Notes on Linear Control Systems: Module V

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***Abstract*—Transient and steady-state response. Steady-state responses to polynomial, sinusoidal, exponential inputs. Time translation and linearity properties of steady-state response.**

I. Transient and steady-state response

The aim of this section is to find for each input function $\mathbf{u}$ a function $\mathbf{x}^{(ss)}(t, \mathbf{u})$, whenever it exists, such that

$$\dot{\mathbf{x}}^{(ss)}(t, \mathbf{u}) \equiv A\mathbf{x}^{(ss)}(t, \mathbf{u}) + B\mathbf{u}(t) \quad (1)$$

and for each $x_0 \in \mathbb{R}^n$

$$\lim_{t\to\infty} \|\mathbf{x}(t, x_0, \mathbf{u}) - \mathbf{x}^{(ss)}(t, \mathbf{u})\| = 0 \quad (2)$$

where $\mathbf{x}(t, x_0, \mathbf{u})$ is the solution of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \quad (3)$$

ensuing from x_0 at $t = 0$ with input function $\mathbf{u}$. The function $\mathbf{x}^{(ss)}(t, \mathbf{u})$ represents the *steady-state* state response of (3), the solution of (3) which any other state response tends to whatever is its initial condition. The steady-state response of (3) depends only on the input function $\mathbf{u}$ (and not on a particular initial condition x_0).

Let us investigate conditions for the existence of $\mathbf{x}^{(ss)}(t, \mathbf{u})$ and how to compute $\mathbf{x}^{(ss)}(t, \mathbf{u})$. To this aim, write the solution $\mathbf{x}(t, t_0, x_0, \mathbf{u})$ of (3) at time t ensuing from x_0 at time $t_0 < 0$ with input function $\mathbf{u}_{[t_0,t]}$. The matrix e^{At} is invertible for each t and its inverse is e^{-At}: indeed, since A always commutes with $-A$

$$e^{At}e^{-At} = e^{At-At} = e^0 = I \quad (4)$$

Set $z = e^{-At}x$. We have

$$\dot{\mathbf{z}}(t) = e^{-At}\mathbf{x}(t) - e^{-At}A\mathbf{x}(t) = e^{-At}B\mathbf{u}(t) \quad (5)$$

By integrating (5) over (t_0, t), it follows that the solution $\mathbf{z}(t)$ of (5) ensuing from z_0 at $t_0 \in \mathbb{R}$ with input function $\mathbf{u}$ satisfies

$$\mathbf{z}(t) - z_0 = \int_{t_0}^{t} e^{-A\tau}B\mathbf{u}(\tau)d\tau \quad (6)$$

Since $x = e^{At}z$ then the solution $\mathbf{x}(t, x_0, \mathbf{u})$ of (3) ensuing from x_0 at $t_0 \in \mathbb{R}$ with input function $\mathbf{u}$ is

$$\mathbf{x}(t, t_0, x_0, \mathbf{u}_{[t_0,t]}) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \quad (7)$$

An important result states that the existence of a solution $\mathbf{x}^{(ss)}(t, \mathbf{u})$ of (3) with the property (2) is guaranteed by the asymptotic stability of (3).

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Theorem *1.1: Assume that (3) is asymptotically stable. For each input function $\mathbf{u}$ such that for all $t \in \mathbb{R}$*

$$\|\mathbf{u}(t)\| \leqslant ce^{kt} \sum_{i=0}^{n} |t|^i \quad (8)$$

for some $k, c \geqslant 0$, the steady-state state response is given by

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \lim_{t_0\to-\infty} \mathbf{x}(t, t_0, x_0, \mathbf{u}_{[t_0,t]}) = \lim_{t_0\to-\infty} \int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \quad (9)$$

Proof (OPTIONAL). Since the eigenvalues of A have negative real parts, there exist a positive real b such that for all $t \geqslant 0$

$$\|e^{At}\| \leqslant be^{-mt} \sum_{i=0}^{n} t^i \quad (10)$$

where $m := -\max_i \mathrm{Re}(\lambda_i)$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Therefore, on account of (8) for $t_0 < 0$

$$\begin{aligned} \| \int_{t_0}^{0} e^{-A\tau}B\mathbf{u}(\tau)d\tau\| &= \| \int_{0}^{-t_0} e^{A\tau}B\mathbf{u}(-\tau)d\tau\| \\ &\leqslant cb\|B\| \int_{0}^{-t_0} e^{-(m+k)\tau} \sum_{i=0}^{n}\sum_{j=0}^{n} \tau^{i+j}d\tau \\ &\leqslant cb\|B\| \sum_{i=0}^{n}\sum_{j=0}^{n} \int_{0}^{-t_0} e^{-(m+k)\tau}\tau^{i+j}d\tau \end{aligned} \quad (11)$$

But integrating by parts for each $h \geqslant 0$

$$\begin{aligned} \int_{0}^{-t_0} e^{-(m+k)\tau}\tau^h d\tau &= -\sum_{l=0}^{h} \frac{h!}{l!} \frac{e^{-(m+k)\tau}\tau^l}{(m+k)^{h-l+1}}\Big|_0^{-t_0} \\ &= \sum_{l=0}^{h} (-1)^{l+1} \frac{l!}{h!} \frac{e^{(m+k)t_0}t_0^l}{(m+k)^{h-l+1}} + \frac{h!}{(m+k)^{h+1}} \end{aligned} \quad (12)$$

