Notes on Linear Control Systems: Module X

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***Abstract*—Root locus. Stabilization and pole placement in frequency domain.**

I. Poles assignment: the root locus

The Nyquist criterion together with the methods based on the stability margins are useful for a qualitative evaluation of the stability properties of a feedback system. On the other hand a more accurate design requires a deeper knowledge of the poles of the closed-loop system and how these poles influence the critical design parameters. To this aim, the *root locus* is a description of how the poles of the closed-loop system vary in terms of a proportional control action K.

As we know, the transfer function of a feedback interconnection of a given process $\mathbf{P}(s)$ with unitary feedback and proportional control action K is

$$\mathbf{W}(s) = \frac{K\mathbf{P}(s)}{1 + K\mathbf{P}(s)} \tag{1}$$

The zeroes of $1 + K\mathbf{P}(s)$ (or, equivalently, the roots of $\mathrm{NUM}(1 + K\mathbf{P}(s))$) are the poles of the feedback system $\mathbf{W}(s)$. The *root locus* of $\mathbf{P}(s)$ is the locus of the zeroes of $1 + K\mathbf{P}(s)$ parametrized by $K \in \mathbb{R}$, i.e. the set of points

$$(s, K) = (s(K), K) \in \mathbb{C} \times \mathbb{R} \tag{2}$$

such that

$$1 + K\mathbf{P}(s) = 0 \tag{3}$$

or, equivalently,

$$\mathrm{NUM}(1 + K\mathbf{P}(s)) = 0 \tag{4}$$

The locus of $\mathbf{P}(s)$ corresponding to positive values of $K \in \mathbb{R}^+$ is called *positive root locus* of $\mathbf{P}(s)$ and the locus corresponding to negative values of $K \in \mathbb{R}^-$ is called *negative root locus*.

In order to establish few basic rules for drawing the root locus of $\mathbf{P}(s)$, we assume that $\mathbf{P}(s)$ is strictly proper and has the following *zero-pole* form

$$\mathbf{P}(s) = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} \tag{5}$$

where $m < n$, $z_1, \ldots, z_m$ are the poles and $p_1, \ldots, p_n$ the poles, respectively, of $\mathbf{P}(s)$. In general a strictly proper function $\mathbf{P}(s)$ can be always re-written as

$$\mathbf{P}(s) = H\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} = H\mathbf{P}'(s) \tag{6}$$

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with $H \in \mathbb{R}$ and $\mathbf{P}'(s)$ in zero-pole form. In this case it is necessary to change K as $K' = KH$ so that, since $1 + K\mathbf{P}(s) = 1 + K'\mathbf{P}'(s)$, what we actually represent is the locus of the zeroes of $1 + K'\mathbf{P}'(s)$ parametrized by $K' \in \mathbb{R}$. The corresponding values of $K \in \mathbb{R}$ are obtained from $K' \in \mathbb{R}$ by inversion of the re-parametrization $K' = KH$ as $K = \frac{H}{K'}$.

Notice that the root locus of $\mathbf{P}(s)$ is given equivalently by the set of points

$$(s, K) = (s(K), K) \in \mathbb{C} \times \mathbb{R} \tag{7}$$

such that

$$\mathbf{P}(s) = -\frac{1}{K} \tag{8}$$

Accrodingly, the positive root locus of $\mathbf{P}(s)$ is equivalently represented by the set of points $(s, K) = (s(K), K) \in \mathbb{C} \times \mathbb{R}^+$ such that

$$\begin{aligned} |\mathbf{P}(s)| &= \frac{1}{K} \\ \mathrm{Arg}\{\mathbf{P}(s)\} &= (2h+1)\pi,\ h \in \mathbb{Z}, \end{aligned} \tag{9}$$

while positive root locus of $\mathbf{P}(s)$ is equivalently represented by the set of points $(s, K) = (s(K), K) \in \mathbb{C} \times \mathbb{R}^-$ such that

$$\begin{aligned} |\mathbf{P}(s)| &= \frac{1}{|K|} \\ \mathrm{Arg}\{\mathbf{P}(s)\} &= 2h\pi,\ h \in \mathbb{Z}. \end{aligned} \tag{10}$$

From the computational point of view, the second equations in (9) and, respectively, (10) (the *module* root locus equations) are instrumental to plot the curve of the locus in $\mathbb{C}$ while the first equations in (9) and (10), respectively, (the *phase* root locus equations) are instrumental to associate the corresponding values of K to each point of the curve.

II. Root locus properties

In this section we want to describe some key properties and technical rules which help drawing the root locus.

(P1) *The positive (resp. negative) locus has as many curves as the number of poles of $\mathbf{P}(s)$. For K increasing from 0 to $+\infty$ (resp. from $-\infty$ to 0) each curve of the positive (resp. negative) locus starts (resp. ends) for $K = 0$ from (resp. in) the poles of $\mathbf{P}(s)$ and tends to (resp. comes from) either one of the m zeroes of $\mathbf{P}(s)$ or the point at infinity.*

Technical explanation. As a consequence of the magnitude condition in (9) (resp. (10)) if $K \to 0^{\pm}$ then $s \to p_i$ for at least one i since $\lim_{s \to p_i} |\mathbf{P}(s)| = +\infty$. It follows that the ponts $(s, K) = (p_i, 0)$, $i = 1, \dots, n$, are points of the locus. Moreover, as a consequence of (9) (resp. (10)) if $K \to \pm\infty$ then either $s \to z_i$ for at least one i or $|s| \to +\infty$ since $\lim_{s \to z_i} |\mathbf{P}(s)| = 0$ and $\lim_{|s| \to \infty} |\mathbf{P}(s)| = 0$. For K increasing from 0 to $+\infty$ (resp. from $-\infty$ to 0) each curve of the root locus runs from a pole of $\mathbf{P}(s)$ (resp. either one of the m zeroes of $\mathbf{P}(s)$ or the point at infinity) to either one of the m zeroes of $\mathbf{P}(s)$ or the point at infinity (resp. a pole of $\mathbf{P}(s)$).

(P2) *The positive (resp. negative) locus is symmetric with respect to the real axis.*

This follows from the fact $\mathbf{P}(s^*) = \mathbf{P}^*(s)$ for all $s \in \mathbb{C}$.

(P3) *A point of the real axis is a point of the positive (resp. negative) locus if it leaves to his left an odd (resp. even) number of poles/zeroes.*

Technical explanation. A point of the real axis is a point of the positive (resp. negative) locus must satisfies the phase condition in (9) (resp. (10)). Notice that

$$\mathrm{Arg}\{\mathbf{P}(s)\} = \sum_{i=1}^{m} \mathrm{Arg}\{s - z_i\} - \sum_{i=1}^{n} \mathrm{Arg}\{s - p_i\} \quad (11)$$

A term $s - z_i$ (resp. $s - p_i$) is a vector on the complex plane which starts from z_i (resp. p_i) and points into s. If s is a point of the real axis, any term $s - z_i$ (resp. $s - p_i$) for real z_i (resp. real p_i) contributes $2h\pi,\ h \in \mathbb{Z}$, to the phase of $\mathbf{P}(s)$ if z_i (resp. p_i) lies on the right of s and $(2h+1)\pi,\ h \in \mathbb{Z}$, if z_i (resp. p_i) lies on the left of s (the right/left of a point is meant to be the left/right of an external observer). On the other hand any term $(s - z_i)(s - z_i^*)$ (resp. $(s - p_i)(s - p_i^*)$) for complex conjugate z_i, z_i^* (resp. p_i, p_i^*) contributes $2h\pi,\ h \in \mathbb{Z}$, to the phase of $\mathbf{P}(s)$. Therefore, for any point s of the real axis, if an odd (resp. even) number of poles/zeroes lie on its left these poles/zeroes contibute $(2h+1)\pi,\ h \in \mathbb{Z}$ (resp. $2h\pi,$) to the phase of $\mathbf{P}(s)$ and s is a point of the positive (resp. negative) locus.

(P4) *A point with multiplicity μ of the positive (resp. negative) locus corresponds to the intersection point of μ curves of the positive (resp. negative) locus. This point is regular if $\mu = 1$ and singular if $\mu > 1$. Any singular point (s, K) (with multiplicity $\mu > 2$) is solution of the root locus equation (3) and its derivative:*

$$\begin{aligned} &1 + K\mathbf{P}(s) = 0, \\ &\frac{\partial}{\partial s}(1 + K\mathbf{P}(s)) = 0 \end{aligned} \quad (12)$$

(the last equation referred to as singular points equation). There are at most $n + m - 1$ singular points.

Technical explanation. This follows from the fact that a zero of $1 + K\mathbf{P}(s)$ with multiplicity μ is also a zero of the first $\mu - 1$ derivatives of $1 + K\mathbf{P}(s)$. Equations (12) are equivalent to

$$\begin{aligned} &\mathrm{NUM}(1 + K\mathbf{P}(s)) = 0, \\ &\frac{\partial}{\partial s}\mathrm{NUM}(1 + K\mathbf{P}(s)) = 0 \end{aligned} \quad (13)$$

From the zero-pole form of $\mathbf{P}(s)$, it turn out that equations (12) (or (13)) are equivalent to

$$\begin{aligned} &1 + K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} = 0, \\ &\sum_{i=1}^{m} \frac{1}{s - z_i} - \sum_{i=1}^{n} \frac{1}{s - p_i} = 0. \end{aligned} \quad (14)$$

Moreover, if we want to determine also the multiplicity of a singular point, for the set of singular points with given multiplicity μ we have to find the set of points s solving the following equations:

$$\begin{aligned} &\frac{\partial^r}{\partial s^r}(1 + K\mathbf{P}(s)) = 0,\ r = 0, 1, \dots, \mu - 1, \\ &\frac{\partial^\mu}{\partial s^\mu}(1 + K\mathbf{P}(s)) \neq 0, \end{aligned} \quad (15)$$

or, equivalently,

$$\begin{aligned} &\frac{\partial^r}{\partial s^r} NUM(1 + K\mathbf{P}(s)) = 0,\ r = 0, 1, \dots, \mu - 1, \\ &\frac{\partial^\mu}{\partial s^\mu} NUM(1 + K\mathbf{P}(s)) \neq 0. \end{aligned} \quad (16)$$

(P5) *If (s, K) is a singular point of the positive (resp. negative) locus with multiplicity μ the directions along which the μ curves approach the point (s, K) and intersect each other at that point form a star, with center in the point (s, K) and radii splitting the angle $360°$ into μ equal angles: the sign (positive or negative) of the locus is such that incoming curves alternate with outcoming curves.*

Technical explanation. Let (s°, K°) be a point of the positive locus with multiplicity μ (the argument is the same for the negative locus). Since (s°, K°) is a point of the locus

$$\mathbf{P}(s^\circ) + \frac{1}{K^\circ} = 0 \quad (17)$$

On the other hand for any point (s, K) of the locus

$$\mathbf{P}(s) + \frac{1}{K} = 0 \quad (18)$$

or what is the same

$$\mathbf{P}(s) + \frac{1}{K^\circ} = \varepsilon \quad (19)$$

with $\varepsilon := \frac{1}{K^\circ} - \frac{1}{K}$.

Since (s°, K°) has multiplicity μ and on account of (17), locally around s°

$$\mathbf{P}(s) + \frac{1}{K^\circ} = (s - s^\circ)^\mu \mathbf{f}(s) \tag{20}$$

for some polynomial $f(s)$. Therefore, locally around (s°, K°) (i.e. for $K \in (-\overline{K} + K^\circ, \overline{K} + K^\circ)$ with sufficiently small $\overline{K} > 0$) a point (s, K) of the positive locus is such that

$$(s - s^\circ)^\mu \mathbf{f}(s^\circ) \approx \varepsilon \tag{21}$$

or equivalently

$$(s - s^\circ)^\mu \approx \frac{\varepsilon}{\mathbf{f}(s^\circ)} \tag{22}$$

Note that, since $\mathbf{f}(s^\circ) = \rho^\circ e^{-j\theta^\circ}$ for some $\theta^\circ \in \mathbb{R}$ and $\rho^\circ > 0$, the above equation has the form $(s - s^\circ)^\mu = He^{j\theta^\circ}$, with

$$H := \frac{\varepsilon}{\rho^\circ} = \frac{\frac{1}{K^\circ} - \frac{1}{K}}{\rho^\circ}$$

varying from $\overline{H}^-$ to $\overline{H}^+$ where

$$\overline{H}^\pm := \frac{\frac{1}{K^\circ} - \frac{1}{\pm\overline{K} + K^\circ}}{\rho^\circ}$$

Notice $\overline{H}^+ > 0$ and $\overline{H}^- < 0$ and that values of $H \in (\overline{H}^-, 0)$ corresponds to values of $K \in (-\overline{K} + K^\circ, K^\circ)$ while values of $H \in (0, \overline{H}^+)$ correspond to values of $K \in (K^\circ, \overline{K} + K^\circ)$. Therefore, the curves which correspond to values of $K \in$ flow out of the point s° and the curves which correspond to values of $K \in (-\overline{K} + K^\circ, K^\circ)(K^\circ, \overline{K} + K^\circ)$ flow into the point s°. Property (iv) follows from the next lemma.

Lemma *2.1: The root locus (s, H) of*

$$(s - s^\circ)^\mu = He^{j\theta^\circ} \tag{23}$$

with $s^\circ \in \mathbb{C}$ and $\theta^\circ \in \mathbb{R}$, describes on the complex plane a star centered at s° with consecutive radii forming angles of $\frac{\pi}{\mu}$. In particular, the radii of the positive (resp. negative) locus form with the real axis an angle

$$\theta = \frac{\theta^\circ + 2h\pi}{\mu},\ h = 0, 1, \ldots, \mu - 1 \tag{24}$$

$$(\text{resp. } \theta = \frac{\theta^\circ + (2h+1)\pi}{\mu},\ h = 0, 1, \ldots, \mu - 1\ ) \tag{25}$$

Remark *2.1:* Notice that the poles of $\mathbf{P}(s)$ with multiplicity $\mu > 1$ are singular points of the locus with the same multiplicity μ and are obtained from the singular points equation by setting $K = 0$ and solving in s. As such these points behave according to **(P4)** and **(P5)** . For this reason and since each curve of the positive (resp. negative) locus exits from a pole for incresing $k > 0$ (resp. enters into a pole for increasing $k < 0$), when a pole of $\mathbf{P}(s)$ is a singular point with multiplicity μ it is the intersection point of μ alternating curves from the positive and negative locus: μ flowing into the point (negative locus) and μ flowing out the point (positive locus).

On the other hand, the zeroes of $\mathbf{P}(s)$ with multiplicity $\mu > 1$ are not singular points of the locus, since these point cannot be obtained as solutions of the singular points equation (they are limit points of the locus for $K \to \pm\infty$). However, the locus around these points behaves according to the same rule **(P4)** and **(P5)** , i.e. the zeroes of $\mathbf{P}(s)$ with multiplicity is $\mu > 1$ behave as singular points with multiplicity is μ. $\triangleleft$

(P6) *m curves of the positive (resp. negative) locus approach the points $z_1, \ldots, z_m$ as $K \to \infty$ (resp. as $K \to -\infty$) while $n-m$ curves tend to the point at infinity along asymptotes which form a star with radii inclined w.r.t. the real positive axis by the angle (positive counterclockwise)*

$$\theta = \frac{(2h+1)\pi}{n - m},\ h = 0, 1, \ldots, n - m - 1 \tag{26}$$

$$(\text{resp. } \theta = \frac{2h\pi}{n - m},\ h = 0, 1, \ldots, n - m - 1\ ) \tag{27}$$

and centered at the point (the asymptotes center)

$$s_0 := \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m} \tag{28}$$

The number $n - m$ is also referred to as zero-pole excess or relative degree of P.

Technical explanation. Note that for all s

$$\prod_{i=1}^n (s - p_i) = (s - s_0)^{n-m} [\prod_{i=1}^m (s - z_i) + \mathbf{n}_0(s)] \tag{29}$$

where $n_0(s)$ is some $(m-2)$-degree polynomial. Since

$$|\frac{\mathbf{n}_0(s)}{\prod_{i=1}^n (s - p_i)}| \to 0 \text{ as } |s| \to +\infty$$

then for $|s| >> 1$ and, therefore, for $K >> 1$ the root locus of

$$\begin{aligned} 1 + K\mathbf{P}(s) &= 1 + K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)} \\ &= 1 + K \Big[\frac{1}{(s - s_0)^{n-m}} - \frac{\mathbf{n}_0(s)}{\prod_{i=1}^n (s - p_i)} \Big] \end{aligned}$$

is approximately the same as the root locus of

$$1 + \frac{K}{(s - s_0)^{n-m}}$$

or, equivalently, the root locus of

$$(s - s_0)^{n-m} + K \tag{30}$$

By lemma 2.1 with $H := K$ and $\theta^\circ = -\pi$ the root locus of (30) describes on the complex plane a star centered at s_0 with consecutive radii forming angles of $\frac{\pi}{n-m}$.

