root with null real part then the Routh table cannot be regular (for example $\mathbf{p}(\lambda) = \lambda^2 + 1$). On the other hand, if the Routh table is not regular this does not necessarily imply that $\mathbf{p}(\lambda)$ has at least one root with null real part (for example $\mathbf{p}(\lambda) = \lambda^2 - 1$).

Exercize *2.1: Discuss the sign of the roots of $\mathbf{p}(\lambda) = \lambda^5 + 3\lambda^4 + 2\lambda^3 - 2\lambda^2 + 2\lambda + 4$.*

Let us construct the Routh table for $\mathbf{p}(\lambda)$. The rows $r^{(5)}$ and $r^{(4)}$ are

$$\begin{array}{r|ccc} r^{(5)} & 1 & 2 & 2 \\ r^{(4)} & 3 & -2 & 4 \end{array} \tag{24}$$

The row $r^{(3)}$ is calculated as

$$r_j^{(3)} := -\frac{1}{r_1^{(k+1)}} \det \begin{pmatrix} r_1^{(k+2)} & r_{j+1}^{(k+2)} \\ r_1^{(k+1)} & r_{j+1}^{(k+1)} \end{pmatrix}, \ j = 1, \ldots, \mu$$

or equivalently

$$r_j^{(3)} = r_{j+1}^{(5)} - \gamma r_{j+1}^{(4)}, \ j = 1, \ldots, \mu \tag{25}$$

where $\gamma = \frac{1}{3}$ and $\mu = 2$ (since the index of the row to be constructed is odd). We obtain

$$\begin{array}{r|ccc} r^{(5)} & 1 & 2 & 2 \\ r^{(4)} & 3 & -2 & 4 \\ r^{(3)} & \frac{8}{3} & \frac{2}{3} & \end{array} \tag{26}$$

Similarly,

$$\begin{array}{r|ccc} r^{(5)} & 1 & 2 & 2 \\ r^{(4)} & 3 & -2 & 4 \\ r^{(3)} & \frac{8}{3} & \frac{2}{3} & \\ r^{(2)} & -\frac{11}{4} & 4 & \end{array} \tag{27}$$

and finally

$$\begin{array}{r|ccc} r^{(5)} & 1 & 2 & 2 \\ r^{(4)} & 3 & -2 & 4 \\ r^{(3)} & \frac{8}{3} & \frac{2}{3} & \\ r^{(2)} & -\frac{11}{4} & 4 & \\ r^{(1)} & \frac{50}{11} & & \\ r^{(0)} & 4 & & \end{array} \tag{28}$$

Therefore, the Routh table is regular. Moreover, $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 3$. By the Routh criterion $\mathbf{p}(\lambda)$ has three roots with negative real part and two roots with positive real part.

It is worth noting that in this case the necessary condition 2.1 can be used since $a_5 = 1$ and $a_2 = -2$ and $a_2 a_5 < 0$ and we can conclude that $\mathbf{p}(\lambda)$ is not Hurwitz. $\lhd$

Exercize *2.2: Discuss the sign of the roots of $\mathbf{p}(\lambda) = -2\lambda^3 - \lambda^2 - 4\lambda - 11$.*

The Routh table generated by $\mathbf{p}(\lambda)$ is

$$\begin{array}{r|cc} r^{(3)} & -2 & -4 \\ r^{(2)} & -1 & -11 \\ r^{(1)} & 18 & \\ r^{(0)} & -11 & \end{array} \tag{29}$$

Therefore, the Routh table is regular. Moreover, $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 1$. By theorem 2.3 $\mathbf{p}(\lambda)$ has one (real) negative root and two roots with positive real part. It is worth noting that in this case the necessary condition 2.1 cannot be used since $a_3 a_i > 0$ for all i. $\lhd$

Exercize *2.3: Discuss the sign of the roots of $\mathbf{p}(\lambda) = \lambda^5 + \lambda^4 + 2\lambda^3 + 2\lambda^2 + 3\lambda + 15$.*

The Routh table generated by $\mathbf{p}(\lambda)$ is

$$\begin{array}{r|ccc} r^{(5)} & 1 & 2 & 3 \\ r^{(4)} & 1 & 2 & 15 \\ r^{(3)} & 0 & -12 & \end{array} \tag{30}$$

Therefore the Routh table is not regular since $r_1^{(3)} = 0$ and by theorem 2.1 it follows that $\mathbf{p}(\lambda)$ is not Hurwitz. It is worth noting that also in this case the necessary condition 2.1 cannot be used since $a_5 a_i > 0$ for all i.