Therefore, from (11) and (12)

$$\lim_{t_0\to-\infty} \| \int_{t_0}^{0} e^{-A\tau}B\mathbf{u}(\tau)d\tau\| \leqslant cb\|B\| \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{(i+j)!}{(m+k)^{i+j+1}} \quad (13)$$

We conclude from (10) that for each $t \geqslant 0$

$$\begin{aligned} &\lim_{t_0\to-\infty} \| \int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\| \\ &\leqslant \|e^{At}\| \lim_{t_0\to-\infty} \| \int_{t_0}^{0} e^{-A\tau}B\mathbf{u}(\tau)d\tau\| \\ &+\| \int_{0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\| = \| \int_{0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\| \end{aligned} \quad (14)$$

which implies that $\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$ is well-defined for each $t \geqslant 0$. Also, using (10) and (13)

$$\lim_{t\to+\infty}\lim_{t_0\to-\infty}\|\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau-\int_{0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\|$$
$$=\lim_{t\to+\infty}\lim_{t_0\to-\infty}\|\int_{t_0}^{0} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\|$$
$$\leqslant \lim_{t\to+\infty}\|e^{At}\|\lim_{t_0\to-\infty}\|\int_{t_0}^{0} e^{-A\tau}B\mathbf{u}(\tau)d\tau\|=0. \quad (15)$$

By (10)

$$\lim_{t\to+\infty}\|e^{At}x_0\| \leqslant \|x_0\|e^{-mt}\sum_{i=0}^{n} t^i \quad (16)$$

for all $x_0 \in \mathbb{R}^n$ and $t \geqslant 0$ so that

$$\lim_{t\to\infty}\|e^{At}x_0\|=0 \quad (17)$$

for all $x_0 \in \mathbb{R}^n$. Using this and (15), we obtain for all $x_0 \in \mathbb{R}^n$

$$\lim_{t\to\infty}\|\mathbf{x}(t,x_0,\mathbf{u})-\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\|$$
$$=\lim_{t\to\infty}\|e^{At}x_0+\int_{0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$
$$-\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\|$$
$$\leqslant \lim_{t\to\infty}\|e^{At}x_0\|+\lim_{t\to\infty}\lim_{t_0\to-\infty}\|\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$
$$-\int_{0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\|=0$$

which proves the claim (9). Moreover as it can be easily seen $\mathbf{x}_{ss}(t,\mathbf{u})$ is a solution of (3), i.e. satisfies (1),

$$\dot{\mathbf{x}}_{ss}(t,\mathbf{u})=A\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau+B\mathbf{u}(t)$$
$$=A\mathbf{x}_{ss}(t,\mathbf{u})+B\mathbf{u}(t).\square \quad (18)$$

The difference

$$\mathbf{x}^{(tr)}(t,x_0,\mathbf{u}):=\mathbf{x}(t,x_0,\mathbf{u})-\mathbf{x}^{(ss)}(t,\mathbf{u}) \quad (19)$$

is called the *transient* state response.

A similar result can be proved for the steady-state output response $\mathbf{y}^{(ss)}(t)(\mathbf{u})$. Taking into account that the output response $\mathbf{y}(t,x_0,\mathbf{u})$ of (3) ensuing from x_0 at $t=0$ with input function $\mathbf{u}$ is $C\mathbf{x}(t,x_0,\mathbf{u})+D\mathbf{u}(t)$, for each $x_0 \in \mathbb{R}^n$ we have

$$\lim_{t\to\infty}\|\mathbf{y}(t,x_0,\mathbf{u})-(C\mathbf{x}^{(ss)}(t,\mathbf{u})+D\mathbf{u}(t))\|$$
$$=\lim_{t\to\infty}\|C\mathbf{x}(t,x_0,\mathbf{u})-C\mathbf{x}^{(ss)}(t,\mathbf{u})\|$$
$$\leqslant \lim_{t\to\infty}\|C\|\|\mathbf{x}(t,x_0,\mathbf{u})-\mathbf{x}^{(ss)}(t,\mathbf{u})\|=0 \quad (20)$$

Therefore, $\mathbf{y}^{(ss)}(t,\mathbf{u}):=C\mathbf{x}^{(ss)}(t,\mathbf{u})+D\mathbf{u}(t)$ is the steady-state output response.

The difference

$$\mathbf{y}^{(tr)}(t,x_0,\mathbf{u}):=\mathbf{y}(t,x_0,\mathbf{u})-\mathbf{y}^{(ss)}(t,\mathbf{u}) \quad (21)$$

is called the *transient* output response.

II. Steady state response to polynomial inputs

In this section we calculate the steady-state response to *polynomial* inputs $\mathbf{u}(t)=\frac{t^k}{k!}$. This class of inputs satisfies the conditions of theorem 1.1 since for all $t \in \mathbb{R}$

$$|\mathbf{u}(t)| \leqslant \frac{1}{k!}|t|^k \quad (22)$$

with $c_k := \frac{1}{k!}$. The steady-state state response is by virtue of theorem 1.1

$$\mathbf{x}^{(ss)}(t,\mathbf{u})=\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \quad (23)$$

Recall that A is nonsingular since A has eigenvalues with negative real part (it is known that a matrix A is singular if and only if has at least one null real eigenvalue) and $\frac{d}{dt}e^{At}=Ae^{At}=e^{At}A$. Also remember that

$$\lim_{t_0\to-\infty}e^{A(t-t_0)}\frac{t_0^i}{i!}=0 \quad (24)$$

for each t and i, since A has eigenvalues with negative real part.