(P7) *The values of K for which the root locus crosses the imaginary axis are (among) the values of K for which the Routh table generated by $\mathrm{NUM}(1 + K\mathbf{P}(s))$ is not regular. Furthermore, the number N_V of sign variations in the first column of the Routh table corresponding to the values of K in some open interval $\mathfrak{I} \subset \mathbb{R}$ is equal to the number of curves $K \in \mathfrak{I} \to s(K) \in \mathbb{C}^+$ (i.e. the curves of the root locus in $\mathbb{C}^+$ corresponding to the values of $K \in I$).*

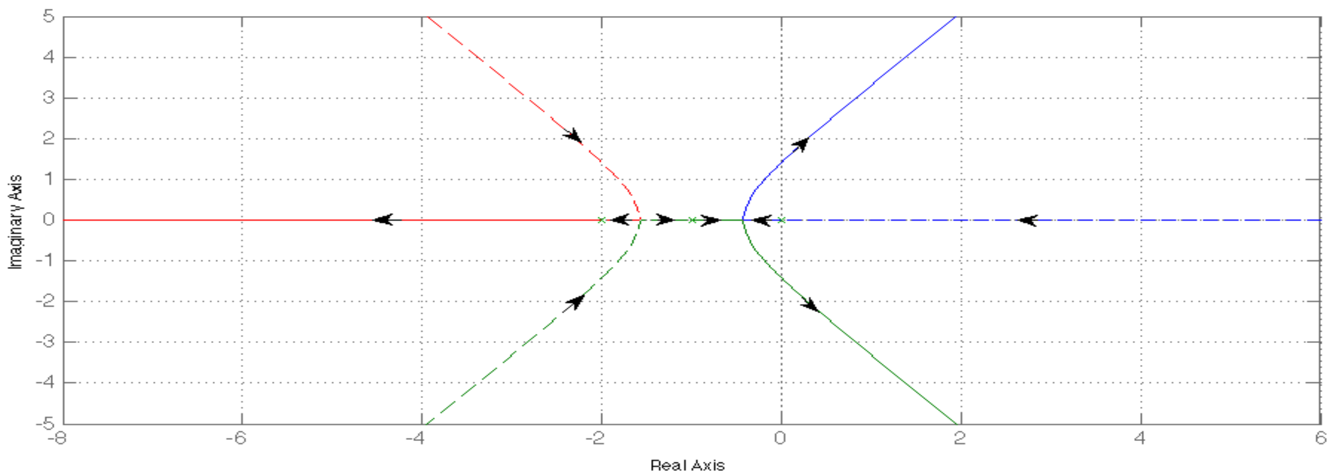


Figure 1. Root locus of $1 + K\mathbf{p}(s)$ with of $P(s) = \frac{1}{s(s+1)(s+2)}$.

Exercize *2.1: Plot the root locus of*

$$\mathbf{P}(s) := \frac{1}{s(s+1)(s+2)} \tag{31}$$

We denote the zeroes of $\mathbf{P}(s)$ on the complex plane by circles and the poles of $\mathbf{P}(s)$ by crosses. The poles of $\mathbf{P}(s)$ are $p_1 := 0$, $p_2 := -1$ and $p_3 := -2$ $(n = 3)$. $\mathbf{P}(s)$ has no zeroes $(m = 0)$. The root locus is shown in Figure 1.

The positive (resp. negative) locus has as many curves as the number of poles of $\mathbf{P}(s)$, i.e. 3 curves. Each curve of the positive (resp. negative) locus starts (resp. ends) for $K = 0$ from (resp. in) the poles of $\mathbf{P}(s)$ and tends to the point at infinity (rule **P1**): the direction of the curves is the one for which K varies from $-\infty$ to $+\infty$. Moreover, the positive (resp. negative) locus is symmetric with respect to the real axis (rule **P2**).

Notice that the points of the real axis are points of the positive (resp. negative) locus if they leave to their left an odd (resp. even) number of poles/zeroes (rule **P3**).

Each curve of the positive (resp. negative) locus approaches the point at infinity along directions (*asymptotes*) which form a star with 3 radii inclined to the real positive axis by $60°, 180°, 300°$ (resp. $0°, 60°, 240°$) and centered at the point

$$s_0 := \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{(-2-1)}{3} = -1 \tag{32}$$

(rule **P5**).

The singular points (with any multiplicity $\mu \geq 2$) are determined by the equations (rule **P4**)

$$\frac{\partial^r}{\partial s^r}[1 + K\mathbf{P}(s)] = 0, \; r = 0, 1 \tag{33}$$

which are also equivalent to

$$\begin{aligned} &1 + K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} = 0, \\ &\sum_{i=1}^{m} \frac{1}{s - z_i} - \sum_{i=1}^{n} \frac{1}{s - p_i} = 0 \end{aligned} \tag{34}$$

i.e.

$$\begin{aligned} &s(s+1)(s+2) + K = 0, \\ &3s^2 + 6s + 2 = 0 \end{aligned} \tag{35}$$

which have solutions

$$\begin{aligned} &s = -0.422, \; K = 0.385 \\ &s = -1.578, \; K = -0.910 \end{aligned} \tag{36}$$

(see Figure 1). In each one of these two points 2 curves intersect each other with alternating incoming/outcoming directions (rule **P4**). Moreover, the singular points (36) have multiplicity $\mu = 2$ since at these points

$$\frac{\partial^2}{\partial s^2}[1 + K\mathbf{P}(s)] \neq 0.$$

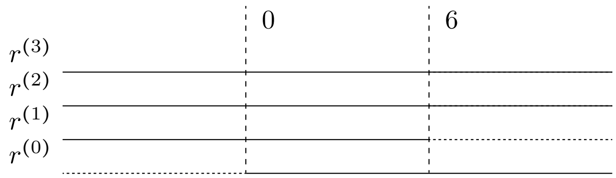
The Routh table generated by the numerator of $1 + K\mathbf{P}(s)$, i.e. the polynomial

$$\begin{aligned} \mathbf{w}(s) &:= \mathrm{NUM}(1 + K\mathbf{P}(s)) \\ &= s(s+1)(s+2) + K = s^3 + 3s^2 + 2s + K, \end{aligned} \tag{37}$$

is

$$\begin{array}{c|cc} r^{(3)} & 1 & 2 \\ r^{(2)} & 3 & K \\ r^{(1)} & 6-K & \\ r^{(0)} & K & \end{array} \tag{38}$$

We can discuss the number of sign variations $N_V(\mathbf{p})$ and permanencies $N_P(\mathbf{p})$ in the first column of the Routh table as follows. By rule **P7** the sign variations $N_V(\mathbf{p})$ for a certain interval $\mathfrak{I}$ of K establish the number of curves $s(K)$ of the root locus which lie in $\mathbb{C}^+$ for $K \in \mathfrak{I}$. The latter on the other hand corresponds to the number of closed-loop poles in $\mathbb{C}^+$.



Therefore, we have

- for $K = 0$ and $K = 6$ the table is not regular
- for $K < 0$ the table is regular and $N_V(\mathbf{w}) = 1$ and $N_P(\mathbf{w}) = 2$
- for $K \in (0, 6)$ the table is regular and $N_V(\mathbf{w}) = 0$ and $N_P(\mathbf{w}) = 3$
- for $K > 6$ the table is regular and $N_V(\mathbf{w}) = 2$ and $N_P(\mathbf{w}) = 1$

We conclude by virtue of the Routh criterion that (see the root locus in Figure 1)

- for $K = 6$ the curves of the positive root locus intersect the imaginary axis: the intersection points are obtained from the roots of

$$\mathrm{NUM}(1 + 6\mathbf{P}(s)) = s(s+1)(s+2) + 6 = 0 \tag{39}$$

  i.e. the root locus equation for $K = 6$. The roots are $s = \pm j\sqrt{2}$ and $s = -3$ and, clearly, $s = \pm j\sqrt{2}$ are the crossing points on the imaginary axis we look for ($s = -3$ is a point of the positive locus but not a crossing point on the imaginary axis).
- for $K < 0$ only one (entire) curve of the negative root locus lies in $\mathbb{C}^+$
- for $K \in (0, 6)$ there is no curve of the positive root locus lying in $\mathbb{C}^+$
- for $K > 6$ two curves of the positive root locus lie in $\mathbb{C}^+$.

□

Exercize *2.2: Plot the root locus of*

$$\mathbf{P}(s) := \frac{s+1}{s^2(s+3)} \tag{40}$$

We denote the zeroes of $\mathbf{P}(s)$ on the complex plane by circles and the poles of $\mathbf{P}(s)$ by crosses. The poles of $\mathbf{P}(s)$ are $p_1 := 0$ (with multiplicity 2) and $p_2 := -3$ ($n = 3$). The zeroes of $\mathbf{P}(s)$ are $z_1 := -1$ ($m = 1$). The root locus is shown in Figure 2.

Notice that the positive (resp. negative) locus has as many curves as the number of poles of $\mathbf{P}(s)$, i.e. 3 curves. Each curve of the positive (resp. negative) locus starts (resp. ends) for $K = 0$ from (resp. in) the poles of $\mathbf{P}(s)$ and tends to the point at infinity (rule **P1**): the direction of the curves is the one for which K varies from $-\infty$ to $+\infty$. The positive (resp. negative) locus is symmetric with respect to the real axis (rule **P2**).

Notice that the points of the real axis are points of the positive (resp. negative) locus if they leave to their left an odd (resp. even) number of poles/zeroes (rule **P3**).

Two curves of the positive (resp. negative) locus approaches the point at infinity along directions which form a star with 2 radii inclined to the real positive axis by $90°, 270°$ (resp. $0°, 180°$) and centered at the point

$$s_0 := \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m} = \frac{-3 - (-1)}{2} = -1 \tag{41}$$

(property **P5**).

The singular points (with any multiplicity $\mu \geq 2$) are determined by the equations (rule **P4**)

$$\frac{\partial^r}{\partial s^r}[1 + K\mathbf{P}(s)] = 0, \ r = 0, 1, \tag{42}$$

equivalent to

$$\begin{aligned} &1 + K\frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)} = 0, \\ &\sum_{i=1}^m \frac{1}{s - z_i} - \sum_{i=1}^n \frac{1}{s - p_i} = 0 \end{aligned} \tag{43}$$

i.e.

$$\begin{aligned} &s^2(s+3) + K(s+1) = 0, \\ &3s^2 + 6s + K = 0 \end{aligned} \tag{44}$$

which have solutions

$$\begin{aligned} &s = 0, \ K = 0 \\ &s = \frac{-3 + j\sqrt{3}}{2}, \ K = j3\sqrt{3} \\ &s = \frac{-3 - j\sqrt{3}}{2}, \ K = -j3\sqrt{3} \end{aligned} \tag{45}$$

The last two solutions must be discarded since they correspond to a complex value of the parameter K. In the point $s = 0$ (which is the pole of the open-loop system $\mathbf{P}(s)$) 2 curves intersect each other with alternating incoming/outcoming directions (rule **P4**: see Figure 2). The intersecting curves alternate from the positive and negative locus. Notice that when a singular points coincide with some open-loop pole, we know its multiplicity directly from the multiplicity as a pole of $\mathbf{P}(s)$. Notice also that indeed $s = 0$ is a singular with multiplicity $\mu = 2$ since for $s = 0$ we have $\frac{\partial^2}{\partial s^2}[1 + K\mathbf{P}(s)] \neq 0$.

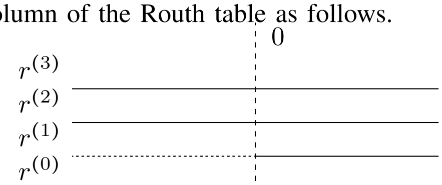
The Routh table generated by the numerator of $1 + K\mathbf{P}(s)$, i.e. the polynomial

$$\begin{aligned} \mathbf{w}(s) &:= \mathrm{NUM}(1 + K\mathbf{P}(s)) \\ &= s^2(s+3) + K(s+1) = s^3 + 3s^2 + Ks + K, \end{aligned} \tag{46}$$

is

$$\begin{array}{c|cc} r^{(3)} & 1 & K \\ r^{(2)} & 3 & K \\ r^{(1)} & 2K & \\ r^{(0)} & K & \end{array} \tag{47}$$

We can discuss the number of variations and permanencies in the first column of the Routh table as follows.



Therefore, we have

- for $K = 0$ the table is not regular
- for $K < 0$ the table is regular and $N_V(\mathbf{w}) = 1$ and $N_P(\mathbf{w}) = 2$
- for $K > 0$ the table is regular and $N_V(\mathbf{w}) = 0$ and $N_P(\mathbf{w}) = 3$

We conclude by virtue of the Routh criterion that (see Figure 2)

- for $K < 0$ only one (entire) curve of the negative root locus lies in $\mathbb{C}^+$
- for $K > 0$ all the curves of the positive root locus lie in $\mathbb{C}^-$

Moreover, there are no intersections with the imaginary axis. □

Exercize *2.3: Plot the root locus of* $1 + K\mathbf{P}(s)$ *with*

$$\mathbf{P}(s) := \frac{(s-1)^2}{s^2(s^2+1)} \tag{48}$$

We denote the zeroes of $\mathbf{P}(s)$ on the complex plane by circles and the poles of $\mathbf{P}(s)$ by crosses. The poles of $\mathbf{P}(s)$ are $p_1 := 0$ (with multiplicity 2), $p_2 := +j$ and $p_3 := -j$ ($n = 4$). The zeroes of $\mathbf{P}(s)$ are $z_1 := 1$ (with multiplicity 2: $m = 2$). The root locus is shown in Figure 3.

Notice that the positive (resp. negative) locus has as many curves as the number of poles of $\mathbf{P}(s)$, i.e. 3 curves. Each curve of the positive (resp. negative) locus starts (resp. ends) for $K = 0$ from (resp. in) the poles of $\mathbf{P}(s)$ and tends to the point at infinity (rule **P1**): the direction of the curves is the one for which K varies from $-\infty$ to $+\infty$. The positive (resp. negative) locus is symmetric with respect to the real axis (rule **P2**).

Notice that the points of the real axis are points of the positive (resp. negative) locus if they leave to their left an odd (resp. even) number of poles/zeroes (rule **P3**).

Two curves of the positive (resp. negative) locus approach the point at infinity along directions which form a star with 2 radii inclined to the real positive axis by $90°, 270°$ (resp. $0°, 180°$) and centered at the point

$$s_0 := \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n-m} = \frac{(j-j)-(1+1)}{2} = -1 \tag{49}$$

(rule ($\mathbf{P5}$)).

The singular points are determined by the equations (rule ($\mathbf{P4}$))

$$\frac{\partial^r}{\partial s^r}[1 + K\mathbf{P}(s)] = 0,\ r = 0, 1 \tag{50}$$

or, equivalently,

$$\begin{aligned} &1 + K\frac{\prod_{i=1}^m (s-z_i)}{\prod_{i=1}^n (s-p_i)} = 0, \\ &\sum_{i=1}^m \frac{1}{s-z_i} - \sum_{i=1}^n \frac{1}{s-p_i} = 0 \end{aligned} \tag{51}$$

i.e.

$$\begin{aligned} &s^2(s^2+1) + K(s-1)^2 = 0, \\ &2s^3 + s + K(s-1) = 0 \end{aligned} \tag{52}$$

which have real solutions

$$\begin{aligned} &s = 0,\ K = 0 \\ &s \approx 2.2,\ K \approx -17.97 \end{aligned} \tag{53}$$

(the other two solutions must be discarded since they correspond to a complex value of the parameter K). In the point $s = 0$ (which is the pole of the open-loop system $\mathbf{P}(s)$) 2 curves intersect each other with alternating incoming/outcoming directions (rule **P4**). Also, note that the intersecting curves in $s = 0$ alternates from the positive and negative locus. A similar situation happens at the point $s \approx 2.2$.