Exercize *2.4: Discuss the sign of the roots of $\mathbf{p}(\lambda) = \lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda + K$ for K varying over $(-\infty, \infty)$.*

The Routh table generated by $\mathbf{p}(\lambda)$ is

$$\begin{array}{r|ccc} r^{(4)} & 1 & 11 & K \\ r^{(3)} & 6 & 6 & \\ r^{(2)} & 10 & K & \\ r^{(1)} & \frac{6}{10}(10-K) & & \\ r^{(0)} & K & & \end{array} \tag{31}$$

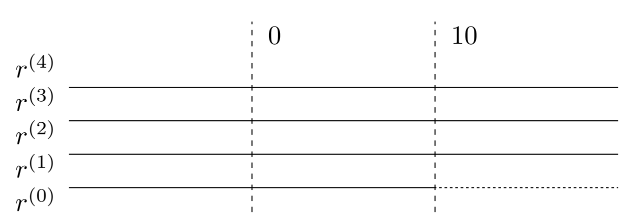
Note that we could have simplified the Routh table by multiplying $r^{(1)}$ by $\frac{10}{6}$ without altering the regularity of the table and the discussion of the sign of the roots of $\mathbf{p}(\lambda)$

$$\begin{array}{r|ccc} r^{(4)} & 1 & 11 & K \\ r^{(3)} & 6 & 6 & \\ r^{(2)} & 10 & K & \\ r^{(1)} & 10-K & & \\ r^{(0)} & K & & \end{array} \tag{32}$$

We can discuss the number of variations and permanencies in the first column of the Routh table as follows. First, we discuss the sign of each $r_1^{(j)}$, $j = 0, \ldots, 4$:

- $r_1^{(0)} = 0$ for $K = 0$ and $r_1^{(1)} = 0$ for $K = 10$
- $r_1^{(4)}, r_1^{(3)}$ and $r_1^{(2)}$ are positive for all K
- $r_1^{(1)} > 0$ for $K < 10$
- $r_1^{(0)} > 0$ for $K > 0$

These results can be visualized in the following table. We will draw a full line if the sign of $r_1^{(j)}$, $j = 0, \ldots, 4$ is positive and a dashed line if its sign is negative:



A variation in the first column of the Routh table corresponds to a variation of line (full or dashed) in the above table. Therefore, we have

- for $K = 0$ or $K = 10$ the table is not regular
- for $K < 0$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 3$
- for $K \in (0, 10)$ the table is regular and $N_V(\mathbf{p}) = 0$ and $N_P(\mathbf{p}) = 4$
- for $K > 10$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 2$

We conclude by virtue of theorem 2.3

- for $k = 0$ or $k = 10$ the table in not regular $\Rightarrow \mathbf{p}(\lambda)$ is not Hurwitz
- for $K < 0$ the table is regular and $N_+(\mathbf{p}) = 1$ and $N_-(\mathbf{p}) = 3 \Rightarrow \mathbf{p}(\lambda)$ is not Hurwitz
- for $K \in (0, 10)$ $N_+(\mathbf{p}) = 0$ and $N_-(\mathbf{p}) = 4 \Rightarrow \mathbf{p}(\lambda)$ is Hurwitz
- for $K > 10$ $N_+(\mathbf{p}) = 2$ and $N_-(\mathbf{p}) = 2 \Rightarrow \mathbf{p}(\lambda)$ is not Hurwitz.

$\lhd$

Exercize *2.5: Discuss the sign of the roots of* $\mathbf{p}(\lambda) = \lambda^4 + \lambda^3 + 2\lambda^2 + (1 + K)\lambda + K$ *for* K *varying over* $(-\infty, \infty)$.

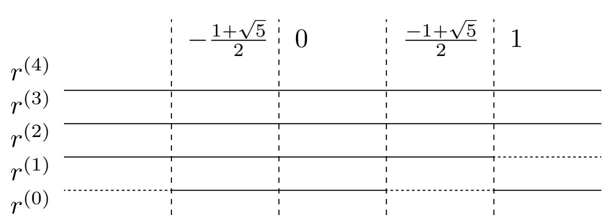
The Routh table generated by $\mathbf{p}(\lambda)$ is

$$\begin{array}{l|ccc} r^{(4)} & 1 & 2 & K \\ r^{(3)} & 1 & 1+K & \\ r^{(2)} & -K+1 & K & \\ r^{(1)} & \frac{K^2+K-1}{K-1} & & \\ r^{(0)} & K & & \end{array} \tag{33}$$

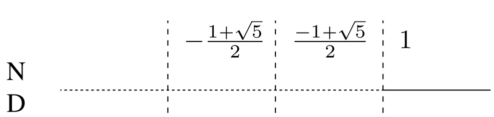
We can discuss the number of variations and permanencies in the first column of the Routh table as follows. First, we discuss the sign of each $r_1^{(j)}$, $j = 0, \ldots, 4$:

- $r_1^{(0)} = 0$ for $K = 0$, $r_1^{(1)} = 0$ for $K = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ and $r_1^{(2)} = 0$ for $K = 1$
- $r_1^{(4)}$ and $r_1^{(3)}$ are positive for all K
- $r_1^{(2)} > 0$ for $K < 1$
- $r_1^{(1)} > 0$ for $K \in (-\frac{1}{2} - \frac{\sqrt{5}}{2}, -\frac{1}{2} + \frac{\sqrt{5}}{2})$ or $K > 1$
- $r_1^{(0)} > 0$ for $K > 0$

These results can be visualized in the following table.