Integrating by parts

$$\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau=\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^k}{k!}d\tau$$
$$=\lim_{t_0\to-\infty}A^{-1}\Big(-e^{A(t-\tau)}B\frac{\tau^k}{k!}\Big|_{t_0}^{t}+\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^{k-1}}{(k-1)!}d\tau\Big)$$
$$=A^{-1}\Big(-B\frac{t^k}{k!}+\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^{k-1}}{(k-1)!}d\tau\Big) \quad (25)$$

Assume that for some $i<k$

$$\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^i}{i!}d\tau=-\sum_{j=0}^{i}(A^{-1})^{j+1}B\frac{t^{i-j}}{(i-j)!} \quad (26)$$

From (24) it follows that

$$\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^{i+1}}{(i+1)!}d\tau$$
$$=A^{-1}\Big(-B\frac{t^{i+1}}{i+1!}+\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^i}{i!}d\tau\Big)$$
$$=A^{-1}\Big(-B\frac{t^{i+1}}{i+1!}-\sum_{j=0}^{i}(A^{-1})^{j+1}B\frac{t^{i-j}}{(i-j)!}\Big)$$
$$=-\sum_{j=0}^{i+1}(A^{-1})^{j+1}B\frac{t^{i+1-j}}{(i+1-j)!} \quad (27)$$

which proves (25) with i replaced by $i+1$. Since on account of (23)

$$\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}Bd\tau$$
$$=-A^{-1}B+A^{-1}\lim_{t_0\to-\infty}e^{A(t-t_0)}B=-A^{-1}B \quad (28)$$

we have proved by induction that

$$\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau=\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\frac{\tau^k}{k!}d\tau$$
$$=-\sum_{j=0}^{k}(A^{-1})^{j+1}B\frac{t^{k-j}}{(k-j)!} \quad (29)$$

Therefore,

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = -\sum_{j=0}^{k} (A^{-1})^{j+1} B \frac{t^{k-j}}{(k-j)!} \tag{30}$$

Recalling that $\mathbf{H}(s) = (sI - A)^{-1}B$,

$$-(A^{-1})^{j+1}B = \frac{1}{j!}\frac{d^j}{ds^j}(sI-A)^{-1}\Big|_{s=0} B = \frac{1}{j!}\frac{d^j}{ds^j} H\Big|_{s=0} \tag{31}$$

so that equivalently

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \sum_{j=0}^{k} \frac{d^j\mathbf{H}}{ds^j}\Big|_{s=0} \frac{t^{k-j}}{(k-j)!j!} \tag{32}$$

The steady -state output response is

$$\mathbf{y}^{(ss)}(t,\mathbf{u}) = -C\sum_{j=0}^{k} (A^{-1})^{j+1} B \frac{t^{k-j}}{(k-j)!} + D\frac{t^k}{k!} \tag{33}$$

or equivalently, recalling that $\mathbf{W}(s) = C(sI - A)^{-1}B + D$ and using (30),

$$\mathbf{y}^{(ss)}(t,\mathbf{u}) = \sum_{j=0}^{k} \frac{d^j\mathbf{W}}{ds^j}\Big|_{s=0} \frac{t^{k-j}}{(k-j)!j!} \tag{34}$$

Theorem *2.1: Assume that (3) is asymptotically stable. For an input function* $\mathbf{u}(t) = \frac{t^k}{k!}$ *the steady-state state and output responses are*

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \sum_{j=0}^{k} \frac{d^j\mathbf{H}}{ds^j}\Big|_{s=0} \frac{t^{k-j}}{(k-j)!j!} \tag{35}$$

$$\mathbf{y}^{(ss)}(t,\mathbf{u}) = \sum_{j=0}^{k} \frac{d^j\mathbf{W}}{ds^j}\Big|_{s=0} \frac{t^{k-j}}{(k-j)!j!} \tag{36}$$

We remark that the function $\mathbf{x}^{(ss)}(t,\mathbf{u})$ could have be computed as the function of the form $\mathbf{x}^{(ss)}(t,\mathbf{u}) = \sum_{j=0}^{k} c_j \frac{t^j}{j!}$, for some $c_0,\dots,c_k \in \mathbb{R}^n$, such that

$$\dot{\mathbf{x}}^{(ss)}(t,\mathbf{u}) = A\mathbf{x}^{(ss)}(t,\mathbf{u}) + B\mathbf{u}(t) \tag{37}$$

for all $t \geqslant 0$ with $\mathbf{u}(t) = \frac{t^k}{k!}$. The coefficients $c_0,\dots,c_k \in \mathbb{R}^n$ are calculated from the identity (36). Therefore, $\mathbf{x}^{(ss)}(t,\mathbf{u})$ is the solution $\mathbf{x}(t,x_0,\mathbf{u})$ of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{38}$$

with $\mathbf{u}(t) = \frac{t^k}{k!}$ and initial condition $x_0 = c_0$.

III. Steady-state response to sinusoidal and cosinusoidal inputs

In this section we calculate the steady-state state response to *sinusoidal* inputs $\mathbf{u}(t) = \sin\omega t$. This class of inputs satisfies the conditions of theorem 1.1 since for all $t \in \mathbb{R}$

$$|\mathbf{u}(t)| \leqslant 1 \tag{39}$$

The steady-state state response $\mathbf{x}^{(ss)}(t,\mathbf{u})$ is by virtue of theorem 1.1

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \tag{40}$$

Recall that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ and $\lim_{t_0\to-\infty} e^{A(t-t_0)}B\cos\omega t_0 = 0$ for each $t \geqslant 0$, since A has eigenvalues with negative real part.