The Routh table generated by the numerator of $1 + K\mathbf{P}(s)$, i.e.

$$\mathrm{NUM}(1 + K\mathbf{P}(s)) = s^2(s^2+1) + K(s-1)^2$$

is not regular for all K (see Figure 3). This also means that for no values of K (positive or negative) the closed-loop poles are all in $\mathbb{C}^-$. □

Exercize *2.4: Plot the root locus of*

$$\mathbf{P}(s) := \frac{s(s^2+2s+2)}{(s+1)^2(s+2)^2} \tag{54}$$

The relative degree $n - m$ of $\mathbf{P}$ is 1 and we have at most $m - m + 1 = 6$ singular points. We have three possible (and equally plausible) root locuses for $\mathbf{P}(s)$ (Figure 4). Including the poles of $\mathbf{P}(s)$ at $s = -1$ and $s = -2$ (which are singular points with multiplicity 2 corresponding to $k = 0$), in Figure a) we have 4 singular points, in Figure b) we have 6 singular points and in Figure c) we have 4 singular points. However, the root locus of $\mathbf{P}(s)$ solved with Matlab shows that the exact one is the second plot in Figure 4. As a matter of fact, excluding the poles of $\mathbf{P}(s)$ at $s = -1$ and $s = -2$, the singular points are solutions of the equation

$$\gamma(s) = s^4 + s^3 - 2s - 4 = 0 \tag{55}$$

i.e. $s_{1,\pm} = \pm\sqrt{2}$ and $s_{2,\pm} = -0.5 \pm j1.32$. While the singular points $s_{1,\pm}$ were predicted from the analysis of the root locus on the real axis, the other two points were actually unpredictable from a simple qualitative analysis of the locus.

The analysis of the root locus shows the existence of an interval $(K_1, +\infty)$, with $K_1 < 0$, such that the closed-loop system is asymptotically stable for $K \in (K_1, +\infty)$. As a matter of fact, by the Routh table of $\mathrm{NUM}(1 + K\mathbf{P}(s))$ we obtain

$$\begin{array}{c|ccc} r^{(4)} & 1 & 13+2K & 4 \\ r^{(3)} & 6+K & 2(6+K) & \\ r^{(2)} & 11+2K & 4 & \\ r^{(1)} & \frac{18+4K}{11+2K} & & \\ r^{(0)} & 4 & & \end{array} \tag{56}$$

By discussing the sign variations, we obtain that $K_1 = -4.5$. □

III. Stabilization and pole placement

The root locus gives important information about the positions of the closed-loop system poles. Moreover, the Routh criterion gives a method for determining the values of K for which the closed-loop system poles have negative real part,

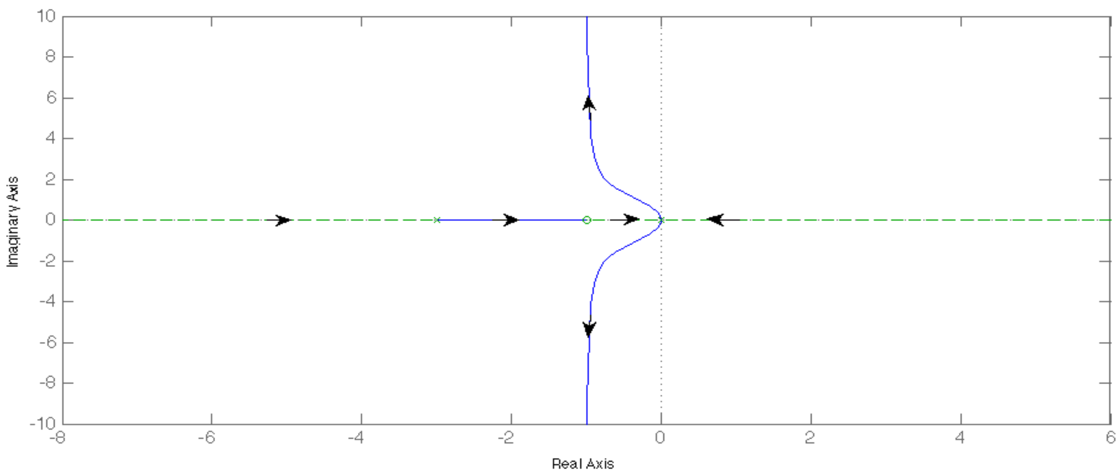


Figure 2. Root locus of $1 + K\mathbf{P}(s)$ with of $\mathbf{P}(s) = \frac{s+1}{s^2(s+3)}$.

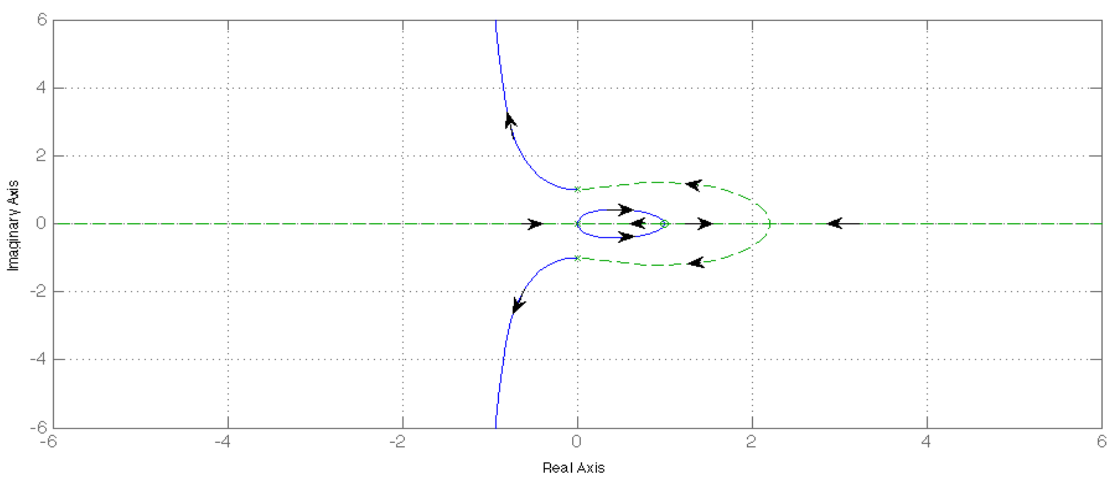


Figure 3. Root locus of $1 + K\mathbf{P}(s)$ with of $\mathbf{P}(s) = \frac{(s-1)^2}{s^2(s^2+1)}$.

i.e. the closed-loop system is asymptotically stable. However, if such values of K do not exist we must think of more general control action (than proportional) in order to stabilize the closed-loop system. The key idea comes from the root locus. Indeed, assume that the zeroes of $\mathbf{P}(s)$ are in $\mathbb{C}^-$ and let n and m be the number of poles and, respectively, zeroes of $\mathbf{P}(s)$. As well-known, m curves of the positive root locus of $1 + K\mathbf{P}(s)$ tend to the zeroes as $K \to +\infty$. This means that for $K >> 1$ the points of the locus which lie on these m curves have negative real part. We also know that $n - m$ curves of the positive root locus of $1 + K\mathbf{P}(s)$ tend to the point at infinity as $K \to +\infty$ along certain directions. If the relative degree $n - m = 1$, the closed-loop system is asymptotically stable for all $K >> 1$. Moreover, if $n - m = 2$ these asymptotes are vertical (form with the positive real axis $90°$ and $270°$, respectively). If, in addition, the asymptotes center s_0 is negative, then for $K >> 1$ also the points of the locus which lie on these $n - m$ curves have negative real part. Summing up, if the zeroes of $\mathbf{P}(s)$ have negative real part, $n-m = 2$ and $s_0 < 0$ the closed-loop system is asymptotically stable for all $K >> 1$. This suggest the following approach to the stabilization of $\mathbf{P}(s)$, as long as its zeroes have negative real part (a process $\mathbf{P}(s)$ with all zeroes in $\mathbb{C}^-$ is called minimum phase):

A. Stabilization of minimum phase $\mathbf{P}(s)$

(Relative degree of $\mathbf{P}(s) = 1$). Apply a proportional control action to $\mathbf{P}(s)$

$$\mathbf{G}(s) := K \tag{57}$$

By means of the Routh criterion on

$$\mathrm{NUM}(1 + K\mathbf{P}(s))$$

find $K > 0$ such that the roots of $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}(s))$ lie in $\mathbb{C}^-$. The stabilizing controller is $\mathbf{G}(s)$.

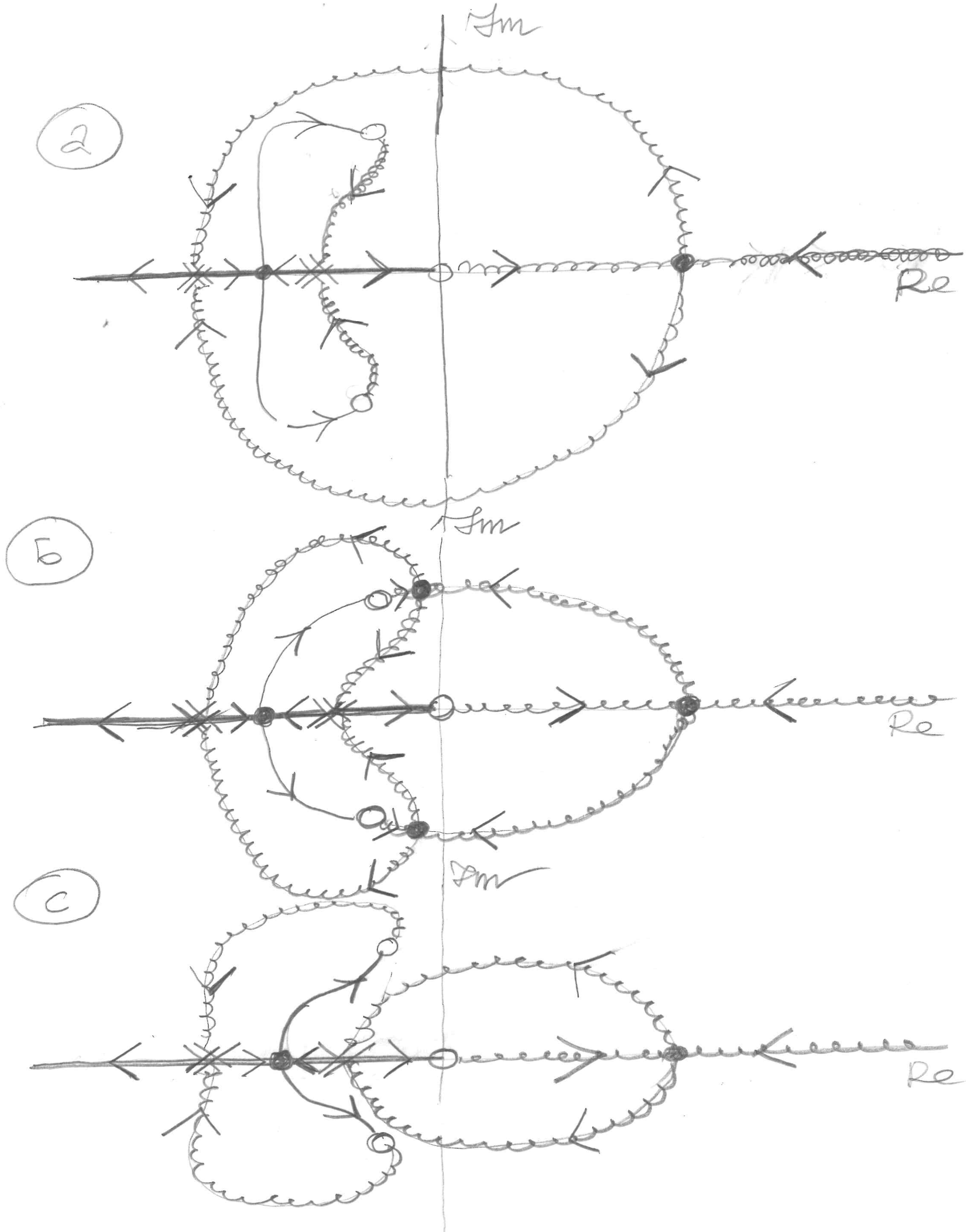


Figure 4. Root locus of $1 + K\mathbf{P}(s)$ with of $\mathbf{P}(s) = \frac{s(s^2+2s+2)}{(s+1)^2(s+2)^2}$.

(Relative degree of $\mathbf{P}(s) = 2$). Assume that the root locus of $\mathbf{P}(s)$ has an asymptote center $s_0 \geq 0$. Apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_1(s) := \frac{s - z'}{s - p'} \tag{58}$$

with $p' < z' < 0$. The asymptotes center s_0' of the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)$ is

$$s_0' := \frac{\sum_{i=1}^{n} p_i + p' - (\sum_{i=1}^{m} z_i + z')}{n - m} = s_0 + \frac{p' - z'}{n - m} \tag{59}$$

where $p_1, \dots, p_n$ and $z_1, \dots, z_m$ are the poles and zeroes of $\mathbf{P}(s)$. Therefore, select p' and z' in such a way that $s_0' < 0$. If $\mathbf{P}(s)$ has an asymptote center $s_0 < 0$, it is possible to set in alternative $\mathbf{G}_1(s) = 1$, by reducing by one dimension the final controller.

Finally, apply a proportional control action to $\mathbf{P}(s)$

$$\mathbf{G}_2(s) := K \tag{60}$$

By means of the Routh criterion applied on

$$\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$$

find K such that the roots of $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$ lie in $\mathbb{C}^-$. The stabilizing controller is $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$.

(Relative degree of $\mathbf{P}(s) > 2$). In this case, apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_1(s) := \prod_{i=1}^{n-m-2} (s - z_i') \tag{61}$$

with $z_1', \dots, z_{n-m-2}' < 0$. In other words, let us add to $\mathbf{P}(s)$ $n - m - 2$ negative zeroes so that the relative degree of $\mathbf{P}(s)\mathbf{G}_1(s)$ is 2. The control action (123) is not physically realizable as such. At this point if the root locus of $1 + K\mathbf{P}(s)\mathbf{G}_1(s)$ has the asymptotes center $s_0 \geq 0$, then apply the following control action to $\mathbf{P}(s)\mathbf{G}_1(s)$

$$\mathbf{G}_2(s) := \frac{s - z'}{s - p'} \tag{62}$$

with $p' < z' < 0$ selected in such a way to place the asymptotes center s_0' of the root locus of $1 + K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$ on the negative real axis (i.e. $s_0' < 0$). If $\mathbf{P}(s)\mathbf{G}_1(s)$ has an asymptote center $s_0 < 0$, it is possible to set in alternative $\mathbf{G}_2(s) = 1$, by reducing by one dimension the final controller. Next, apply a proportional control action to $\mathbf{P}(s)$

$$\mathbf{G}_3(s) := K \tag{63}$$

By means of the Routh criterion applied on

$$\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s))$$

find K such that that the roots of $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s))$ lie in $\mathbb{C}^-$. Finally, apply to $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)$ the control action

$$\mathbf{G}_4(s) := \frac{1}{(1 + sT)^{n-m-2}}, \tag{64}$$

which makes the overall control action $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)$ physically realizable, and by means of the Routh criterion find $T > 0$ such that the roots of $\mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s))$ lie in $\mathbb{C}^-$. The stabilizing controller is $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)$.

Exercize *3.1: Plot the root locus of $1 + K\mathbf{P}(s)$ with*

$$\mathbf{P}(s) := \frac{1}{s^2} \tag{65}$$

and find, if possible, a controller $\mathbf{G}(s)$ such that the feedback interconnection with unitary feedback of $\mathbf{P}(s)\mathbf{G}(s)$ is asymptotically stable.

Let us plot the root locus of $\mathbf{P}(s)$. We denote the zeroes of $\mathbf{P}(s)$ on the complex plane by circles and the poles of $\mathbf{P}(s)$ by crosses. The poles of $\mathbf{P}(s)$ are $p_1 := 0$ (with multiplicity 2). $\mathbf{P}(s)$ has no zeroes ($m = 0$). The root locus is shown in Figure 5.

Two curves of the positive (resp. negative) locus approaches the point at infinity along directions which form a star with 2 radii inclined to the real positive axis by $90°, 270°$ (resp. $0°, 180°$) and centered at the point

$$s_0 := \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m} = \frac{0}{2} = 0 \tag{66}$$

(rule **P5**).

The singular points are determined by the equations (rule (**P4**))

$$\frac{\partial^r}{\partial s^r}[1 + K\mathbf{P}(s)] = 0, \; r = 0, 1 \tag{67}$$

In particular, the singular points (s, K) with multiplicity 2 satisfy the equations

$$\begin{aligned} 1 + K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} &= 0, \\ \sum_{i=1}^{m} \frac{1}{s - z_i} - \sum_{i=1}^{n} \frac{1}{s - p_i} &= 0 \end{aligned} \tag{68}$$

i.e.

$$\begin{aligned} s^2 + K &= 0, \\ s &= 0 \end{aligned} \tag{69}$$

which have the unique solution

$$s = 0, \; K = 0$$

In the point $s = 0$ (which is the pole of the open-loop system $\mathbf{P}(s)$) 4 curves (two curves of the positive locus and two curves of the negative locus) intersect each other with alternating directions (rule **P4**).

The numerator of $1 + K\mathbf{P}(s)$, i.e. the polynomial

$$\mathbf{p}(s) := s^2 + K \tag{70}$$

has roots $\pm j\sqrt{K}$ for $K > 0$ and $\pm\sqrt{|K|}$ for $K < 0$ (see Figure 5). Therefore, for no value of K the root locus lies in $\mathbb{C}^-$, i.e. the feedback interconnection of $\mathbf{P}(s)$ is not stabilizable with proportional control action.