The sign of $r_1^{(1)} > 0$ has been discussed as follows



where N denotes the numerator of $r_1^{(1)}$ and D denotes the denominator of $r_1^{(1)}$ and we draw a full line if the sign of N (resp. D) is positive and a dashed line if its sign is negative.

Therefore, we have

- for $K = 0$ and $K = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ the table is not regular
- for $K < -\frac{1}{2} - \frac{\sqrt{5}}{2}$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 3$
- for $K \in (-\frac{1}{2} - \frac{\sqrt{5}}{2}, 0)$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 3$
- for $K \in (0, -\frac{1}{2} + \frac{\sqrt{5}}{2})$ the table is regular and $N_V(\mathbf{p}) = 0$ and $N_P(\mathbf{p}) = 4$
- for $K \in (-\frac{1}{2} + \frac{\sqrt{5}}{2}, 1)$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 2$
- for $K > 1$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 2$

We conclude by virtue of theorem 2.3

- for $K = 0$ and $K = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ the table in not regular $\Rightarrow \mathbf{p}(\lambda)$ is not Hurwitz
- for $K < 0$ the table is regular and $N_+(\mathbf{p}) = 1$ and $N_-(\mathbf{p}) = 3 \Rightarrow \mathbf{p}(\lambda)$ is not Hurwitz
- for $K \in (0, -\frac{1}{2} + \frac{\sqrt{5}}{2})$ the table is regular and $N_+(\mathbf{p}) = 0$ and $N_-(\mathbf{p}) = 4 \Rightarrow \mathbf{p}(\lambda)$ is Hurwitz
- for $K > -\frac{1}{2} + \frac{\sqrt{5}}{2}$ the table is regular and $N_+(\mathbf{p}) = 2$ and $N_-(\mathbf{p}) = 2 \Rightarrow \mathbf{p}(\lambda)$ is not Hurwitz.

$\lhd$

III. Extensions of the Routh criterion

The Routh criterion can be used also for determining if the roots of a polynomial are inside a given region of the complex plane. In particular, any half-plane

$$\mathfrak{S}(\alpha) := \{\lambda \in \mathbb{C}^n : \mathrm{Re}(\lambda) < -\alpha\} \tag{34}$$

We may require that the modes tend asymptotically to zero as fast as possible. This exactly correspond to require that the eigenvalues of A be inside a region $\mathfrak{S}(\alpha)$ with given α. The following proposition holds true.

Proposition *3.1: The polynomial* $\mathbf{p}(\lambda)$ *has all roots in* $\mathfrak{S}(\alpha)$ *if and only if* $\mathbf{p}(\lambda - \alpha)$ *has all roots in* $\mathfrak{S}(0)$*, i.e.* $\mathbf{p}(\lambda - \alpha)$ *is Hurwitz.*

Proof. This simply follows from the fact that $\mathbf{p}(\lambda)$ has a root $\overline{\lambda}$ if and only if $\mathbf{p}(\lambda - \alpha)$ has a root $\overline{\lambda} + \alpha$. The root $\overline{\lambda} + \alpha$ has negative real part if and only if $\mathrm{Re}(\overline{\lambda}) < -\alpha$. $\square$

Exercize *3.1: Discuss for which values of* $K \in (-\infty, \infty)$ *the roots of* $\mathbf{p}(\lambda) = \lambda^3 + 6\lambda^2 + (12 + K)\lambda + 2K + 8$ *are in* $\mathfrak{S}(1)$.