By iterated integration by parts

$$\begin{aligned}
&\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau = \int_{t_0}^{t} e^{A(t-\tau)}B\sin\omega\tau d\tau \\
&= \lim_{t_0\to-\infty}\frac{1}{\omega}(-e^{A(t-\tau)}B\cos\omega\tau\Big|_{t_0}^{t} \\
&-\frac{A}{\omega}\int_{t_0}^{t} e^{A(t-\tau)}B\cos\omega\tau d\tau \\
&= -\frac{B}{\omega}\cos\omega t - \frac{A}{\omega}\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\cos\omega\tau d\tau \\
&= -\frac{B}{\omega}\cos\omega t - \frac{A}{\omega}\Big(\frac{B}{\omega}\sin\omega t \\
&+\frac{A}{\omega}\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\sin\omega\tau d\tau\Big)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big(I + \Big(\frac{A}{\omega}\Big)^2\Big)\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \\
&= -\frac{B}{\omega}\cos\omega t - \frac{AB}{\omega^2}\sin\omega t
\end{aligned}$$

and finally

$$\begin{aligned}
&\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \\
&= -\Big(I + \Big(\frac{A}{\omega}\Big)^2\Big)^{-1}\frac{B}{\omega}\cos\omega t - (I + \Big(\frac{A}{\omega}\Big)^2)^{-1}\frac{AB}{\omega^2}\sin\omega t
\end{aligned}$$

Recalling that $\mathbf{H}(s) = (sI - A)^{-1}B$, note that since

$$\begin{aligned}
\mathbf{H}(j\omega) &= (jI - \frac{A}{\omega})^{-1}\frac{B}{\omega} = -(jI + \frac{A}{\omega})(I + \Big(\frac{A}{\omega}\Big)^2)^{-1}\frac{B}{\omega} \\
&= -\frac{A}{\omega}\Big(I + \Big(\frac{A}{\omega}\Big)^2\Big)^{-1}\frac{B}{\omega} - j(I + \Big(\frac{A}{\omega}\Big)^2\Big)^{-1}\frac{B}{\omega} \\
&= -(I + \Big(\frac{A}{\omega}\Big)^2)^{-1}\frac{AB}{\omega^2} - j(I + \Big(\frac{A}{\omega}\Big)^2)^{-1}\frac{B}{\omega}
\end{aligned} \tag{41}$$

then

$$\begin{aligned}
\mathrm{Re}(\mathbf{H}(j\omega)) &= -\Big(I + \Big(\frac{A}{\omega}\Big)^2\Big)^{-1}\frac{AB}{\omega^2} \\
\mathrm{Im}(\mathbf{H}(j\omega)) &= -\Big(I + \Big(\frac{A}{\omega}\Big)^2\Big)^{-1}\frac{B}{\omega}
\end{aligned} \tag{42}$$

and

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \mathrm{Im}(\mathbf{H}(j\omega))\cos\omega t + \mathrm{Re}(\mathbf{H}(j\omega))\sin\omega t \tag{43}$$

On account of

$$\begin{aligned}
\mathrm{Re}(\mathbf{H}_i(j\omega)) &= |\mathbf{H}_i(j\omega)|\cos(\mathrm{Arg}(\mathbf{H}_i(j\omega))) \\
\mathrm{Im}(\mathbf{H}_i(j\omega)) &= |\mathbf{H}_i(j\omega)|\sin(\mathrm{Arg}(\mathbf{H}_i(j\omega)))
\end{aligned}$$

for each $i = 1, \dots, n$ ($\mathbf{H}_i(j\omega)$ denotes the i-th element of the vector $\mathbf{H}(j\omega)$), we obtain

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \begin{pmatrix} |\mathbf{H}_1(j\omega)| \sin(\omega t + \text{Arg}(\mathbf{H}_1(j\omega))) \\ \cdots \\ |\mathbf{H}_n(j\omega)| \sin(\omega t + \text{Arg}(\mathbf{H}_n(j\omega))) \end{pmatrix} \quad (44)$$

Recalling that $\mathbf{W}(s) = (sI - A)^{-1}B + D$, note also that

$$\begin{aligned} &C\text{Re}(\mathbf{H}(j\omega)) + D = \text{Re}(C\mathbf{H}(j\omega) + D) = \text{Re}(\mathbf{W}(j\omega)) \\ &= |\mathbf{W}(j\omega)| \cos(\text{Arg}(\mathbf{W}(j\omega))) \\ &C\text{Im}(\mathbf{H}(j\omega)) = \text{Im}(C\mathbf{H}(j\omega) + D) = \text{Im}(\mathbf{W}(j\omega)) \\ &= |\mathbf{W}(j\omega)| \sin(\text{Arg}(\mathbf{W}(j\omega))) \end{aligned}$$

and from (42)

$$\begin{aligned} &\mathbf{y}^{(ss)}(t, \mathbf{u}) = C\text{Im}(\mathbf{H}(j\omega)) \cos\omega t + C\text{Re}(\mathbf{H}(j\omega)) \sin\omega t \\ &+ D \sin\omega t \ = |\mathbf{W}(j\omega)| \sin(\text{Arg}(\mathbf{W}(j\omega))) \cos\omega t \\ &+ |\mathbf{W}(j\omega)| \cos(\text{Arg}(\mathbf{W}(j\omega))) \sin\omega t \\ &= |\mathbf{W}(j\omega)| \sin(\omega t + \text{Arg}(\mathbf{W}(j\omega))) \end{aligned} \quad (45)$$