Since $\mathbf{P}(s)$ has no zeroes, $n - m = 2$ and the root locus of $1 + K\mathbf{P}(s)$ has $s_0 = 0$ (see case $n - m = 2$ above), we apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_1(s) := \frac{s - z'}{s - p'} \tag{71}$$

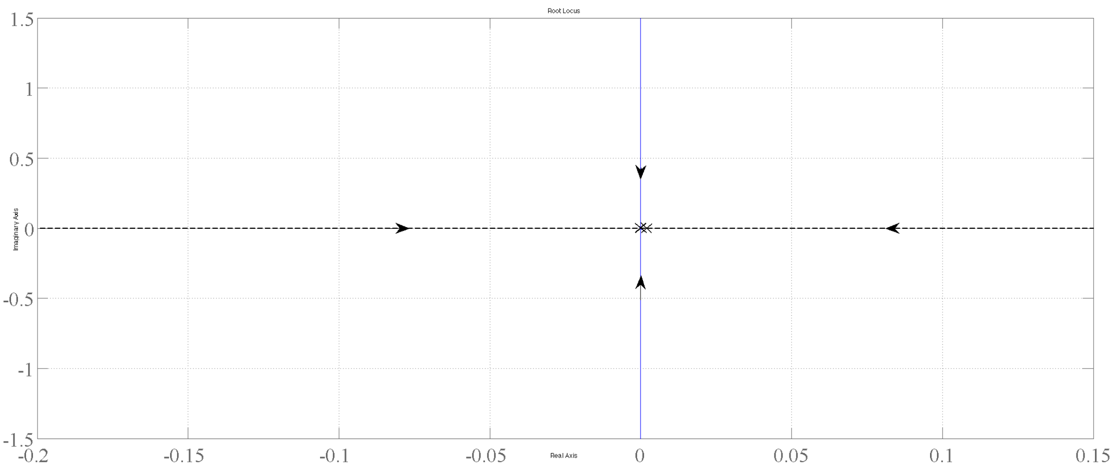


Figure 5. Root locus of $1 + K\mathbf{P}(s)$ with of $\mathbf{P}(s) = \frac{1}{s^2}$.

with $p' < z' < 0$ selected in such a way that the asymptotes center s_0' of the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)$

$$s_0' := s_0 + \frac{p' - z'}{n - m} \tag{72}$$

is negative. For example, $p' = -10$ and $z' = -1$. The positive root locus of $1 + \mathbf{P}(s)\mathbf{G}_1(s)$ is shown in Figure 6.

Finally, by means of the Routh criterion find $\mathbf{G}_2(s) = K$ such that the roots of $\mathrm{NUM}(1+K\mathbf{P}(s)\mathbf{G}_1(s))$ lie in $\mathbb{C}^-$. We apply the Routh criterion to $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$, i.e.

$$\begin{aligned}\mathbf{p}(s) &:= \mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s)) \\ &= s^2(s + 10) + K(s + 1) = s^3 + 10s^2 + Ks + K\end{aligned} \tag{73}$$

The Routh table is

$$\begin{array}{c|cc} r^{(3)} & 1 & K \\ r^{(2)} & 10 & K \\ r^{(1)} & 9K & \\ r^{(0)} & K & \end{array} \tag{74}$$

Clearly, any $K > 0$ is such that the number of variations in the first column of the Routh table is zero. Any controller

$$\mathbf{G}(s) := \mathbf{G}_1(s)\mathbf{G}_2(s) = \overline{K}\frac{s + 1}{s + 10} \tag{75}$$

with $\overline{K} > 0$ stabilizes the feedback interconnection of $\mathbf{G}(s)\mathbf{P}(s)$ with unitary feedback. □

Exercize *3.2: Plot the root locus of* $1 + K\mathbf{P}(s)$ *with*

$$\mathbf{P}(s) := \frac{1}{s^n} \tag{76}$$

$n > 2$, *and find, if possible, a controller* $\mathbf{G}(s)$ *such that the feedback interconnection with unitary feedback of* $\mathbf{P}(s)\mathbf{G}(s)$ *is asymptotically stable.*

Let us plot the root locus. We denote the zeroes of $\mathbf{P}(s)$ on the complex plane by circles and the poles of $\mathbf{P}(s)$ by crosses. The poles of $\mathbf{P}(s)$ are $p_1 := 0$ (with multiplicity n). $\mathbf{P}(s)$ has no zeroes ($m = 0$).

In this example, n curves of the positive (resp. negative) locus approaches the point at infinity along directions which form a star with n radii inclined to the real positive axis by

$$\theta = \frac{\theta^\circ + 2h\pi}{n},\ h = 0, 1, \ldots, n - 1 \tag{77}$$

$$(\text{resp. } \theta = \frac{\theta^\circ + (2h + 1)\pi}{n},\ h = 0, 1, \ldots, n - 1) \tag{78}$$

and center at the point

$$s_0 := \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n} = \frac{0}{n} = 0 \tag{79}$$

(rule (**P5**)).

In the point $s = 0$ (which is the pole of the open-loop system $\mathbf{P}(s)$) $2n$ curves (n curves of the positive locus and n curves of the negative locus) intersect each other with alternating directions (rule (**P4**)).

The numerator of $1 + K\mathbf{P}(s)$, i.e. the polynomial

$$\mathbf{p}(s) := s^n + K \tag{80}$$

has no roots in $\mathbb{C}^-$. Therefore, the feedback interconnection of $\mathbf{P}(s)$ is not stabilizable with proportional control action.

Since $\mathbf{P}(s)$ has no zeroes and $n - m = n > 2$ (see case $n - m > 2$ above), we apply to $\mathbf{P}(s)$ the following control action

$$\mathbf{G}_1(s) := \prod_{i=1}^{n-2}(s - z_i') \tag{81}$$

with $z_1', \ldots, z_{n-2}' < 0$. For example, $z_1' = z_2' = \cdots = z_{n-2}' = -1$. Secondly, apply to $\mathbf{P}(s)\mathbf{G}_1(s)$ the following control action

$$\mathbf{G}_2(s) := \frac{s - z'}{s - p'} \tag{82}$$

with $p' < z' < 0$ selected in such a way that the asymptotes center s_0' of the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$

$$s_0' := \frac{p' - z' - \sum_{i=1}^{n-2} z_i'}{2} = \frac{p' - z' + n - 2}{2} \tag{83}$$

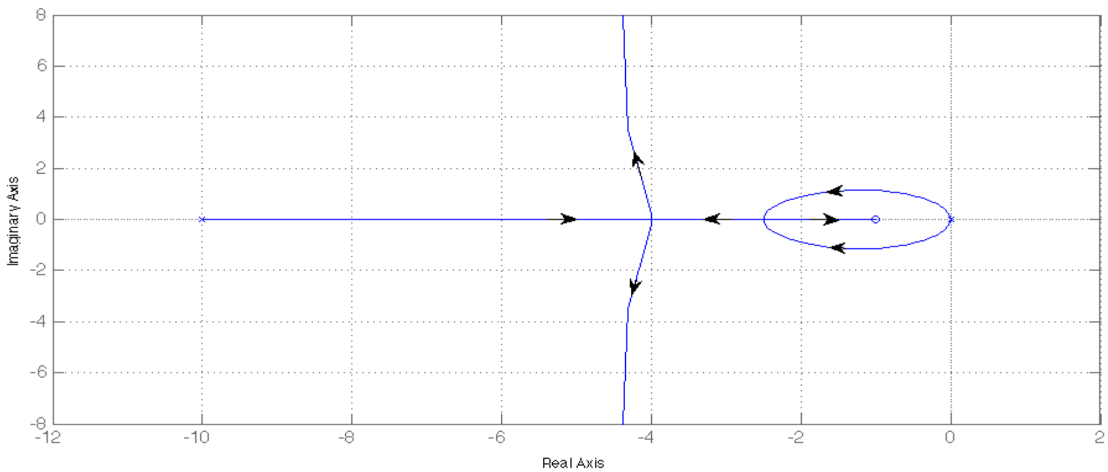


Figure 6. The positive root locus of $1 + K\mathbf{P}(s)$ with $\mathbf{P}(s) = \frac{s+1}{s^2(s+10)}$.

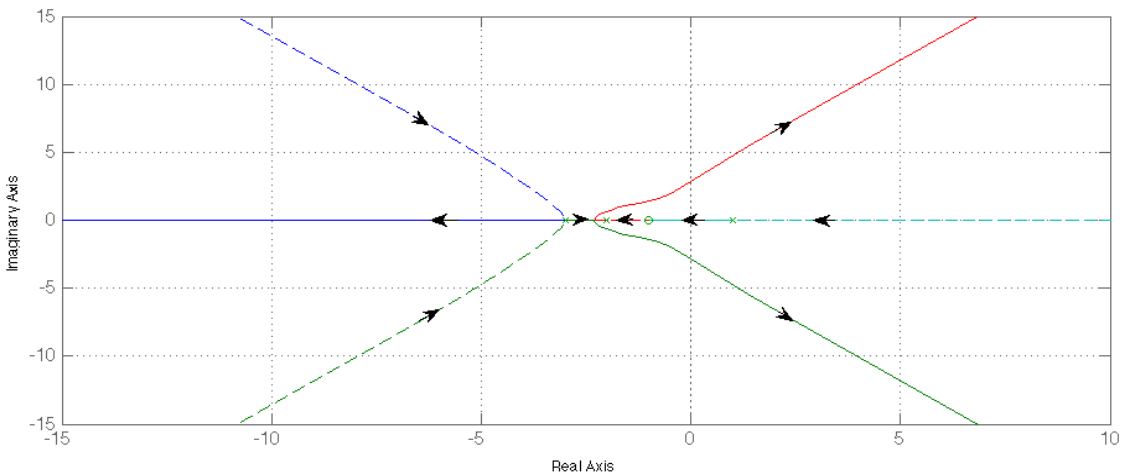


Figure 7. Root locus of $1 + K\mathbf{P}(s)$ with $\mathbf{P}(s) = \frac{s+1}{(s+2)(s-1)(s+3)^2}$.

is negative. For example, $p' = -n - 1$ and $z' = -1$. Finally, by means of the Routh criterion find $\mathbf{G}_3(s) = K$ such that the roots of $\mathrm{NUM}(1+K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s))$ lie in $\mathbb{C}^-$. We apply the Routh criterion to $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s))$, i.e.

$$\begin{aligned} \mathbf{p}(s) &:= \mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)) \\ &= s^n(s+n+1) + K(s+1)^{n-1} \end{aligned} \tag{84}$$

For example, for $n = 3$ the Routh table is

$$\begin{array}{l|ccc} r^{(4)} & 1 & K & K \\ r^{(3)} & 4 & 2K & \\ r^{(2)} & K & 2K & \\ r^{(1)} & \frac{2K^2-8K}{K} & & \\ r^{(0)} & 2K & & \end{array} \tag{85}$$

Clearly, any $K > 4$ is such that the number of variations in the first column of the Routh table is zero. Set $\mathbf{G}_3(s) = K = 5$. Finally, apply to $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)$ the following control action

$$\mathbf{G}_4(s) := \frac{1}{(1+sT)^{n-2}} \tag{86}$$

which makes physically realizable the overall control action $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)$ and by means of the Routh criterion find $T > 0$ such that the roots of $\mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s))$ lie in $\mathbb{C}^-$. We apply the Routh criterion to the numerator of $\mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s))$, i.e.

$$\mathbf{p}(s) := s^n(s+n+1)(1+sT)^{n-2} + \overline{K}(s+1)^{n-1} \tag{87}$$

For example, for $n = 3$ the Routh table is:

$$
\begin{array}{c|ccc}
r^{(5)} & T & 4 & 10 \\
r^{(4)} & 1+4T & 5 & 5 \\
r^{(3)} & 4+11T & 10+35T & \\
r^{(2)} & 10+75T+140T^2 & 5(4+11T) & \\
r^{(1)} & (10+75T+140T^2)(10+35T)-5(4+11T)^2 & & \\
r^{(0)} & 5(4+11T) & &
\end{array}
\tag{88}
$$

Clearly, any $T \in (0, 0.1)$ is such that the number of variations in the first column of the Routh table is zero.

The controller

$$\mathbf{G}(s) := \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s) = \frac{5}{s+n+1}\frac{(s+1)^n}{(1+0.01s)^{n-1}}$$

stabilizes the feedback interconnection of $\mathbf{G}(s)\mathbf{P}(s)$ with unitary feedback. □

Exercize *3.3: Plot the root locus of* $1 + K\mathbf{P}(s)$ *with*

$$\mathbf{P}(s) := \frac{s+1}{(s+2)(s-1)(s+3)^2} \tag{89}$$

and find, if possible, a controller $\mathbf{G}(s)$ *such that the feedback interconnection with unitary feedback of* $\mathbf{P}(s)\mathbf{G}(s)$ *is asymptotically stable.*

Let us plot the root locus. The poles of $\mathbf{P}(s)$ are $p_1 := -3$ (with multiplicity 2), $p_2 := -2$ and $p_3 := 1$ ($n = 4$). The zeroes of $\mathbf{P}(s)$ are $z_1 := -1$ ($m = 1$). The root locus is shown in Figure 7.

Three curves of the positive (resp. negative) locus approaches the point at infinity along directions which form a star with 3 radii inclined to the real positive axis by $60°, 180°, 320°$ (resp. $0°, 120°, 240°$) and centered at the point

$$s_0 := \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n-m} = \frac{(-2+1-3-3)-(-1)}{3} = -2 \tag{90}$$

(rule ($\mathbf{P5}$)).

At the point $s = -3$ (which is the pole of the open-loop system $\mathbf{P}(s)$) 4 curves (2 of the positive locus and 2 of the negative locus) intersect each other with alternating directions (property (IV)). A similar situation takes place at a singular point located between -3 and -2 (see Figure 7).

The numerator of $1 + K\mathbf{P}(s)$, i.e. the polynomial

$$
\begin{aligned}
&\mathbf{p}(s) := \mathrm{NUM}(1 + K\mathbf{P}(s)) \\
&(s+2)(s-1)(s+3)^2 + K(s+1)
\end{aligned}
\tag{91}
$$

From the Routh table it turns out that for no value of K the root locus lies s in $\mathbb{C}^-$, i.e. the feedback interconnection of $\mathbf{P}(s)$ is not stabilizable with proportional control action.

Since $\mathbf{P}(s)$ has a negative zero and its relative degree $n - m = 3 > 2$, we apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_1(s) := s + 3 \tag{92}$$

Notice that the zero $s = -3$ of $\mathbf{G}_1(s)$ cancels out the pole $s = -3$ of $\mathbf{P}(s)$. This cancellation makes the convergent mode corresponding to the pole $s = -3$ unobservable from the output/unexcitable from the input: however, this not change the stability of the closed-loop system. Notice also that the asymptotes center s_0' of the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)$

$$s_0' := \frac{(-2+1-3)-(-1)}{2} = -\frac{3}{2} \tag{93}$$

is negative. In other words, it is not necessary to change the asymptotes center. Finally, by means of the Routh criterion find $\mathbf{G}_2(s) = K$ such that the roots of $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$ lie in $\mathbb{C}^-$. We apply the Routh criterion to the numerator of $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$, i.e.

$$
\begin{aligned}
&\mathbf{p}(s) := \mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s)) \\
&= (s+2)(s-1)(s+3) + K(s+1) \\
&= s^3 + 4s^2 + (K+1)s + K - 6
\end{aligned}
\tag{94}
$$

The Routh table is

$$
\begin{array}{c|cc}
r^{(3)} & 1 & 1+K \\
r^{(2)} & 4 & K-6 \\
r^{(1)} & 3K+10 & \\
r^{(0)} & K-6 &
\end{array}
\tag{95}
$$

Clearly, any $K > 6$ is such that the number of variations in the first column of the Routh table is zero. Set $\mathbf{G}_2(s) = K = 8$. Finally, introduce the control action

$$\mathbf{G}_3(s) := \frac{1}{1+sT} \tag{96}$$

which makes physically realizable the control action (92) and by means of the Routh criterion find $T > 0$ such that the roots of $\mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s))$ lies in $\mathbb{C}^-$. We apply the Routh criterion to the numerator of $\mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s))$, i.e.

$$
\begin{aligned}
\mathbf{p}(s) &:= \mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)) \\
&= (s+2)(s-1)(s+3)(1+sT) + \overline{K}(s+1)
\end{aligned}
\tag{97}
$$

The Routh table is

$$
\begin{array}{c|ccc}
r^{(4)} & T & 4+T & 2 \\
r^{(3)} & 1+4T & 9-6T & \\
r^{(2)} & \frac{10T^2+8T+4}{1+4T} & 2 & \\
r^{(1)} & \frac{-60T^3+74T^2+64T+36}{10T^2+8T+4} & & \\
r^{(0)} & 2 & &
\end{array}
\tag{98}
$$

Clearly, any $T \in (0, 0.1)$ is such that the number of variations in the first column of the Routh table is zero. The controller

$$\mathbf{G}(s) := \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s) = 8\frac{s+3}{1+0.01T} \tag{99}$$

stabilizes the feedback interconnection of $\mathbf{G}(s)\mathbf{P}(s)$ with unitary feedback. □

Exercize *3.4: Let*

$$\mathbf{P}_1(s) := \frac{1}{s(s-2)}, \ \mathbf{P}_2(s) := \frac{s-2}{s+3} \tag{100}$$

Find controllers $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ such that the dimension of $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ is 1 and the feedback system in Figure 8 is asymptotically stable. Plot the root locus of the transfer function $\mathbf{F}(s)$ on the direct path of the feedback system in Figure 8.