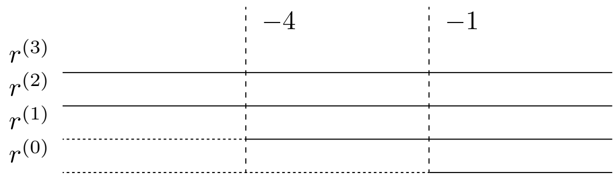
By proposition 3.1 it is sufficient to discuss for which values of $K \in (-\infty, \infty)$

$$\begin{aligned} \mathbf{p}(\lambda - 1) &:= (\lambda - 1)^3 + 6(\lambda - 1)^2 + (12 + K)(\lambda - 1) \\ &+2K + 8 = \lambda^3 + 3\lambda^2 + (3 + K)\lambda + K + 1 \end{aligned} \tag{35}$$

is Hurwitz. The Routh table generated by $\mathbf{p}(\lambda - \alpha)$ is

$$\begin{array}{l|cc} r^{(3)} & 1 & K+3 \\ r^{(2)} & 3 & K+1 \\ r^{(1)} & K+4 & \\ r^{(0)} & K+1 & \end{array} \tag{36}$$

We can discuss the number of variations and permanencies in the first column of the Routh table as follows.



Therefore, we have

- for $K = -4$ and $K = -1$ the table is not regular
- for $K < -4$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 2$
- for $K \in (-4, -1)$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 2$
- for $K > -1$ the table is regular and $N_V(\mathbf{p}) = 0$ and $N_P(\mathbf{p}) = 3$

We conclude by virtue of theorem 2.3

- for $K = -4$ and $K = -1$ the table in not regular $\Rightarrow$ $\mathbf{p}(\lambda - 1)$ is not Hurwitz
- for $K \in (-\infty, -1)$ the table is regular and $N_+(\mathbf{p}(\lambda - 1)) = 1$ and $N_-(\mathbf{p}(\lambda - 1)) = 2 \Rightarrow \mathbf{p}(\lambda - 1)$ is not Hurwitz
- for $K > -1$ the table is regular and $N_+(\mathbf{p}(\lambda - 1)) = 0$ and $N_-(\mathbf{p}(\lambda - 1)) = 3 \Rightarrow \mathbf{p}(\lambda - 1)$ is Hurwitz

On account of proposition 3.1

- for $K \in (-\infty, -1] \Rightarrow$ the roots of $\mathbf{p}(\lambda)$ are not in $\mathfrak{S}(1)$
- for $K > -1 \Rightarrow$ the roots of $\mathbf{p}(\lambda)$ are in $\mathfrak{S}(1)$.

$\lhd$

(OPTIONAL)************************************

The Routh criterion can be used also for determining if the roots of a polynomial are inside the following region of the complex plane

$$\mathfrak{T}(\theta) := \{\lambda \in \mathbb{C}^n : \frac{\pi}{2} + \theta < \operatorname{Arg}(\lambda) < \frac{3\pi}{2} - \theta\} \tag{37}$$

where $\theta \in [0, \frac{\pi}{2})$. We may require that the modes show as less oscillations as possible. This exactly correspond to require that the eigenvalues of A be inside a region $\mathfrak{T}(\theta)$ with given θ, i.e. with a damping as close to 1 as possible. The following proposition holds true.

Proposition 3.2: *The polynomial $\mathbf{p}(\lambda)$ has all roots in $\mathfrak{T}(\theta)$ if and only if $\mathbf{p}(e^{j\theta}\lambda)\mathbf{p}(e^{-j\theta}\lambda)$ has all roots in $\mathfrak{S}(0)$, i.e. $\mathbf{p}(e^{j\theta}\lambda)\mathbf{p}(e^{-j\theta}\lambda)$ is Hurwitz.*

Proof. Indeed, $\mathbf{p}(\lambda)$ has a root $\overline{\lambda}$ if and only if $\mathbf{p}(e^{j\theta}\lambda)\mathbf{p}(e^{-j\theta}\lambda)$ has two complex conjugate roots $e^{\mp j\theta}\overline{\lambda}$. Moreover, $\operatorname{Arg}(e^{\mp j\theta}\overline{\lambda}) = \mp\theta + \operatorname{Arg}(\overline{\lambda})$. The root $e^{\mp j\theta}\overline{\lambda}$ has negative real part if and only if $\operatorname{Arg}(e^{\mp j\theta}\overline{\lambda}) \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and, therefore, if and only if $\operatorname{Arg}(\overline{\lambda}) \in (\frac{\pi}{2} + \theta, \frac{3\pi}{2} - \theta)$. $\square$

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Exercize 3.2: *Discuss for which values of $K \in (-\infty, \infty)$ the roots of $\mathbf{p}(\lambda) = \lambda^2 + \lambda + K$ are in $\mathfrak{T}(\frac{\pi}{6})$.*