Theorem *3.1: Assume that (3) is asymptotically stable. For an input function $\mathbf{u}(t) = \sin\omega t$ the steady-state state and output responses to the input $\mathbf{u}(t)$ are*

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \begin{pmatrix} |\mathbf{H}_1(j\omega)| \sin(\omega t + \text{Arg}(\mathbf{H}_1(j\omega))) \\ \cdots \\ |\mathbf{H}_n(j\omega)| \sin(\omega t + \text{Arg}(\mathbf{H}_n(j\omega))) \end{pmatrix} \quad (46)$$

$$\mathbf{y}^{(ss)}(t, \mathbf{u}) = |\mathbf{W}(j\omega)| \sin(\omega t + \text{Arg}(\mathbf{W}(j\omega))) \quad (47)$$

We remark that the function $\mathbf{x}^{(ss)}(t, \mathbf{u})$ could have be computed as the function of the form $\mathbf{x}^{(ss)}(t, \mathbf{u}) = c\cos\omega t + d\sin\omega t$, for some $c, d \in \mathbb{R}^n$, such that

$$\dot{\mathbf{x}}^{(ss)}(t, \mathbf{u}) = A\mathbf{x}^{(ss)}(t, \mathbf{u}) + B\mathbf{u}(t) \quad (48)$$

for all $t \geqslant 0$ with $\mathbf{u}(t) = \sin\omega t$. The coefficients c, d are calculated from the identity (47). Therefore, $\mathbf{x}^{(ss)}(t, \mathbf{u})$ is the solution of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \quad (49)$$

with $\mathbf{u}(t) = \sin\omega t$ and initial condition $x_0 = c$.

We can prove a similar result for cosinusoidal inputs.

Theorem *3.2: Assume that (3) is asymptotically stable. For an input function $\mathbf{u}(t) = \cos\omega t$ the steady-state state and output responses are*

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \begin{pmatrix} |\mathbf{H}_1(j\omega)| \cos(\omega t + \text{Arg}(\mathbf{H}_1(j\omega))) \\ \cdots \\ |\mathbf{H}_n(j\omega)| \cos(\omega t + \text{Arg}(\mathbf{H}_n(j\omega))) \end{pmatrix} \quad (50)$$

$$\mathbf{y}^{(ss)}(t, \mathbf{u}) = |\mathbf{W}(j\omega)| \cos(\omega t + \text{Arg}(\mathbf{W}(j\omega))) \quad (51)$$

IV. Steady-state response to exponential inputs

In this section we calculate the steady-state response to *exponential* inputs $\mathbf{u}(t) = e^{at}$ for some $a \in \mathbb{R}$. This class of inputs satisfies the conditions of theorem 1.1 since form all $t \in \mathbb{R}$

$$|\mathbf{u}(t)| \leqslant e^{|a|t} \quad (52)$$

The steady-state state response $x^{(ss)}(\cdot)$ is by virtue of theorem 1.1

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau \quad (53)$$

If $a \notin \sigma(A)$ we have that $(A - aI)$ is nonsingular and

$$\begin{aligned} &\lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau = \int_{t_0}^{t} e^{A(t-\tau)} B e^{a\tau} d\tau \\ &= e^{At} \lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{-(A-aI)\tau} B d\tau) \\ &= -e^{At} \lim_{t_0 \to -\infty} e^{-(A-aI)\tau} (A - aI)^{-1} B \Big|_{t_0}^{t} \\ &= -e^{at} (A - aI)^{-1} B = e^{at} \mathbf{H}(a) \end{aligned} \quad (54)$$

If $a \in \sigma(A)$, since (3) is asymptotically stable then $\sigma(A) \subset \mathbb{C}^-$ and $a < 0$. We will not consider this case, since for any input $\mathbf{u}(t) = e^{at}$ with $a < 0$ it is easy to see that

$$\lim_{t \to \infty} \mathbf{x}^{(ss)}(t, \mathbf{u}) = 0. \quad (55)$$

Theorem *4.1: Assume that (3) is asymptotically stable. For each input function $\mathbf{u}(t) = e^{at}$ for which $a \notin \sigma(A)$ the steady-state state and output responses*

$$\begin{aligned} \mathbf{x}^{(ss)}(t, \mathbf{u}) &= e^{at} \mathbf{H}(a) \\ \mathbf{y}^{(ss)}(t, \mathbf{u}) &= e^{at} \mathbf{W}(a) \end{aligned} \quad (56)$$

We remark that the function $\mathbf{x}^{(ss)}(t, \mathbf{u})$ could have be computed as the function of the form $\mathbf{x}^{(ss)}(t, \mathbf{u}) = ce^{at}$, for some $c \in \mathbb{R}^n$, such that

$$\dot{\mathbf{x}}^{(ss)}(t, \mathbf{u}) = A\mathbf{x}^{(ss)}(t, \mathbf{u}) + B\mathbf{u}(t) \quad (57)$$

for all $t \geqslant 0$ with $\mathbf{u}(t) = e^{at}$. The coefficient c is calculated from the identity (56). Therefore, $\mathbf{x}^{(ss)}(t, \mathbf{u})$ is the solution $\mathbf{x}(t, x_0, \mathbf{u})$ of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \quad (58)$$

with $\mathbf{u}(t) = e^{at}$ and initial condition $x_0 = c$.