Notice that we have one internal feedback loop and one external feedback loop. Moreover, the dimension of $\mathbf{G}(s)$ is exactly the number of its poles. We will design a one-dimensional $\mathbf{G}_1(s)$ for stabilizing the internal feedback loop and a one-dimensional $\mathbf{G}_2(s)$ for stabilizing the external feedback loop. For stabilizing the internal feedback loop, notice that $\mathbf{P}_1(s)$ has all zeroes in $\mathbb{C}^-$. Moreover, its relative degree is 2 and the asymptote center is in $s_0 = 1$. Therefore, a zero-pole control action

$$\mathbf{G}_1(s) := K_{G_1}\frac{s-z}{s-p} \tag{101}$$

with $z, p < 0$, will move the asymptote center in $\mathbb{R}^-$ and increasing the gain K_{G_1} we will stabilize the internal loop with $\mathbf{P}_1(s)$. By selecting $z = 3$ and $p = 7$ we place the asymptote center in

$$s_0' := s_0 + \frac{-7+3}{2} = -1 \tag{102}$$

To determine the value of K_{G_1} which stabilizes the internal feedback loop, we will apply the Routh criterion to the denominator of the internal loop I/O transfer function $\mathbf{W}_1(s) = \frac{\mathbf{P}_1(s)\mathbf{G}_1(s)}{1+\mathbf{P}_1(s)\mathbf{G}_1(s)}$:

$$\begin{aligned} &\mathrm{NUM}(1+\mathbf{P}_1(s)\mathbf{G}_1(s)) \\ &= s^3 + 5s^2 + (K_{G_1} - 14)s + 3K_{G_1} \end{aligned} \tag{103}$$

Constructing the associated Routh table and discussing the sign variations in the elements of its first column we obtain that $\mathbf{W}_1(s)$ is asymptotically stable for $K_{G_1} > 35$. For instance, choose $K_{G_1} = 72$. With this choice

$$\begin{aligned} \mathbf{W}_1(s) &= \frac{\mathbf{P}_1(s)\mathbf{G}_1(s)}{1+\mathbf{P}_1(s)\mathbf{G}_1(s)} \\ &= 72\frac{s+3}{(s+4)(s+0.5+j7.33)(s+0.5-j7.33)} \end{aligned} \tag{104}$$

Since the dimension of $\mathbf{G}_1(s)$ is 1 and it is required that the dimension of $\mathbf{G}_1(s)\mathbf{G}_2(s)$ be 1, we necessarily have that

$$\mathbf{G}_2(s) = K_{G_2} \tag{105}$$

(i.e. a proportional controller). The controller $\mathbf{G}_2(s)$ will stabilize the external feedback loop around the process

$$\begin{aligned} \mathbf{F}_1(s) &:= \frac{1}{s}\mathbf{W}_1(s) \\ &= 72\frac{s-2}{s(s+4)(s+0.5+j7.33)(s+0.5-j7.33)} \end{aligned} \tag{106}$$

From the analysis of the root locus of $\mathbf{F}_1(s)$ (recall that for the locus of $\mathbf{F}_1(s)$ it is necessary to re-parametrize K_{G_2} as $K'_{G_2} = 72K_{G_2}$) it can be seen that for small negative values of K'_{G_2} the closed-loop poles are all in $\mathbb{C}^-$. This conclusion is also accounted for by noticing that $\mathbf{F}_1(s)$ has one pole at $s = 0$, all the other poles in $\mathbb{C}^-$ and one zero in $\mathbb{C}^+$. We will apply the Routh criterion to the denominator of the external loop I/O transfer function $\mathbf{W}_2(s) := \frac{\mathbf{F}_1(s)\mathbf{G}_2(s)}{1+\mathbf{F}_1(s)\mathbf{G}_2(s)}$ (it is important to notice that for the Routh table can be constructed using either K'_{G_2} or K_{G_2} with the obvious changes):

$$\begin{aligned} &\mathrm{NUM}(1+\mathbf{F}_1(s)\mathbf{G}_2(s)) \\ &= s^4 + 5s^3 + 58s^2 + (216 + K'_{G_2})s - 2K'_{G_2} \end{aligned} \tag{107}$$

Constructing the associated Routh table and discussing the sign variations in the elements of its first column we obtain that $\mathbf{W}_2(s)$ is asymptotically stable for $K'_{G_2} \in (-180.54, 0)$. Therefore, getting back to the original parametrization, $\mathbf{W}_2(s)$ is asymptotically stable for $K_{G_2} \in (-180.54/72, 0)$.

The transfer function on the direct path of the feedback system 8 is

$$\mathbf{F}(s) = \mathbf{F}_1(s)\mathbf{G}_2(s) \tag{108}$$

The root locus of $\mathbf{F}(s)$ is drawn in Figure 9. □

Exercize *3.5: Let*

$$\mathbf{P}(s) := \frac{1}{(s+9)(s^2+a^2)}, \ \mathbf{G}(s) := K. \tag{109}$$

Find the values of $a \in \mathbb{R}$ for which, choosing properly the parameter K, the feedback interconnection in Figure 10 has all real negative poles.

The root locus of $\mathbf{P}(s)$ gives useful information on the solution of the problem. The relative degree of $\mathbf{P}(s)$ is $n - m = 3$ and the locus has at most $n + m - 1 = 2$ singular points. The asymptote center is in

$$s_0 = \frac{-9}{3} = -3$$

The singular points equation is

$$3s^2 + 18s + a^2 = 0$$

with solutions

$$s_\pm = \frac{-9 \pm \sqrt{81 - 3a^2}}{3}$$

These solutions are real if and only if

$$81 - 3a^2 > 0 \Leftrightarrow a^2 \geq 27$$

In particular, for $a^2 \in [0, 27]$ the position of the first singular point s_+ varies between 0 (for $a = 0$) and -3 (for $a^2 = 27$), while the second point s_- varies between -6 (for $a = 0$) and -3 (fro $a^2 = 27$). In any case, these two point are in the negative locus.

If on the other hand the solutions $s_\pm$ are complex conjugate (which is for $a^2 > 27$) they are not points of the locus, since as it can be easily checked these points do not satisfy the root locus equation.

Consequently, the positive locus is as drawn in Figures 11-a for $a^2 > 27$ and Figures 11-b for $a^2 < 27$. In the second case there exist values of K for which the closed-loop system has three negative real poles, while in the first case we always have one real pole and two complex conjugate poles.

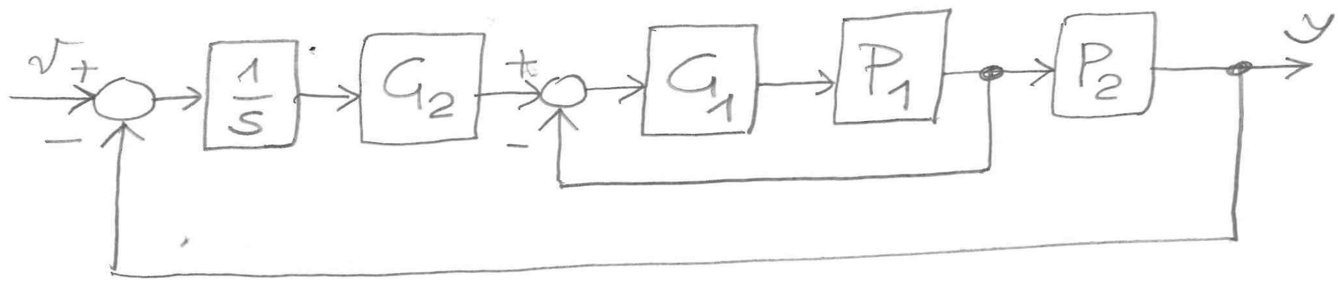


Figure 8. The feedback loop of exercize 3.4.

Next, set for instance $a^2 = 24$ for which we obtain $s_+ = -2$ and $s_- = -4$. The transfer function $\mathbf{F}(s) := \mathbf{G}(s)\mathbf{P}(s)$ is

$$\mathbf{F}(s) = K\frac{1}{(s+9)(s^2+24)}$$

For finding the interval of values of K for which the closed-loop poles are all real negative, it is sufficient to compute the values of K corresponding to the origin of $\mathbb{C}$ and to the points s_+ and s_-. Let denote these values with, respectively, K_0, K_1 and K_2. We conclude from the root locus in Figure 12 that the three closed-loop poles are all real negative for

$$K \in [\max\{K_2, K_0\}, K_1]$$

For determining K_0, K_1 and K_2 we replace in the root locus equation

$$(s^2+24)(s+9) + K = 0$$

the corresponding values of s. One finds

$$K_0 = -216,\ K_1 = -196,\ K_2 = -200.$$

Therefore, for $a^2 = 24$ the admissible values of K are inside the interval $[-200, -196]$.

Notice that if $a^2 = 27$ the two singular points coincide in $s_\pm = -3$. Consequently, the latter is a triple root of the locus. In the point $(s, K) = (-3, 0)$ flow in and out (alternatively) six arcs of the negative locus (Figure 12). The unique value of K satisfying the problem is $K = -216$ and indeed the root locus equation is $(s+3)^3 = 0$ for $K = -216$. □

Exercize *3.6: Consider the feedback interconnection in Figure 13 with*

$$\mathbf{P}(s) := \frac{s+2}{s^2+1},\ \mathbf{G}_1(s) := \frac{1}{s}. \tag{110}$$

(i) Draw the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)$ and design a controller $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ with minimal dimension such that the closed-loop system is asymptotically stable.
(ii) Is it possible, mantaining the same structure for $\mathbf{G}(s)$, to arrange all closed-loop poles with the same negative real part $-\alpha < 0$? If yes determine α.
(iii) Design a strictly proper controller $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ such that the closed-loop system is asymptotically stable.

(i) For a minimal dimensional controller $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ it is convenient first to see if we can take a proportional controller $\mathbf{G}_2(s) = K_{G_2}$. Let draw the root locus of $\mathbf{G}_1(s)\mathbf{P}(s)$. The relative degree of $\mathbf{G}_1(s)\mathbf{P}(s)$ is $n-m = 2$ and the singular points are at most $n+m-1 = 3$. By examination of the flow directions of the curves, we discover a real singular point in the negative locus. The asymptote center is in $s_0 = 1$. The root locus is drawn in Figure 14. The inspection of the locus, confirmed by the Routh table associated to the denominator of the I/O closed-loop transfer function

$$\mathrm{NUM}(1 + K_{G_2}\mathbf{G}_1(s)\mathbf{P}(s)) = s(s^2+1) + K_{G_2}(s+2),$$

reveals that it is not possible to stabilize the closed-loop system with a proportional controller $\mathbf{G}_2(s) = K_{G_2}$.

In order to keep the dimension of $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ as small as possible, since $\mathbf{G}_1(s)\mathbf{P}(s)$ has all zeroes in $\mathbb{C}^-$, we explore the possibility of stabilizing the closed-loop system with

$$\mathbf{G}_2(s) = K(s-z)$$

with $z < 0$. Since the relative degree of $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s)$ is 1 and its zeroes are all in $\mathbb{C}^-$, the closed-loop system will be asymptotically stable for a sufficiently large value of K. For this reason, for the denominator of the closed-loop I/O transfer function

$$\begin{aligned}&\mathrm{NUM}(1 + \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s))\\&= s^3 + Ks^2 + (K(-z+2)+1)s - 2Kz\end{aligned} \tag{111}$$

we construct the Routh table:

$$\begin{array}{c|cc} r^{(3)} & 1 & K(-z+2) \\ r^{(2)} & K & -2Kz \\ r^{(1)} & K(z+2)+1+2z & \\ r^{(0)} & -2Kz & \end{array} \tag{112}$$

We obtain that the closed-loop system is asymptotically stable for $K > \frac{2z+1}{z-2}$. For instance, if $z = -5$ we obtain $K > 9/7$. We choose

$$\mathbf{G}_2(s) = 2(s+5) \tag{113}$$

(ii) For obtaining three closed-loop poles with the same real part $-\alpha < 0$ and with the same structure of $\mathbf{G}(s) =$

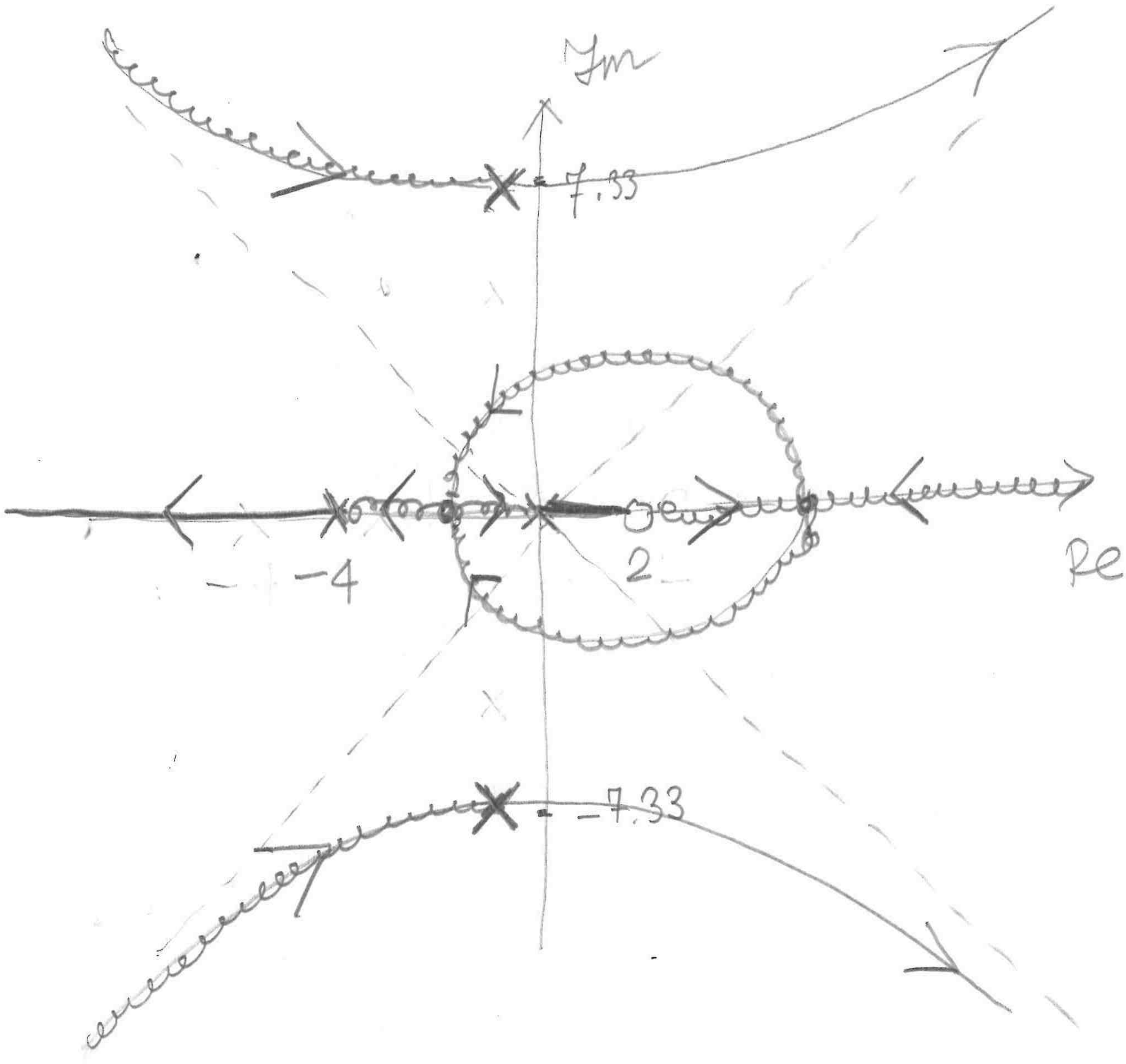


Figure 9. The root locus of exercize 3.4.