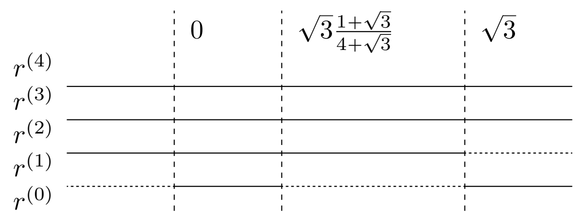
By proposition 3.1 it is sufficient to discuss for which values of $K \in (-\infty, \infty)$

$$\begin{aligned}
&\mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda) \\
&:= [(e^{j\frac{\pi}{6}}\lambda)^2 + (e^{j\frac{\pi}{6}}\lambda) + K][(e^{-j\frac{\pi}{6}}\lambda)^2 + (e^{-j\frac{\pi}{6}}\lambda) + K] \\
&\lambda^4 + \sqrt{3}\lambda^3 + (K+1)\lambda^2 + K(1+\sqrt{3})\lambda + K^2
\end{aligned} \tag{38}$$

is Hurwitz (recall that $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$). The Routh table generated by $\mathbf{p}(\lambda - \alpha)$ is

$$\begin{array}{c|ccc}
r^{(4)} & 1 & K+1 & K^2 \\
r^{(3)} & \sqrt{3} & K(1+\sqrt{3}) & \\
r^{(2)} & -K+\sqrt{3} & \sqrt{3}K^2 & \\
r^{(1)} & \frac{K(K(4+\sqrt{3})-\sqrt{3}(1+\sqrt{3}))}{K-\sqrt{3}} & & \\
r^{(0)} & \sqrt{3}K^2 & &
\end{array} \tag{39}$$

We can discuss the number of variations and permanencies in the first column of the Routh table as follows.



Therefore, we have

- for $K = 0$, $K = \sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}}$ and $K = \sqrt{3}$ the table is not regular
- for $K < 0$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 2$
- for $K \in (0, \sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}})$ the table is regular and $N_V(\mathbf{p}) = 0$ and $N_P(\mathbf{p}) = 4$
- for $K \in (\sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}}, \sqrt{3})$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 2$
- for $K > \sqrt{3}$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 2$

We conclude by virtue of theorem 2.3

- for $K = 0$, $K = \sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}}$ and $K = \sqrt{3}$ the table in not regular $\Rightarrow \mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)$ is not Hurwitz
- for $K < 0$ or $K > \sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}}$ the table is regular and $N_+(\mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)) = 2$ and $N_-(\mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)) = 2 \Rightarrow \mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)$ is not Hurwitz
- for $K \in (\sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}}, \sqrt{3})$ the table is regular and $N_+(\mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)) = 0$ and $N_-(\mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)) = 4 \Rightarrow \mathbf{p}(e^{j\frac{\pi}{6}}\lambda)\mathbf{p}(e^{-j\frac{\pi}{6}}\lambda)$ is Hurwitz

On account of proposition 3.2

- for $K \leqslant 0$ or $K \geqslant \sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}} \Rightarrow$ the roots of $\mathbf{p}(\lambda)$ are not in $\mathfrak{T}(\frac{\pi}{6})$
- for $K \in (\sqrt{3}\frac{1+\sqrt{3}}{4+\sqrt{3}}, \sqrt{3}) \Rightarrow$ the roots of $\mathbf{p}(\lambda)$ are in $\mathfrak{T}(\frac{\pi}{6})$.

$\lhd$

IV. I/S AND I/O STABILITY

Let us consider the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t),\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t),\end{aligned} \tag{40}$$

with initial state $\mathbf{x}_0 = x_0$. Let $\mathbf{x}^{(0)}, \mathbf{y}^{(0)}$ denote the unforced responses and $\mathbf{x}^{(u)}, \mathbf{y}^{(u)}$ the forced responses. The asymptotic behaviour of its solutions, i.e. when t tends to infinity, can be analyzed with respect to the input $\mathbf{u}(t)$ when $x_0 = 0$.

Definition *4.1: A system (40) is input-to-state stable in the zero state (0-I/S stable) if for each input function* $\boldsymbol{u}$ *such that* $\sup_{t\geqslant 0}\|\boldsymbol{u}(t)\| < M$ *for some* $M > 0$ *there exists* $N > 0$ *such that*

$$\sup_{t\geqslant 0}\|\boldsymbol{x}^{(u)}(t,\boldsymbol{u})\| < N. \tag{41}$$

A system (40) is input-to-output stable in the zero state (0-I/O stable) if for each input function $\boldsymbol{u}$ *such that* $\sup_{t\geqslant 0}\|\boldsymbol{u}(t)\| < M$ *for some* $M > 0$ *there exists* $N > 0$ *such that*

$$\sup_{t\geqslant 0}\|\boldsymbol{y}^{(u)}(t,\boldsymbol{u})\| < N. \tag{42}$$

The above definitions require that the forced (state or output) responses be bounded with bounded inputs. Non-zero initial conditions x_0 are taken into account in the next definitions.