V. Delayed inputs

In this section we calculate the steady-state response to *delayed* inputs $\mathbf{u}(t) = \mathbf{u}^{(0)}_{t-T}$ with real $T > 0$ and the function $\mathbf{u}^{(0)}(t)$ satisfy the assumptions of theorem 1.1. Also the inputs $\mathbf{u}(t)$ satisfies the conditions of theorem 1.1. Indeed, for all $t \geqslant 0$

$$\begin{aligned} &|\mathbf{u}(t)| = |\mathbf{u}^{(0)}_{t-T}| \\ &\leqslant ce^{k(t-T)} \sum_{j=0}^{n} |t - T|^j \leqslant c_T e^{kt} \sum_{j=0}^{n} |t|^j \end{aligned} \quad (59)$$

for some positive real c_T (depending on T). The steady-state state response is by virtue of theorem 1.1

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau \quad (60)$$

We have

$$\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau = \int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}^{(0)}(\tau-T)d\tau$$
$$= \lim_{t_0\to-\infty}\int_{t_0-T}^{t-T} e^{A(t-\theta-T)}B\mathbf{u}^{(0)}(\theta)d\theta$$
$$= \lim_{t_0\to-\infty}\int_{t_0}^{t-T} e^{A(t-T-\theta)}B\mathbf{u}^{(0)}(\theta)d\theta = \mathbf{x}^{(ss,0)}(t-T,\mathbf{u}^{(0)}) \tag{61}$$

where $\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)})$ is the steady-state state response to the input $\mathbf{u}^{(0)}(t)$.

Theorem *5.1: Assume that (3) is asymptotically stable. For an input function $\mathbf{u}(t) = \mathbf{u}^{(0)}_{t-T}$, where $\mathbf{u}^{(0)}(t)$ satisfy the assumptions of theorem 1.1 and for which the steady-state state and output responses are $\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)})$ and $\mathbf{y}^{(ss,0)}(t,\mathbf{u}^{(0)})$, the steady-state state and output responses are*

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \mathbf{x}^{(ss,0)}(t-T,\mathbf{u}^{(0)}),$$
$$\mathbf{y}^{(ss)}(t,\mathbf{u}) = \mathbf{y}^{(ss,0)}(t-T,\mathbf{u}^{(0)}). \tag{62}$$

VI. Linear combinations of inputs admitting a steady-state response

In this section we calculate the steady-state state response to *linear combinations of inputs* admitting each a steady-state state response, i.e. $\mathbf{u}(t) = \sum_{i=0}^{r} c^{(i)}\mathbf{u}^{(i)}(t)$ for some $c^{(0)},\dots,c^{(r)} \in \mathbb{R}$ and input functions $\mathbf{u}^{(0)}(t),\dots,\mathbf{u}^{(r)}(t)$ satisfying the assumptions of theorem 1.1, i.e. for all $t\in\mathbb{R}$

$$|\mathbf{u}(t)^{(i)}| \leqslant c^{(i)}e^{k_i t}\sum_{j=0}^{n^{(i)}}|t|^j \tag{63}$$

for some $c^{(i)}, k^{(i)} > 0$ and integers $n^{(i)}$, $i = 0,\dots,r$. Also the input $\mathbf{u}(t)$ satisfies the conditions of theorem 1.1, as it can be easily proved. We have

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$
$$= \lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\sum_{i=0}^{r}c^{(i)}\mathbf{u}^{(i)}(\tau)d\tau$$
$$= \sum_{i=0}^{r}c^{(i)}\lim_{t_0\to-\infty}\int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}^{(i)}(\tau)d\tau = \sum_{i=0}^{r}c^{(i)}\mathbf{x}^{(ss,i)}(t,\mathbf{u}^{(i)})$$

where $\mathbf{x}^{(ss,i)}(t,\mathbf{u}^{(i)})$ is the steady-state response to the input $\mathbf{u}^{(i)}(t)$.

Theorem *6.1: Assume that (3) is asymptotically stable. For an input function $\mathbf{u}(t) = \sum_{i=0}^{r} c^{(i)}\mathbf{u}^{(i)}(t)$, where $c^{(0)},\dots,c^{(r)} \in \mathbb{R}$ and the input functions $\mathbf{u}(t)^{(0)},\dots,\mathbf{u}(t)^{(r)}$ satisfy the assumptions of theorem 1.1, with steady-state state and output responses, respectively, $\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)}),\dots,\mathbf{x}^{(ss,r)}(t,\mathbf{u}^{(r)})$ and $\mathbf{y}^{(ss,0)}(t,\mathbf{u}^{(0)}),\dots,\mathbf{y}^{(ss,r)}(t,\mathbf{u}^{(r)})$, the steady-state state and output responses are*

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \sum_{i=0}^{r}c^{(i)}\mathbf{x}^{(ss,i)}(t,\mathbf{u}^{(i)})$$
$$\mathbf{y}^{(ss)}(t,\mathbf{u}) = \sum_{i=0}^{r}c^{(i)}\mathbf{y}^{(ss,i)}(t,\mathbf{u}^{(i)}). \tag{64}$$

Exercize *6.1: Consider the model of the simple pendulum with $m = k$ and $l = g$, linearized around null angular position and velocity,*

$$\dot{\mathbf{x}}_1(t) = \mathbf{x}_2(t)$$
$$\dot{\mathbf{x}}_2(t) = -\mathbf{x}_1(t) - \mathbf{x}_2(t) \tag{65}$$

Calculate the steady-state response to $\mathbf{u}(t) = \sin(t-1)$ and $\mathbf{u}(t) = t+2$.