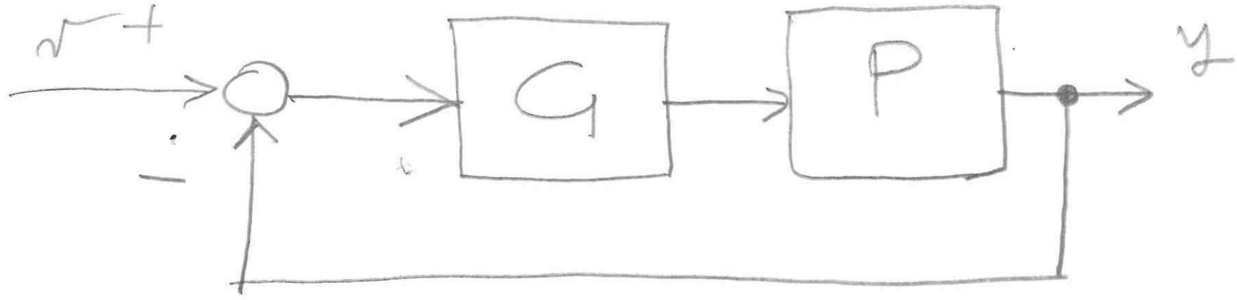


Figure 10. The feedback loop of exercize 3.5.

$\mathbf{G}_1(s)\mathbf{G}_2(s) = \frac{K(s-z)}{s}$, the denominator of the closed-loop I/O transfer function

$$\begin{aligned} &\mathrm{NUM}(1+\mathbf{G}(s)\mathbf{P}(s)) \\ &= s^3 + Ks^2 + (K(-z+2)+1)s - 2Kz \end{aligned} \tag{114}$$

must coincide with

$$(s+\alpha)(s+\alpha+j\beta)(s+\alpha-j\beta) = s^3 = 3\alpha s^2 + (3\alpha^2+\beta^2)s + \alpha\beta^2 + \alpha^3$$

for some $\alpha, \beta > 0$. By a comparison method we obtain the equations in the unknowns K, α, β:

$$\begin{aligned} &K = 3\alpha \\ &K(-z+2)+1 = 3\alpha^2 + \beta^2 \\ &-2Kz = \alpha\beta^2 + \alpha^3 \end{aligned} \tag{115}$$

One obtains two solutions for α

$$\alpha_{\pm} = \frac{3(-z+2) \pm \sqrt{9z^2+12z+44}}{4} \tag{116}$$

and

$$\beta^2 = -6z - \alpha^2 \tag{117}$$

Since $9z^2 + 12z + 44 > 0$ for all $z < 0$, $\alpha_{\pm}$ are both real, moreover, $\alpha_+ > 0$ while α_- may be either positive or negative. If both $\alpha_{\pm}$ are positive we choose the one for which $\beta^2 = -6z - \alpha^2 > 0$, i.e. α_-, otherwise we must take α_+. Finally, we set $K = 3\alpha$. For instance, if $z = -5$ we have $\alpha_- = 1.63$ and $K = 4.90$, correspondingly we have the three closed-loop poles $p_1 = -1.63$, $p_{2,\pm} = -1.63 \pm .j5.22$, which all have the same real part.

(iii) The most direct way of finding a solution is to consider the controller $\mathbf{G}(s)$ of (i) and add one negative pole $s = -\frac{1}{T}$ with $T << 1$:

$$\mathbf{G}'(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)\frac{1}{1+sT} = 2\frac{s+5}{s(1+sT)}$$

As a matter of fact, for a system such that its closed-loop is asymptotically stable by adding a pole $\frac{1}{1+sT}$ with $T > 0$ the resulting closed-loop system remains asymptotically stable if $T << 1$. The value of T can be determined with the Routh table. For this reason, we will construct the Routh table of

$$\begin{aligned} &\mathrm{NUM}(1+\mathbf{G}'(s)\mathbf{P}(s)) \\ &= 2(s+2)(s+5) + s(s^2+1)(1+sT) \\ &= Ts^4 + s^3 + (T+2)s * 2 + 15s + 20 \end{aligned} \tag{118}$$

We obtain

$$\begin{array}{c|ccc} r^{(4)} & T & T+2 & 20 \\ r^{(3)} & 1 & 15 & \\ r^{(2)} & 1-7T & 10 & \\ r^{(1)} & \frac{1-21T}{1-7T} & & \\ r^{(0)} & 10 & & \end{array} \tag{119}$$

For excluding sign variations in the first column, we choose for instance $T = 1/25$. □

B. *Pole placement for minimum phase $\mathbf{P}(s)$.*

When the closed-loop poles are required to lie in certain subregions of $\mathbb{C}^-$ as for instance $\mathfrak{S}(\alpha) := \{s \in \mathbb{C}^- : \mathrm{Re}(s) < -\alpha\}$ and $\mathbf{P}(s)$ is minimum phase, we have to modify the procedure of stabilization for minimum phase $\mathbf{P}(s)$ given at the beginning of this section, in particular

- cancel all the zeroes of $\mathbf{P}(s)$ which are not $\mathfrak{S}(\alpha)$ and replacing them with zeroes in $\mathfrak{S}(\alpha)$,

- move the asymptote center of $\mathbf{P}(s)$ inside the portion of the real axis contained in $\mathfrak{S}(\alpha)$.

(Relative degree of $\mathbf{P}(s) = 1$). Using

$$\mathbf{G}_1(s) := \frac{\prod_{i=1}^{r}(s - z'_{j_i})}{\prod_{i=1}^{r}(s - z_{j_i})} \tag{120}$$

cancel all the zeroes $z_{j_1}, \ldots, z_{j_r}$ of $\mathbf{P}(s)$ which are not in $\mathfrak{S}(\alpha)$ and replace them with zeroes $z'_{j_1}, \ldots, z'_{j_r}$ in $\mathfrak{S}(\alpha)$. Apply a proportional control action to $\mathbf{P}(s)\mathbf{G}_1(s)$

$$\mathbf{G}_2(s) := K \tag{121}$$

By means of the Routh criterion on

$$\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))|_{s-\alpha}$$

find K such that the roots of $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$ lie in $\mathfrak{S}(\alpha)$.

(Relative degree of $\mathbf{P}(s) = 2$). Using $\mathbf{G}_1(s)$ as in (120), cancel all the zeroes $z_{j_1}, \ldots, z_{j_r}$ of $\mathbf{P}(s)$ which are not $\mathfrak{S}(\alpha)$ and replace them with zeroes $z'_{j_1}, \ldots, z'_{j_r}$ in $\mathfrak{S}(\alpha)$. Assume that the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)$ has the asymptotes center $s_0 \notin \mathfrak{S}(\alpha)$. Apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_2(s) := \frac{s - z'}{s - p'} \tag{122}$$

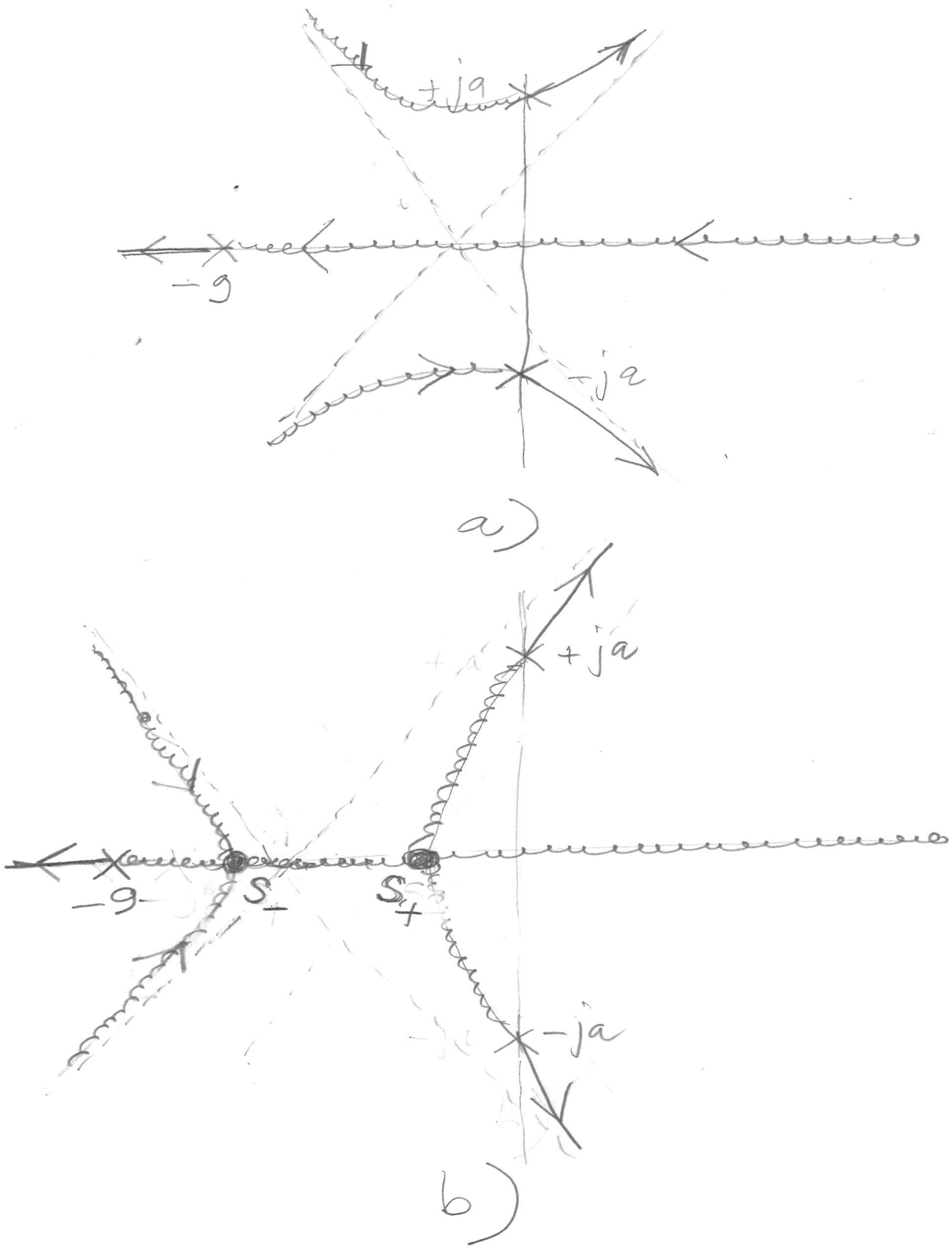


Figure 11. Root locus with $a^2 > 27$ and $a^2 < 27$ for exercize 3.5.

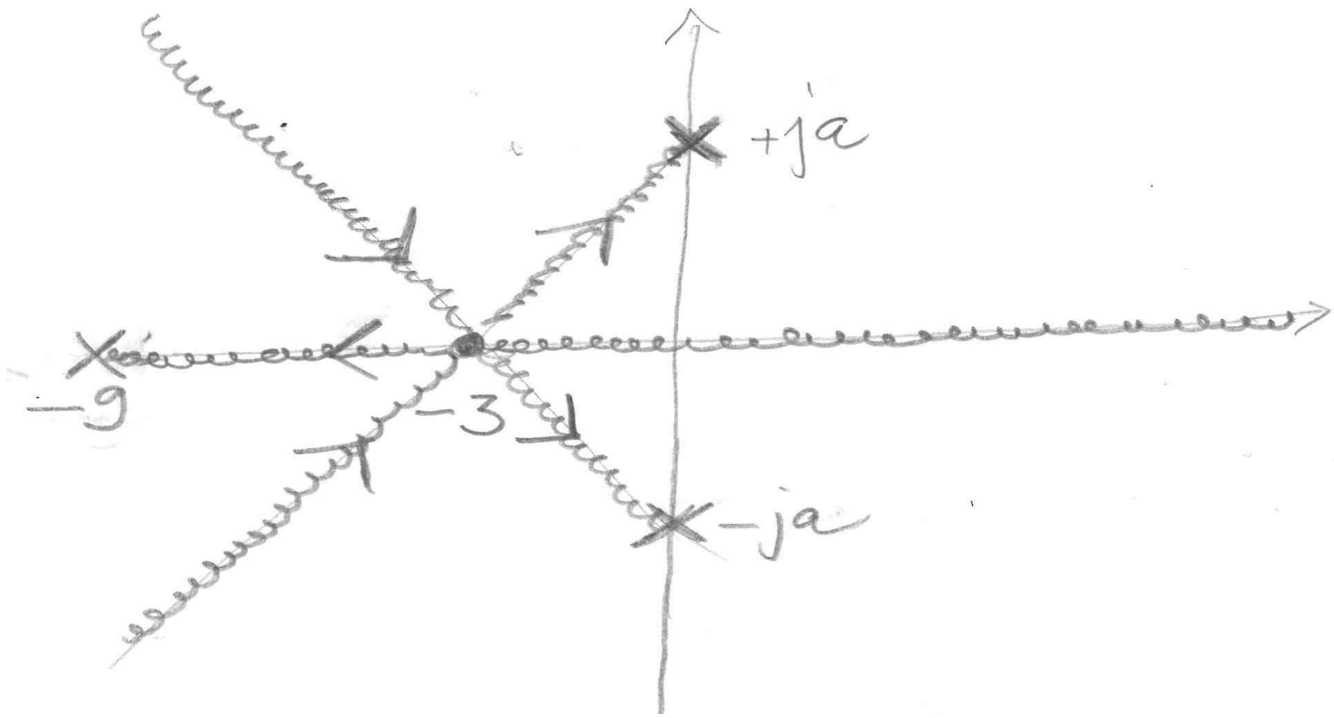


Figure 12. Root locus with $a^2 = 27$ for exercize 3.5.

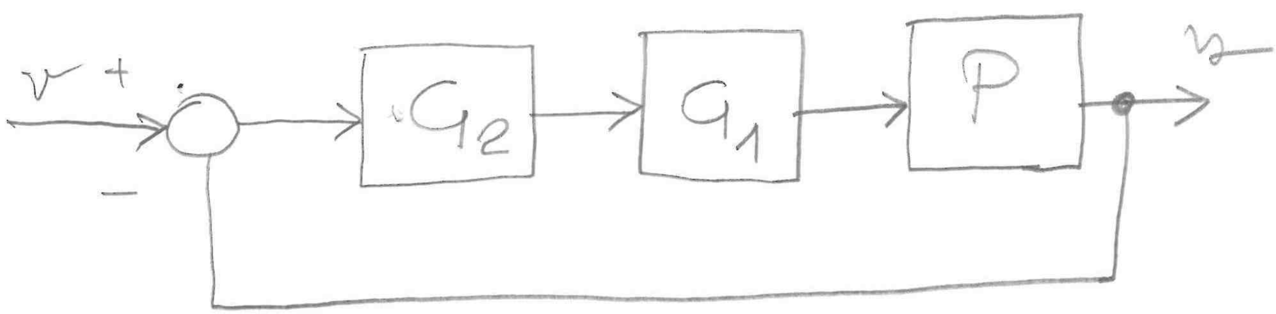


Figure 13. The feedback loop of exercize 3.8.

with $p' < z'$ and $z' \in \mathfrak{S}(\alpha)$. Select p' and z' in such a way that the asymptotes center s_0' of the root locus of $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s)$ is in $\mathfrak{S}(\alpha)$. If the root locus of $\mathbf{P}(s)\mathbf{G}_1(s)$ has the asymptotes center $s_0 \in \mathfrak{S}(\alpha)$ we can take in alternative $\mathbf{G}_2(s) = 1$ so that to reduce by one the dimension of the overall controller $\mathbf{G}(s)$. Finally, by means of the Routh criterion, applied on

$$\mathrm{NUM}(1 + K\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s))|_{s-\alpha},$$

find K such that the roots of $\mathrm{NUM}(1 + K\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s))$ lie in $\mathfrak{S}(\alpha)$.

(Relative degree of $\mathbf{P}(s) > 2$). Using $\mathbf{G}_1(s)$ as in (120), cancel all the zeroes $z_{j_1}, \dots, z_{j_r}$ of $\mathbf{P}(s)$ which are not $\mathfrak{S}(\alpha)$ and replace them with zeroes $z'_{j_1}, \dots, z'_{j_r}$ in $\mathfrak{S}(\alpha)$. Apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_2(s) := \prod_{i=1}^{n-m-2} (s - z_i') \tag{123}$$

with $z_1', \dots, z'_{n-m-2} \in \mathfrak{S}(\alpha)$, so that the relative degree of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$ is 2.