Definition *4.2: A system (40) is input-to-state stable (I/S stable) if for each* $x_0 \in \mathbb{R}^n$ *and input function* $\boldsymbol{u}$ *such that* $\sup_{t\geqslant 0}\|\boldsymbol{u}(t)\| < M$ *for some* $M > 0$ *there exists* $N > 0$ *such that*

$$\begin{aligned}&\sup_{t\geqslant 0}\|\boldsymbol{x}(t,x_0,\boldsymbol{u})\| < N,\\ &\lim_{t\to+\infty}\|\boldsymbol{x}^{(0)}(t,x_0)\| = 0.\end{aligned} \tag{43}$$

A system (40) is input-to-output stable (I/O stable) if for each $x_0 \in \mathbb{R}^n$ *and input function* $\boldsymbol{u}$ *such that* $\sup_{t\geqslant 0}\|\boldsymbol{u}(t)\| < M$ *for some* $M > 0$ *there exists* $N > 0$ *such that*

$$\begin{aligned}&\sup_{t\geqslant 0}\|\boldsymbol{y}(t,x_0,\boldsymbol{u})\| < N,\\ &\lim_{t\to+\infty}\|\boldsymbol{y}^{(0)}(t,x_0)\| = 0.\end{aligned} \tag{44}$$

The above definition of I/S require that the state response be bounded with bounded inputs and for each initial condition x_0 (the bound depends also on x_0) with asymptotic stability. Hence, a necessary condition for I/S stability is asymptotic stability. Moreover, I/S stability implies I/O stability (the converse is false) and I/S (resp. O/S) stability implies 0-I/S (resp. 0-I/O) stability. Surprisingly, asymptotic stability is also a sufficient condition for I/S stability.

Theorem *4.1: A system (40) is I/S stable if and only if it is asymptotically stable.*
It turns out that 0-I/S, resp. 0-I/O, stability depends exclusively on the poles of $\mathbf{H}(s) = (sI - A)^{-1}B$, resp. $\mathbf{W}(s) = C(sI - A)^{-1}B + D$.

Theorem *4.2: A system (40) is 0-I/S stable (resp. 0-I/O stable) if and only if the poles of* $\boldsymbol{H}(s)$ *(resp.* $\boldsymbol{W}(s)$*) are all in* $\mathbb{C}^-$.

Proof. This follows easily in the Laplace domain from

$$\begin{aligned}\mathfrak{L}[\mathbf{x}^{(u)}(t,\mathbf{u})](s) &= \mathbf{H}(s)u(s),\\ \mathfrak{L}[\mathbf{y}^{(u)}(t,\mathbf{u})](s) &= \mathbf{W}(s)u(s)\end{aligned} \tag{45}$$

and the residuals theorem. □

As for asymptotic stability it is necessary and sufficient that the eigenvalues of A be all in $\mathbb{C}^-$, for I/O stability it is necessary and sufficient that the poles of $\mathbf{W}(s)$ be all in $\mathbb{C}^-$. These are two crucial issues in control design and should be always guaranteed by the designer, according to the mathematical model (in time or Laplace domain) we work with.

V. EIGENVALUES AND POLES PLACEMENT VERSUS SYSTEM'S PERFORMANCES

The position of poles of $\mathbf{W}(s)$ or the eigenvalues of A in $\mathbb{C}^-$ is a crucial issue in control design. In particular, it is important to place this eigenvalues or poles in some subregions of $\mathbb{C}^-$ like for example $\mathfrak{S}(\alpha)$ or $\mathfrak{I}(\theta)$ to guarantee certain performances of the state or output responses. In this section we will try to explain this point, referring to systems with an I/O transfer function characterized by one real pole (first order systems) and, respectively, by a couple of complex conjugate poles (second order systems). A reason for the analysis being restricted to first and second order systems is that the I/O transfer function $\mathbf{W}(s)$ of a system (with n poles) is comparable with a good approximation to that of a second order system (with 2 complex conjugate poles, which are also the *dominant* poles). Moreover, we consider the response of the system to step inputs, which are usually adopted by the designer for testing the system's performances.

A. First order systems

Consider the system with I/O transfer function

$$\mathbf{W}(s) = \frac{1}{1+s\tau} \tag{46}$$

with $\tau > 0$. The parameter τ is the *time constant* associated to the pole (or eigenvalue) $s = \lambda := -\frac{1}{\tau}$ of (46). The step output response to an input $\mathbf{u}(t) := \delta^{(-1)}(t)$ is

$$\begin{aligned}\mathbf{y}^{(u)}(t,\mathbf{u}) &= \mathfrak{L}^{-1}[\mathbf{W}(s)\mathfrak{L}[\mathbf{u}(t)](s)](t)\\ &= \mathfrak{L}^{-1}[\frac{1}{s(1+s\tau)}] = (1-e^{-\frac{t}{\tau}})\delta^{(-1)}(t)\end{aligned}$$