We first calculate the steady-state response to $\mathbf{u}(t) = \sin(t-1)$. Note that (64) and the input functions $\mathbf{u}^{(0)}(t) = \sin(t)$ satisfy the assumptions of theorems 3.2 and 5.1. The steady-state response of (64) to $\mathbf{u}^{(0)}(t) = \sin(t)$ is by theorem 3.1 (with $\omega = 1$)

$$\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)}) = \begin{pmatrix}|\mathbf{H}_1(j)|\sin(t+\mathrm{Arg}(\mathbf{H}_1(j)))\\|\mathbf{H}_2(j)|\sin(t+\mathrm{Arg}(\mathbf{H}_2(j)))\end{pmatrix}$$

Since $A = \begin{pmatrix}0 & 1\\-1 & -1\end{pmatrix}$ and $B = \begin{pmatrix}0\\1\end{pmatrix}$,

$$\mathbf{H}(s) = \begin{pmatrix}\mathbf{H}_1(s)\\\mathbf{H}_2(s)\end{pmatrix} = (sI-A)^{-1}B = \frac{1}{s^2+s+1}\begin{pmatrix}1\\s\end{pmatrix}$$

and

$$\mathbf{H}(j) = \begin{pmatrix}\mathbf{H}_1(j)\\\mathbf{H}_2(j)\end{pmatrix} = \begin{pmatrix}-j\\1\end{pmatrix}$$

Therefore,

$$\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)}) = \begin{pmatrix}\sin(t-\frac{\pi}{2})\\\sin t\end{pmatrix} = \begin{pmatrix}-\cos t\\\sin t\end{pmatrix}$$

By theorem 5.1 (with $T = 1$)

$$\mathbf{x}^{(ss)}(t,\mathbf{u}) = \mathbf{x}^{(ss,0)}(t-1,\mathbf{u}^{(0)}) = \begin{pmatrix}-\cos(t-1)\\\sin(t-1)\end{pmatrix}$$

We finally calculate the steady-state response to $\mathbf{u}(t) = t+2$. Note that (64) and also the input functions $\mathbf{u}^{(0)}(t) = t$ and $\mathbf{u}^{(1)}(t) = 1$ satisfy the assumptions of theorems 3.2 and 6.1. The steady-state response of (64) to $\mathbf{u}^{(0)}(t) = t$ is by theorem 2.1

$$\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)}) = \sum_{j=0}^{1}\frac{d^j\mathbf{H}}{ds^j}\Big|_{s=0}\frac{t^{1-j}}{(1-j)!j!}$$

Since

$$\mathbf{H}(s) = \begin{pmatrix}\mathbf{H}_1(s)\\\mathbf{H}_2(s)\end{pmatrix} = (sI-A)^{-1}B = \frac{1}{s^2+s+1}\begin{pmatrix}1\\s\end{pmatrix}$$
$$\frac{d}{ds}\mathbf{H}(s) = \begin{pmatrix}\frac{d}{ds}\mathbf{H}_1(s)\\\frac{d}{ds}\mathbf{H}_2(s)\end{pmatrix} = -\frac{1}{s^2+s+1}^2\begin{pmatrix}2s+1\\s^2-1\end{pmatrix} \tag{66}$$

Therefore,

$$\mathbf{x}^{(ss,0)}(t,\mathbf{u}^{(0)}) = \mathbf{H}(0)t+\frac{d\mathbf{H}}{ds}\Big|_{s=0} = \begin{pmatrix}1\\0\end{pmatrix}t-\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}t-1\\1\end{pmatrix}$$

Next, calculate the steady-state response of (64) to $\mathbf{u}^{(1)}(t) = 2$. The steady-state response of (64) to $\mathbf{u}(t)^{(1,0)} = 1$ is by theorem 2.1

$$\mathbf{x}^{(ss,1,0)}(t,\mathbf{u}^{(1,0)}) = \mathbf{H}(0)$$

Therefore, by theorem 6.1

$$\mathbf{x}^{(ss,1)}(t,\mathbf{u}^{(1)}) = 2\mathbf{H}(0) = \begin{pmatrix}2\\0\end{pmatrix}$$

Finally, by theorem 6.1 (with $r = 2$)

$$\mathbf{x}^{(ss)}(t, \mathbf{u}) = \sum_{i=0}^{1} \mathbf{x}^{(ss,i)}(t)(\mathbf{u}^{(i)}) = \begin{pmatrix} t-1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} t+1 \\ 1 \end{pmatrix} . \lhd \tag{67}$$