If the root locus of $1 + K\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$ has the asymptotes center $s_0 \notin \mathfrak{S}(\alpha)$, apply the following control action to $\mathbf{P}(s)$

$$\mathbf{G}_3(s) := \frac{s - z'}{s - p'} \tag{124}$$

with $p' < z'$ and $z' \in \mathfrak{S}(\alpha)$. Select p' and z' in such a way that the asymptotes center s_0' of the root locus of $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{P}(s)$ is in $\mathfrak{S}(\alpha)$. If $s_0' \in \mathfrak{S}(\alpha)$ we can take in alternative $\mathbf{G}_2(s) = 1$ so that to reduce by one the dimension of the overall controller $\mathbf{G}(s)$. Next, take

$$\mathbf{G}_4(s) := K \tag{125}$$

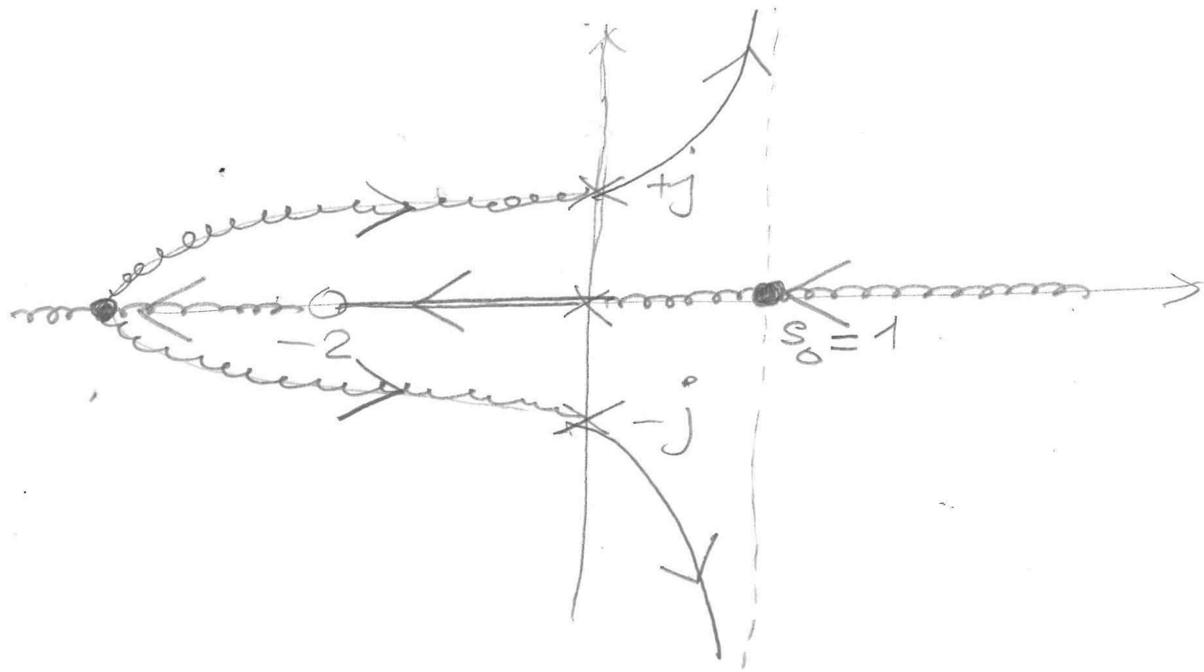


Figure 14. The root locus of exercize 3.8.

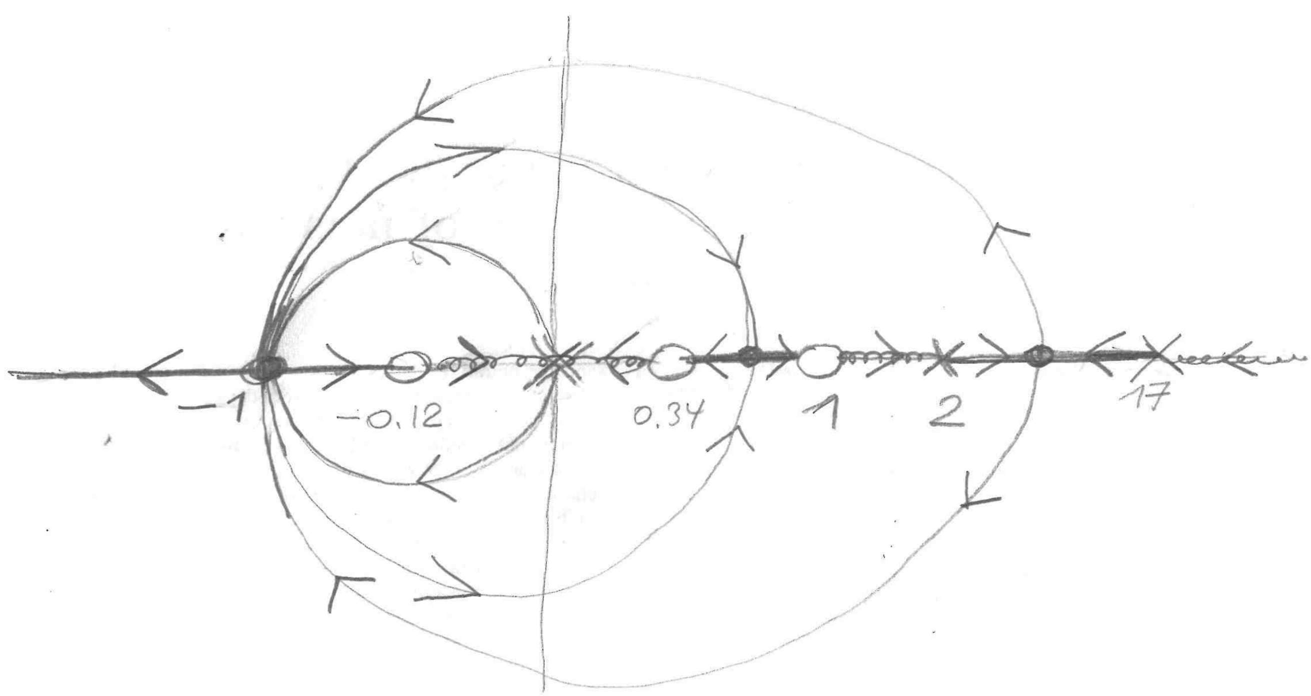


Figure 15. The root locus of exercize 3.9.

and by means of the Routh criterion, applied on

$$\mathrm{NUM}(1 + K\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{P}(s))|_{s-\alpha},$$

find $K > 0$ *such that the roots of* $\mathrm{NUM}(1 + K\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{P}(s))$ *lie in* $\mathfrak{S}(\alpha)$*. Finally, apply to* $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)\mathbf{P}(s)$ *the control action*

$$\mathbf{G}_5(s) := \frac{1}{(1+sT)^{n-m-2}} \tag{126}$$

which makes physically realizable the overall control action $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)\mathbf{G}_5(s)$ *and by means of the Routh criterion, applied on*

$$\mathrm{NUM}(1 + \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)\mathbf{G}_5(s)\mathbf{P}(s))|_{s-\alpha},$$

find $T > 0$ *such that the roots of* $\mathrm{NUM}(1 + \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)\mathbf{G}_5(s)\mathbf{P}(s))$ *lie in in* $\mathfrak{S}(\alpha)$*.*

Exercize *3.7: Let*

$$\mathbf{P}_1(s) := \frac{1}{s(s^2+1)} \tag{127}$$

and

$$\mathbf{P}_2 : \dot{\mathbf{x}}_t = A\mathbf{x}_t + B\mathbf{u}_t,\ \mathbf{y}_t = C\mathbf{x}_t,$$
$$A = \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ C = \begin{pmatrix} -1 & -2 \end{pmatrix} \tag{128}$$

Find a minimal dimensional controller $\mathbf{G}(s)$ *such that the feedback interconnection in Figure 16 is asymptotically stable with poles in* $\mathfrak{S}(-0.3)$*.*

Since the controller $\mathbf{G}(s)$ must have minimal dimension, we check first if we can simply select a proportional controller $\mathbf{G}(s) = K_G$ for stabilizing the feedback interconnection in Figure 16. For this reason, we draw the root locus of $\mathbf{P}(s) = \mathbf{P}_1(s)\mathbf{P}_2(s)$. First of all, we compute $\mathbf{P}_2(s)$:

$$\mathbf{P}_2(s) = C(sI - A)^{-1}B = \frac{1}{s+2} \tag{129}$$

We have

$$\mathbf{P}(s) = \frac{1}{s(s^2+1)(s+2)} \tag{130}$$

The relative degree is $n - m = 4$ and the root locus has at most $n + m - 1 = 3$ singular points. The asymptote center is in

$$s_0 = \frac{-2}{4} = -0.5$$

The singular points equation is

$$4s^3 + 6s^2 + 2s + 2 = 0$$

with solutions

$$s_1 \approx -1.4,\ s_{2,\pm} \approx -0.05 \pm j0.59$$

While s_1 is a point of the positive locus, we should check if $s_{2,\pm}$ are points of the locus, i.e. if they are roots of the root locus equation

$$\mathrm{NUM}(1 + K_G\mathbf{P}_1(s)\mathbf{P}_2(s)) = s(s+2)(s^2+1) + K_G$$

Replacing the points $s_{2,\pm}$ in the above equation, we see that these points cannot satisfy the equation and therefore $s_{2,\pm}$ are not points of the locus.

The analysis of the root locus of $\mathbf{P}_1(s)\mathbf{P}_2(s)$ shows that there are no values of K_G for which the the closed-loop systems is asymptotically stable. However, the asymptote center is in $\mathbb{R}^-$ and, in particular, more negative than the point -0.3. Since the relative degree of $\mathbf{P}_1(s)\mathbf{P}_2(s)$ is $n - m = 4$, this suggests to reduce it to 2 without moving the asymptote center. By doing this we guarantee also the minimality of the dimension of the controller $\mathbf{G}(s)$. Since the $\mathbf{P}_1(s)\mathbf{P}_2(s)$ has no zeroes, this procedure will be sufficient to guarantee for sufficiently large values of the gain K_G the asymptotic stability of the closed-loop system with poles in $\mathfrak{S}(-0.3)$ (as required by the exercize).

For reducing the the relative degree of $\mathbf{P}_1(s)\mathbf{P}_2(s)$ to 2 we introduce a first controller

$$\mathbf{G}_1(s) = (s - z)^2$$

with parameter $z < 0$. By choosing $z = -0.5$ the asymptote center is not changing at all. At this point we will use the Routh criterion for choosing K for which the closed-loop poles are in $\mathfrak{S}(-0.3)$. To this aim, we will apply the Routh criterion to $\mathrm{NUM}(1 + K\mathbf{G}_1(s)\mathbf{P}_1(s)\mathbf{P}_2(s))$ translated by -0.3:

$$\begin{aligned}&\mathrm{NUM}(1 + K\mathbf{G}_1(s)\mathbf{P}_1(s)\mathbf{P}_2(s))|_{s-0.3}\\ &= s^4 + 0.8s^3 + (K - 0.26)s^2 + (0.4K + 1.832)s\\ &+(0.04K - 0.5559)\end{aligned} \tag{131}$$

From the Routh table we obtain that for $K > 13.90$ we have no sign variations in the first column. Therefore, we select $\mathbf{G}_2(s) = K = 15$. Finally, we realize the controller $\mathbf{G}_1(s) = (s + 0.5)^2$ with

$$\mathbf{G}_3(s) = \frac{1}{(1+sT)^2}$$

where $T > 0$ is obtained from the Routh table for $\mathrm{NUM}(1 + \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{P}_1(s)\mathbf{P}_2(s)$ translated by -0.3 with parameter T:

$$\mathrm{NUM}(1 + \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{P}_1(s)\mathbf{P}_2(s))|_{s-0.3} \tag{132}$$

in particular, the value $T > 0$ for which we have no sign variations in the first column of the associated Routh table.

C. Non minimum phase and asymptotically stable systems

If $\mathbf{P}(s)$ *is not minimum phase, we do not have a general procedure for assigning the closed-loop poles in* $\mathfrak{S}(\alpha)$ *or in particular subregions of* $\mathbb{C}^-$*. The situation for which* $\mathbf{P}(s)$ *has all poles in* $\mathbb{C}^-$ *(i.e. it is asymptotically stable) makes exception.*

If $\mathbf{P}(s)$ *has all poles in* $\mathbb{C}^-$ *and not all zeroes in* $\mathbb{C}^-$ *(i.e. it is non minimum phase), then using*

$$\mathbf{G}_1(s) := \frac{\prod_{i=1}^{n}(s - p_{j_i})}{\prod_{i=1}^{r}(s - p'_{j_i})} \tag{133}$$

cancel all the poles $p_1, \ldots, p_n$ *of* $\mathbf{P}(s)$ *which are not* $\mathfrak{S}(\alpha)$ *and replace them with poles* $p'_1, \ldots, p'_n$ *in* $\mathfrak{S}(\alpha)$*. Apply a proportional control action to* $\mathbf{P}(s)\mathbf{G}_1(s)$

$$\mathbf{G}_2(s) := K \tag{134}$$

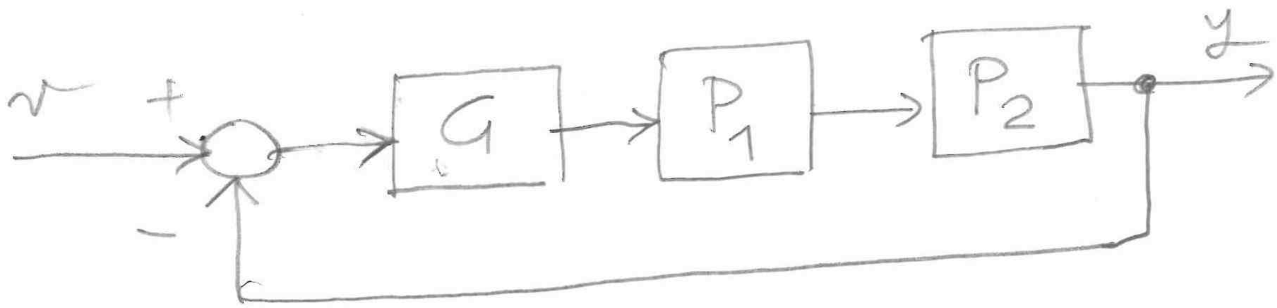


Figure 16. The feedback loop of exercize 3.7.

By means of the Routh criterion on

$$\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))|_{s-\alpha}$$

find $K > 0$ *(sufficiently small) such that the roots of* $\mathrm{NUM}(1 + K\mathbf{P}(s)\mathbf{G}_1(s))$ *are in* $\mathfrak{S}(\alpha)$.

Exercize *3.8: Consider the feedback interconnection in Figure 13 with*

$$\mathbf{P}(s) := \frac{s+1}{s^2+1},\ \mathbf{G}_1(s) := \frac{1}{s}. \tag{135}$$

Design a controller $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ *such that the closed-loop system is asymptotically stable with poles in* $\mathfrak{S}(2)$.

First of all, since $\mathbf{G}_1(s)\mathbf{P}(s)$ is minimum phase, we cancel the zero at $s = -1$ and place another zero at $s = -3$ for instance. To this aim, we use a controller

$$\mathbf{G}_1(s) = \frac{s+3}{s+1}$$

The relative degree of $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s)$ is 2 and its asymptotes center is $s_0 = \frac{-1-(-3-1)}{2} = 3/2$. We move the asymptotes center inside $\mathfrak{S}(2)$ with a zero-pole control action

$$\mathbf{G}_2(s) = \frac{s-z}{s-p}$$

where $z, p < 0$. Since under this control action the new asymptotes center becomes

$$s_0' = \frac{-1+p-(-3-1+z)}{2} = \frac{3+p-z}{2}$$

if we choose $z = -1$ and $p = z - 9 = -10$ the asymptotes center s_0' will be positioned in $-2 \in \mathfrak{S}(3)$.