Note that

$$\mathbf{y}^{(ss)} = \lim_{t\to+\infty}\mathbf{y}^{(u)}(t,\mathbf{u}) = 1 = \mathbf{W}(0)$$

which is the asymptotic value or steady state value of the output response $\mathbf{y}^{(forced)}(t)$. The difference

$$\mathbf{y}^{(tr)}(t) = \mathbf{y}^{(u)}(t,\mathbf{u}) - \mathbf{y}^{(ss)} = e^{-\frac{t}{\tau}}$$

is the transient output response, which tends to asymptotically vanish: i.e. $\lim_{t\to+\infty}\mathbf{y}^{(tr)}(t) = 0$. For $t = \tau$ sec the transient response is within 36.8% of its steady-state value, for $t = 2\tau$ sec the transient response is within 13.5% of its steady-state value $\mathbf{y}^{(ss)}$ and for $t = 3\tau$ sec the transient response is within 5% of its steady-state value $\mathbf{y}^{(ss)}$. Therefore, it takes 3τ sec for the output response to remain within $\%5$ of its steady-state value. After 7τ sec the output response remains within $\%0.09$ of its steady-state value.

The $(\%5)$-*settling time* (denoted by $T_a^{(\%5)}$) is the time instant for which the output response $\mathbf{y}^{(forced)}(t)$ remains for all subsequent times within $\%5$ of its steady-state value:

$$|\mathbf{y}^{(tr)}(t)| \leqslant 0.05|\mathbf{y}^{(ss)}|,\ \forall t \geqslant T_a^{(\%5)}.$$

As mentioned above it takes 3τ sec for the output response to remain within $\%5$ of its steady-state value and therefore we have $T_a^{(\%5)} = 3\tau$. In order to guarantee a fast response of the system, we need to have small values of $T_a^{(\%5)} = 3\tau$ and therefore of τ. This corresponds to place the real pole (or eigenvalue) $s = \lambda := -\frac{1}{\tau}$ in a region of the form $\mathfrak{S}(\alpha)$ for a large $\alpha > 0$.

On the other hand, it takes $t \approx 2.2\tau$ sec for the output response response going from $\%10$ to $\%90$ of its steady-state value $\mathbf{y}^{(ss)}$. The $(\%10 \to \%90)$-*rise time* ($T_r^{(\%10\to\%90)}$) is the period of time needed for the output response $\mathbf{y}^{(u)}(t, \mathbf{u})$ passing from $\%10$ to $\%90$ of its steady-state value $\mathbf{y}^{(ss)}$:

$$T_r^{(\%10\to\%90)} := T^{(\%90)} - T^{(\%10)}$$

where $T^{(\%90)}$ and $T^{(\%10)}$ are the times for which $\mathbf{y}^{(u)}(T^{(\%90)}, \mathbf{u}) = 0.9\mathbf{y}^{(ss)}$ and, respectively, $\mathbf{y}^{(u)}(T^{(\%10)}, \mathbf{u}) = 0.1\mathbf{y}^{(ss)}$.

B. Second order systems

Consider the system with I/O transfer function

$$\mathbf{W}(s) = \frac{1}{1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}} \tag{47}$$

with $\omega_n > 0$ and $\zeta \in (0, 1)$. The parameters ω_n and ζ are the *natural frequency* and, respectively, the *damping* associated to the pair of complex conjugate poles (or eigenvalues)

$$\begin{aligned} s &= \lambda = -\omega_n(\zeta + j\sqrt{1-\zeta^2}), \\ s^* &= \lambda^* = -\omega_n(\zeta - j\sqrt{1-\zeta^2}), \end{aligned} \tag{48}$$

which are the roots of the polynomial $1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}$, the denominator of (47). The step output response is after some calculations

$$\begin{aligned} &\mathbf{y}^{(u)}(t, \mathbf{u}) = \mathfrak{L}^{-1}[\mathbf{W}(s)\mathfrak{L}[\mathbf{u}(t)](s)](t) \\ &= \mathfrak{L}^{-1}[\frac{1}{s(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2})}](t) \\ &= (1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t} \sin(\omega_n\sqrt{1-\zeta^2}t + \phi))\delta^{(-1)}(t) \end{aligned} \tag{49}$$

with

$$\phi := \arctan \frac{\sqrt{1-\zeta^2}}{\zeta} = \arcsin\sqrt{1-\zeta^2} = \arccos\zeta.$$