VII. Steady state responses from frequency domain

As an alternative procedure, it is possible to obtain the steady state responses in Laplace domain and, using inverse transformation, finally in time domain. Let $\mathbf{P}(s)$ be the I/O transfer function with all poles in $\mathbb{C}^-$ and of the form

$$\mathbf{P}(s) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \frac{a_{i,j}}{(s-p_i)^j} \tag{68}$$

with distinct poles $p_1, \ldots, p_r \in \mathbb{C}^-$ and residuals $a_{i,j}$. Morever, consider an input $\mathbf{u}(t)$ such that its transform has the form

$$\mathfrak{L}[\mathbf{u}(t)](s) = \mathbf{u}(s) = \sum_{h=1}^{r'} \sum_{l=1}^{\mu'_h} \frac{a'_{h,l}}{(s-p'_h)^l} \tag{69}$$

with distinct poles $p'_1, \ldots, p'_{r'} \notin \mathbb{C}^-$ and residuals $a'_{i,j}$. The forced output response is

$$\mathfrak{L}[\mathbf{y}^{(u)}(t, \mathbf{u})](s) = \mathbf{P}(s)\mathbf{u}(s) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \frac{a_{i,j}}{(s-p_i)^j} \sum_{h=1}^{r'} \sum_{l=1}^{\mu'_h} \frac{a'_{h,l}}{(s-p'_h)^l} \tag{70}$$

According to the residuals theorem and since $\{p_1, \ldots, p_r\} \cap \{p'_1, \ldots, p'_r\} = \{\varnothing\}$

$$\mathfrak{L}[\mathbf{y}^{(u)}(t, \mathbf{u})](s) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \frac{R_{j,i}}{(s-p_i)^j} + \sum_{i=1}^{r'} \sum_{j=1}^{\mu'_i} \frac{R'_{j,i}}{(s-p'_i)^j} \tag{71}$$

with distinct poles $p_1, \ldots, p_r, p'_1, \ldots, p'_{r'}$ and residuals $a_{i,j}, a'_{i,j}$. Since $p_1, \ldots, p_r \in \mathbb{C}^-$

$$\mathfrak{L}^{-1}[\sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \frac{R_{j,i}}{(s-p_i)^j}](t) \to 0 \tag{72}$$

as $t \to +\infty$ and morever since the system $\mathbf{P}(s)$ is asymptotically stable, the unforced response tends to 0

$$\mathbf{y}^{(0)}(t, x_0) \to 0 \tag{73}$$

as $t \to +\infty$. It follows that

$$|\mathbf{y}^{(u)}(t, \mathbf{u}) - \mathfrak{L}^{-1}[\sum_{i=1}^{r'} \sum_{j=1}^{\mu'_i} \frac{R'_{j,i}}{(s-p'_i)^j}](t)| \to 0 \tag{74}$$

so that

$$\mathbf{y}^{(ss)}(t, \mathbf{u}) = \mathfrak{L}^{-1}[\sum_{i=1}^{r'} \sum_{j=1}^{\mu'_i} \frac{R'_{j,i}}{(s-p'_i)^j}](t) \tag{75}$$

Exercize *7.1: Given the system*

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= \mathbf{u}(t) - a^2\mathbf{x}_1(t) - 2a\mathbf{x}_2(t) \\ \mathbf{y}_1(t) &= \mathbf{x}_1(t) \end{aligned} \tag{76}$$

determine for which values of $a \in \mathbb{R}$ it has a well-defined steady state regime and calculate the output response to inputs $\mathbf{u}(t) = \cos(t)$.

The state space representation of the system is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \tag{77}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -a^2 & -2a \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{78}$$

The characteristic polynomial of A is $\mathbf{p}(\lambda) = (s+a)^2$ and its eigenvalues are all equal to $\lambda = -a$. Therefore, the system has a well-defined steady state regime for $a > 0$. The I/O transfer function is

$$\mathbf{P}(s) = C(sI - A)^{-1}B = \frac{1}{(s+a)^2} \tag{79}$$

The forced output response to $\mathbf{u}(t) = \sin(t)$ is

$$\mathfrak{L}[\mathbf{y}^{(u)}(t, \mathbf{u})](s) = \mathbf{P}(s)\mathfrak{L}[\cos(t)](s) = \frac{1}{(s+a)^2}\frac{s}{s^2+1} \tag{80}$$

By the residuals theorem and comparison method

$$\frac{1}{(s+a)^2}\frac{s}{s^2+1} = \frac{R_{11}}{s+a} + \frac{R_{12}}{(s+a)^2} + \frac{As+B}{s^2+1} \tag{81}$$

with

$$\begin{aligned} R_{11} &= \frac{1-a^2}{(1+a^2)^2}, \ R_{12} = -\frac{a}{1+a^2}, \\ A &= \frac{a^2-1}{(1+a^2)^2}, \ B = \frac{2a}{(1+a^2)^2}. \end{aligned} \tag{82}$$

Back to time domain

$$\mathbf{y}^{(u)}(t, \mathbf{u}) = R_{11}e_+^{-at} + R_{12}te_+^{-at} + A\cos_+(t) + B\sin_+(t) \tag{83}$$

Since

$$R_{11}e_+^{-at} + R_{12}te_+^{-at} \to 0 \tag{84}$$

as $t \to +\infty$,

$$\begin{aligned} \mathbf{y}(t)^{(ss)} &= A\cos(t) + B\sin(t) \\ &= \frac{a^2-1}{(1+a^2)^2}\cos(t) + \frac{2a}{(1+a^2)^2}\sin(t) \\ &= M\cos(t+N) \end{aligned} \tag{85}$$

with

$$\begin{aligned} M &= \frac{1}{1+a^2} = |\mathbf{P}(j)|, \\ N &= \text{Atan}(\frac{-2a}{a^2-1}) = \text{Arg}(\mathbf{P}(j)). \end{aligned} \tag{86}$$