Finally, we apply a proportional action $\mathbf{G}_3(s) = K$. The closed-loop system will be asymptotically stable with all poles in $\mathfrak{S}(2)$ for a sufficiently large value of K. For this reason, for the denominator of the closed-loop I/O transfer function translated by $s - 2$

$$\begin{aligned}&\mathrm{NUM}(1 + \mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{P}(s))|_{s-2}\\ &= (s-2)^4 + 10(s-2)^3 + (s-2)^2(K+1)\\ &+(s-2)(3k+10) + 2k\end{aligned} \tag{136}$$

we construct the Routh table and determine the values of K for which there are no sign variations in the first column of the table (do this for homework!). □

D. Direct pole assignment I

With a controlled process

$$\mathbf{P}(s) = \frac{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n} \tag{137}$$

we may ask if it possible to place the closed-loop poles into specific locations with an r*-dimensional parametrized controller*

$$\mathbf{G}(s) = \frac{c_0 + c_1 s + \cdots + c_r s^r}{d_0 + d_1 s + \cdots + d_{r-1}s^{r-1} + s^r} \tag{138}$$

where $c_0, \dots, c_r, d_0, \dots, d_{r-1} \in \mathbb{R}$ *are the parameters to be determined. Let's say that*

$$p_1^*, \dots, p_{n+r}^* \in \mathbb{C}^-$$

are the poles to be assigned to the closed-loop system so that

$$\mathbf{p}^*(s) = \Pi_{j=1}^n (s - p_j^*)$$

will be the denominator of the closed-loop system. Comparing the denominator of the closed-loop system

$$\mathrm{NUM}(1 + \mathbf{P}(s)\mathbf{G}(s))$$

with the target polynomial

$$\mathbf{p}^*(s) = \Pi_{j=1}^n (s - p_j^*)$$

and equating the coefficients of the corresponding powers of s*, we obtain a certain number of equations in the unknowns* $c_0, \dots, c_r, d_0, \dots, d_{r-1} \in \mathbb{R}$ *for which we are not guaranteed in general that a solution exists. If minimal dimension is required for* $\mathbf{G}(s)$*, it is convenient to proceed by steps, increasing the dimension of* $\mathbf{G}(s)$ *by one at each step until a solution is found.*

Exercize *3.9: Consider the feedback interconnection in Figure 13 with*

$$\mathbf{P}(s) := \frac{s-1}{s(s-2)},\ \mathbf{G}_1(s) := \frac{1}{s}. \tag{139}$$

Design a controller $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ *with minimal dimension such that the closed-loop system is asymptotically stable with all real equal poles. By drawing the root locus of* $\mathbf{G}(s)\mathbf{P}(s)$*, check the stability of the closed-loop system and the position of the closed-loop poles.*

For a minimal dimensional controller $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ it is convenient first to see if we can take a proportional controller $\mathbf{G}_2(s) = K_{G_2}$ for stabilizing the closed-loop system.

This can be qualitatively checked by inspection of the root locus of $\mathbf{G}_1(s)\mathbf{P}(s)$ and confirmed by the Routh table of the closed-loop denominator

$$\mathrm{NUM}(1 + K_{G_2}\mathbf{G}_1(s)\mathbf{P}(s)) = s(s-2) + K_{G_2}(s-1)$$

We conclude that it is not possible to stabilize (no matter where the closed-loop poles are) the closed-loop system with a proportional controller $\mathbf{G}_2(s) = K_{G_2}$.

It is also possible to see that not even

$$\mathbf{G}_2(s) = K(s+z) \tag{140}$$

is able to stabilize the closed-loop system (no matter where the closed-loop poles are). Indeed,

$$\mathrm{NUM}(1+\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)) = s^3+(K-2)s^2+K(z-1)-Kz$$

and there are no values of K and z for which the coefficients of the above polynomial have the same sign (i.e. the necessary condition for the polynomial being Hurwitz is violated).

Since $\mathbf{G}_1(s)\mathbf{P}(s)$ has a positive zero, although the relative degree of $\mathbf{G}_1(s)\mathbf{P}(s)$ is 2 it is not possible to design the controller $G_2(s)$ in such a way to move the asymptote center in the negative real axis and finally increase the gain to stabilize the closed-loop.

Since $\mathbf{G}(s) = \mathbf{G}_1(s)\mathbf{G}_2(s)$ must have minimal dimension, we try a parametric controller

$$\mathbf{G}_2(s) = \frac{as^2+bs+c}{s+d}$$

The denominator of the closed-loop I/O transfer function is

$$\begin{aligned}&\mathrm{NUM}(1+\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s))\\&= s^2(s+d)(s-2) + (as^2+bs+c)(s-1)\\&= s^4 + (a+d-2)s^3 + (b-a-2d)s^2 + (c-b)s - c\end{aligned}$$

We make the above polynomial be equal to

$$(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1$$

By doing this, we guarantee that the closed-loop poles are all real and equal to -1, as required by the exercize. We obtain

$$\begin{aligned}a+d-2&=4\\ b-a-2d&=6\\ c-b&=4\\ -c&=1\end{aligned} \tag{141}$$

which gives $a = 23$, $b = -5$, $c = -1$ and $d = -17$. Therefore,

$$\mathbf{G}_2(s) = \frac{23(s+0.1264)(s-0.3438)}{s(s-17)} \tag{142}$$

The root locus is drawn in Figure 15. As it can be seen for $K = 1$ the closed-loop system is asymptotically stable with three equal negative poles. □

Exercize *3.10: Given*

$$\mathbf{P}(s) = \frac{(s+1)^2}{(s-1)(s-2)(s-4)}$$

(i) design a first controller $G(s)$ such that the closed-loop system is asymptotically stable with all real poles in $s = -1$,

(iii) draw the root locus of $PG(s)$ using the Routh criterion to determine the exact picture on the imaginary axis.

For stabilizing the closed-loop system we should need at least one zero-pole action for moving the asymptotes center in the negative real axis. Therefore, $\mathbf{G}(s)$ should introduce at least one zero-pole action. Also, note that because $\mathbf{P}(s)$ has two zeros at -1, it might be convenient to cancel them under feedback in such a way to simplify the process to be stabilized. Since we have to place the closed-loop at the exact location $s = -1$, we rather adopt a direct design method by using a parametric structure for the controller $\mathbf{G}(s)$. Thus, we assume $\mathbf{G}(s)$ of the form

$$\mathbf{G}(s) = \frac{as^3+bs^2+cs+d}{(s+1)^2(es+f)}$$

where the term

$$\frac{1}{(s+1)^2}$$

cancels the two zeros of $\mathbf{P}(s)$ at -1 and the term

$$\frac{as^3+bs^2+cs+d}{es+f}$$

introduces one proportional action (i.e. one parameter), three zeroes (i.e. three parameters) and one pole (i.e. one parameter): the motivation behind this is that the pole $es+f$ increases the overall number of poles by one (i.e. the closed-loop poles are 4) and, as a consequence, for changing all the closed-loop poles (in our case, they should be placed at $s = -1$) we need at least four parameters which are given by the proportional action and the three zeroes $as^3 + bs^2 + cs + d$. Also, notice that the structure of $\mathbf{G}(s)$ also includes the zero-pole action needed for stabilization.

We have

$$\mathbf{P}(s)\mathbf{G}(s) = \frac{as^3+bs^2+cs+d}{(s-1)(s-2)(s-4)(es+f)}. \tag{143}$$

Denoting by $\mathbf{p}(s)$ the denominator of the closed-loop I/O transfer function $W(s) = \frac{\mathbf{P}(s)\mathbf{G}(s)}{1+\mathbf{P}(s)\mathbf{G}(s)}$

$$\begin{aligned}&\mathbf{p}(s) = \mathrm{NUM}(1+\mathbf{P}(s)\mathbf{G}(s))\\&= (s-1)(s-2)(s-4)(es+f) + as^3 + bs^2 + cs + d\end{aligned}$$

one now proceeds by seeking for $a, b, c, d, e, f \in \mathbb{R}$ such that

$$\mathbf{p}(s) = (s+1)^4$$

that is

$$\begin{aligned}&es^4 + (a-7e+f)s^3 + (b+14e-7f)s^2\\&+(c-8e+14f)s + d - 8f\\&= s^4 + 4s^3 + 6s^2 + 4s + 1\end{aligned} \tag{144}$$

Thus, by equating the terms with the same powers of s one gets

$$a = 11, \quad b = -8, \quad c = 12, \quad d = 1, \quad e = 1 \quad \text{and} \quad f = 0$$

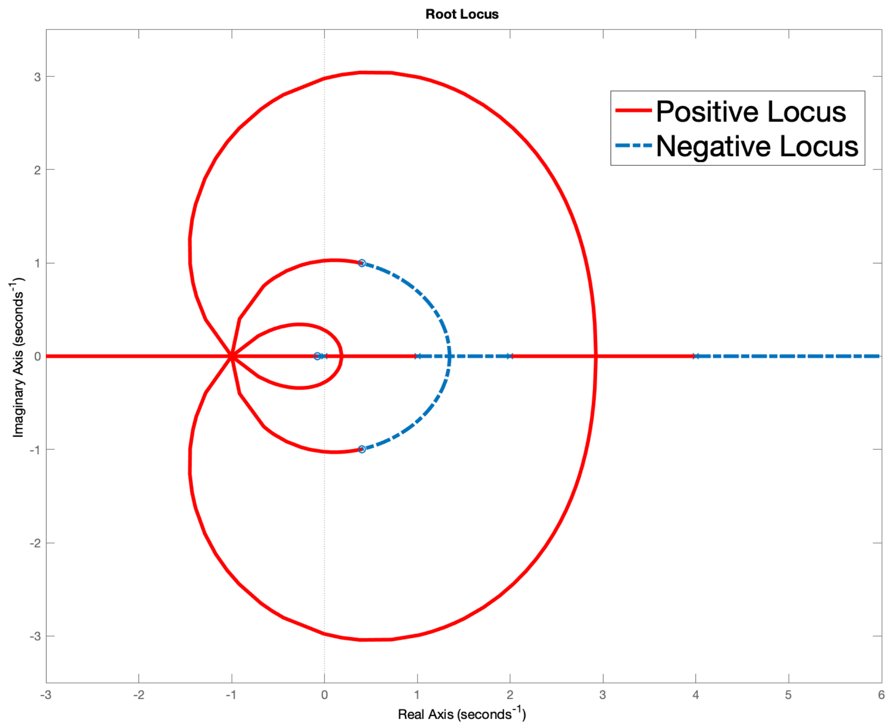


Figure 17. Root locus of $\mathbf{G}(s)\mathbf{P}(s) = \frac{11s^3-8s^2+12s+1}{s(s-1)(s-2)(s-4)}$.

and thus

$$\begin{aligned}\mathbf{G}(s) &= \frac{11s^3 - 8s^2 + 12s + 1}{s(s+1)^2} \\ \mathbf{G}(s)\mathbf{P}(s) &= \frac{11s^3 - 8s^2 + 12s + 1}{s(s-1)(s-2)(s-4)} \\ &= 11\frac{(s+0.07875)(s^2 - 0.806s + 1.154)}{s(s-1)(s-2)(s-4)}. \end{aligned} \tag{145}$$

Next, we draw the root locus of $\mathbf{G}(s)\mathbf{P}(s)$. The relative degree of $\mathbf{G}(s)\mathbf{P}(s)$ is $4 - 3 = 1$ and the asymptotes center

$$s_0 \approx 1 + 2 + 4 - 0.8060 + 0.07875 = 6.2728.$$

with one horizontal asymptote for each locus (negative real axis for positive locus, positive real axis for negative locus). The singular points equations are

$$\begin{aligned}&s^4 + (11K-7)s^3 + (14-8K)s^2 + (12K-8)s + K \\ &4s^3 + 3(11K-7)s^2 + 2(14-8K)s + 12K - 8\end{aligned}$$

It results that the positive locus has a singular point with multiplicity $\mu = 4$ at the point $s = -1$ corresponding to $k = 1$ (i.e. the four closed-loop poles obtained with $\mathbf{G}(s)$ above). Also, the positive locus has two further singular points (both with multiplicity $\mu = 2$) at $(s,k) \approx (0.188, 0.345)$ and $(s,k) \approx (2.92, 0.023)$; the negative locus exhibits one singular point at $(s,k) \approx (1.35, -0.0274)$.

Denote by $\mathbf{p}(s)$ the denominator of the I/O closed-loop transfer function $\mathbf{W}(s) = \frac{K\mathbf{G}(s)\mathbf{P}(s)}{1+K\mathbf{G}(s)\mathbf{P}(s)}$:

$$\mathbf{p}(s) = \mathrm{NUM}(1 + K\mathbf{G}(s)\mathbf{P}(s))$$

The Routh table of $\mathbf{p}(s)$ is

$$\begin{array}{c|ccc} r^{(4)} & 1 & 14-8K & K \\ r^{(3)} & 11K-7 & 12K-8 & \\ r^{(2)} & \frac{-88K^2+198K-90}{11K-7} & K & \\ r^{(1)} & \frac{1177K^3-3234K^2+2713K-720}{88K^2-198K+90} & & \\ r^{(0)} & K & & \end{array}$$

The positive locus crosses the imaginary axis at the points $s \approx \{0, \pm i2.98, \pm i0.297, \pm i1.02\}$ corresponding to $K \approx \{0, 0.6321, 0.669, 1.4465\}$. The root locus is drawn in Figure 17. □

E. Direct pole assignment II

From a general point of view and disregarding dimensionality constraints, the direct pole assignment problem can be tackled as follows, although at the price of an over-parametrized controller. Let the open loop process be described by

$$\mathbf{P}(s) = \frac{\mathbf{b}(s)}{\mathbf{a}(s)} = \frac{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n}$$

and the controller by

$$\mathbf{G}(s) = \frac{\mathbf{c}(s)}{\mathbf{d}(s)} = \frac{c_0 + c_1 s + \cdots + c_{n-1}s^{n-1}}{d_0 + d_1 s + \cdots + d_{n-1}s^{n-1} + s^n} \tag{146}$$

Notice that we are considering a dimension of the controller $\mathbf{G}(s)$ *equal to the the dimension of the process* $\mathbf{P}(s)$*, despite of any dimensional constraint (such as the ones considered in the previous exercizes). Our problem is, given a* $2n$*-degree polynomial*

$$\mathbf{p}^*(s) = \prod_{j=1}^{2n}(s - p_j^*) = a_0^* + a_1^* s + \cdots + a_{2n-1}^* s^{2n-1} + s^{2n} \tag{147}$$

to find unknowns $c_0, \cdots, c_{n-1}, d_0 \cdots, d_{n-1}$ *such that*

$$\mathbf{p}^*(s) = \mathrm{NUM}(1 + \mathbf{G}(s)) = \mathbf{a}(s)\mathbf{d}(s) + \mathbf{b}(s)\mathbf{c}(s) \tag{148}$$

Notice that while the open loop poles are n *we will end up with a closed loop having* $2n$ *poles. By equating the coefficients of equal powers in* (148) *we get*

$$Sw = z \tag{149}$$

where

$$z = \begin{pmatrix} a_{2n-1}^* - a_{n-1} \\ \vdots \\ a_n^* - a_0 \\ a_{n-1}^* \\ \vdots \\ a_0^* \end{pmatrix} \in \mathbb{R}^{2n}, w = \begin{pmatrix} d_{n-1} \\ \vdots \\ d_0 \\ c_{n-1} \\ \vdots \\ c_0 \end{pmatrix} \in \mathbb{R}^{2n}, \tag{150}$$

and

$$S = \begin{pmatrix} \mathfrak{A}_- & \mathfrak{B}_- \\ \mathfrak{A}_+ & \mathfrak{B}_+ \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{151}$$

with

$$\mathfrak{A}_- = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & 1 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \end{pmatrix},$$

$$\mathfrak{A}_+ = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & a_0 \end{pmatrix} \tag{152}$$

and

$$\mathfrak{B}_- = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ b_{n-2} & b_{n-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \cdots & 0 & 0 \\ b_1 & b_2 & b_3 & \cdots & b_{n-1} & 0 \end{pmatrix},$$

$$\mathfrak{B}_+ = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \\ 0 & b_0 & b_1 & \cdots & b_{n-3} & b_{n-2} \\ 0 & 0 & b_0 & \cdots & b_{n-4} & b_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_0 \end{pmatrix} \tag{153}$$

The matrix S *in* (151) *is the so-called Sylvester matrix and its determinant* $\det S$ *is known as Sylvester's resultant. Clearly, we can find a (unique) solution to* (151) *if and only if Sylvester's resultant is nonzero. It can be proven that the Sylvester's resultant is nonzero if and only if* $\mathbf{a}(s)$ *and* $\mathbf{b}(s)$ *are coprime, i.e.* $\mathbf{a}(s)$ *and* $\mathbf{b}(s)$ *have no common roots.*

For instance, let $\mathbf{a}(s) = s^3$, $\mathbf{b}(s) = 1$ *(which are coprime) and* $\mathbf{p}^*(s) = a_0^* + a_1^* s + \cdots + a_5^* s^5 + s^6$ *with given* $a_1^*, \ldots, a_5^* > 0$*. Following* (148)*, we require that*

$$\begin{aligned} & s^3(s^3 + d_2 s^2 + d_1 s + d_0) + c_2 s^2 + c_1 s + c_0 \\ & = s^6 + a_5^* s^5 + \cdots + a_1^* s + a_0^* \end{aligned} \tag{154}$$

and solving in the unknowns $d_2, d_1, d_0, c_2, c_1, c_0 \in \mathbb{R}$

$$d_2 = a_5^*, d_1 = a_4^*, d_0 = a_3^*, c_2 = a_2^*, c_1 = a_1^*, c_0 = a_0^*. \tag{155}$$