The function $\mathbf{y}^{(u)}(t, \mathbf{u})$ has local minimum and maximum points and tends to $\mathbf{W}(0) = 1$ as $t \to +\infty$ (its steady-state value $\mathbf{y}^{(ss)}$). The local maximum and minimum points of $\mathbf{y}^{(u)}(t, \mathbf{u})$ are found by seeking for the zeroes of its first order derivative:

$$\begin{aligned} 0 &= \frac{d}{dt}\mathbf{y}^{(u)}(t, \mathbf{u}) \\ &= -\frac{e^{-\omega_n\zeta t}}{\sqrt{1-\zeta^2}}[-\omega_n\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2}t + \phi) \\ &+\zeta \sin(\omega_n\sqrt{1-\zeta^2}t + \phi)] \end{aligned} \tag{50}$$

or equivalently

$$\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan(\omega_n\sqrt{1-\zeta^2} + \phi) \tag{51}$$

which, on account of the definition of ϕ, has the roots

$$t_h^* = \frac{h\pi}{\omega_n\sqrt{1-\zeta^2}},\ h = 0, 1, \ldots \tag{52}$$

From here we obtain the values of the local minimum (for even h) and maximum (for odd h) points

$$\begin{aligned} \mathbf{y}^{(u)}(t_h^*, \mathbf{u}) &= 1 - \frac{e^{-\frac{h\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(h\pi + \phi) \\ &= 1 - (-1)^h e^{-\frac{h\pi\zeta}{\sqrt{1-\zeta^2}}},\ h = 0, 1, \ldots \end{aligned} \tag{53}$$

Note that the response $\mathbf{y}^{(u)}(t, \mathbf{u})$ has a global minimum at $t_0^* = 0$ with $\mathbf{y}^{(u)}(t_0^*, \mathbf{u}) := y_{min} := 0$ and a global maximum at $t_1^* = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$, with

$$\mathbf{y}^{(u)}(t_1^*, \mathbf{u}) := y_{max} := 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{54}$$

The quantity

$$\hat{s} := y_{max} - 1 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{55}$$

is the *maximal overshooting* of the response $\mathbf{y}^{(u)}(t, \mathbf{u})$. The maximal overshooting is the maximal displacement in excess of the response $\mathbf{y}^{(u)}(t, \mathbf{u})$ from its steady-state value $\mathbf{y}^{(ss)}$, after the first time $T^{(ss)}$ for which $\mathbf{y}^{(u)}(t, \mathbf{u})$ crosses its steady state value $\mathbf{y}^{(ss)}$. In general, the maximal overshoot is given in $\%$ units: $\hat{s}(\%) := (y_{max} - 1)100$. For not stressing the system too much, in practical situations it is convenient to have the smallest as possible maximal overshooting $\hat{s}$. From (55) we see that $\hat{s}$ is a decreasing function of the damping ζ and $\hat{s}$ decreases from 1 to 0 as the damping ζ increases from 0 to 1. Therefore, in order to minimize the maximal overshooting it is necessary to have the damping of the poles as close as possible to 1, i.e. to have the poles of $\mathbf{W}(s)$ as close as possible to the real negative axis. This corresponds to place the pair of complex conjugate poles (or eigenvalues) (48) in a region of the form $\mathfrak{I}(\theta)$ for θ as close as possible to $\pi/2$.

Another important parameter for the analysis of the forced response is the (5%)-settling time $T_s^{(5\%)}$. For designing a control system with prompt output response in the sense that the transient response $\mathbf{y}^{(tr)}(t) = \mathbf{y}^{(u)}(t, \mathbf{u}) - \mathbf{y}^{(ss)}$ has a high convergence rate, it is convenient to have the smallest possible values of $T_s^{(5\%)}$. Since

$$\begin{aligned}
&\frac{|\mathbf{y}^{(u)}(t,\mathbf{u}) - \mathbf{y}^{(ss)}|}{|\mathbf{y}^{(ss)}|} \\
&= |\frac{1}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi)| \\
&\leqslant \frac{1}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t} \qquad (56)
\end{aligned}$$

an upper bound $\overline{T}_s^{(5\%)}$ for the settling time $T_s^{(5\%)}$ (i.e. $\overline{T}_s^{(5\%)} \geqslant T_s^{(5\%)}$) is obtained from the equation

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta \overline{T}_s^{(5\%)}} = 0.05 \qquad (57)$$

from which

$$\overline{T}_s^{(5\%)} = \frac{\ln 20 + \ln \frac{1}{\sqrt{1-\zeta^2}}}{\omega_n \zeta} \qquad (58)$$

and it is a decreasing function of the product $\zeta\omega_n$. Since from (48) the product $\omega_n\zeta$ is the absolute value of the real part of the poles (s, s^*) of $\mathbf{W}(s)$ and since ζ is picked to determine the maximal overshooting (55), in order to minimize $\overline{T}_s^{(5\%)}$ we can maximize ω_n.