A Proofs

Theorem 1 *For any objective sentence about situation s, $\phi(s)$,*[5]

$$Axioms \cup \{Sensed[\sigma]\} \models \phi(end[\sigma])$$

if and only if

$$Axioms \cup \{Sensed[\sigma]\} \models \textbf{Know}(\phi(now), end[\sigma]).$$

Proof Sketch: $\Leftarrow$ Follows trivially from the reflexivity of K in the initial situation, and the fact that it is preserved by the successor state axiom for K.

$\Rightarrow$ From the successor state axiom for K it follows that:

$$Axioms \cup \{Sensed[\sigma'] \cdot (a,1)\} \models \textbf{Know}(SF_a(now), end[\sigma' \cdot (a,1)]) \quad (*)$$
$$Axioms \cup \{Sensed[\sigma'] \cdot (a,0)\} \models \textbf{Know}(\neg SF_a(now), end[\sigma' \cdot (a,0)])(**)$$

Suppose not, i.e., there exists a model M of $Axioms \cup \{Sensed[\sigma]\}$ such that for some s' such that $M \models K(s', end[\sigma]), M \models \neg\phi(s')$.

Then take the structure M' obtained from M by intersecting the objects of sort situation with those that in the situation tree rooted in the initial ancestor situation of s', say s'_0. M' satisfies all axioms in $Axioms$ except the reflexivity axiom, the successor state axiom for K, and the initial state axiom, which is of the form $\textbf{Know}(\Psi(now), S_0)$ (note that the other axioms involve neither K nor S_0). Observe that $Trans$ and $Final$ for the situation in the tree are defined by considering relations involving only situation in the same tree.

Now consider the M'' obtained from M' by adding the constant S_0 and making it denote s'_0. Although M' and M'' does not satisfy $\textbf{Know}(\Psi(now), S_0)$, we have that $M'' \models \Psi(S_0)$. Moreover, (*) and (**) and the fact that the successor state axiom for K in M ensure that all predecessor of s' where K alternatives, imply $M'' \models Sensed[\sigma]$.

Finally let us define M''' by adding to M'' the predicate K and making denote the identity relation on situations. Then $M''' \models Axioms \cup \{Sensed[\sigma]\}$. On the other hand since $M' \models \neg\phi(s')$ so does M'''. Thus getting a contradiction. ■

Theorem 2 *Let dp be such that $Axioms \cup \{Sensed[\sigma]\} \models EFDP(dp, end[\sigma])$. Then, $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dp, end[\sigma], s_f)$ if and only if all online executions of (dp, σ) are terminating.*

[5]Note that K cannot appear in the $\phi(s)$, however $Trans$ and $Final$ can, since they are predicates, although axiomatized using a second-order formula.

Proof Sketch: First of all we observe that dp is a deterministic program and its possible online executions from σ are completely determined by the sensing outcomes. We also observe that in each model there will be a single execution of dp, since the sensing outcomes are fully determined in the model. Moreover, in all models where with the same sensing outcomes up to a given configuration (dp_i, s_i), the next transition of dp from $end[\sigma]$ is the same.

$\Rightarrow$ If $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dp, end[\sigma], s_f)$ then in every model of $Axioms \cup \{Sensed[\sigma]\}$ the only execution of dp from $end[\sigma]$ terminates. Consider an online execution reaching (dp_i, σ_i). Then, in all models of $Axioms \cup \{Sensed[\sigma]\}$ with sensing outcomes as determined by σ_i, the next configuration (dp_{i+1}, s_{i+1}) is the same, given that $LEFDP(dp_i, end[\sigma_i])$ requires the next transition to be known in each of these models, and hence by reflexivity of K we have that such a transition is true as well in each of them. Then, for all a possible online transitions from $(dp_i, end[\sigma_i])$ to $dp'_i, end[\sigma'_i]$ it must be the case that $dp'_i = dp_{i+1}$ and $end[\sigma'_i] = s_{i+1}$, i.e. the next online transitions can differ only wrt the new sensing outcome acquired.

$\Leftarrow$ If an online execution of dp from σ terminates it means that the program dp, from $end[\sigma]$, terminates in all models of $Axioms \cup \{Sensed[\sigma]\}$ with the sensing outcome as in the online execution. Since by hypothesis all online executions terminate, thus covering all possible sensing outcome, then dp, from $end[\sigma]$, terminates in all models. ■

Theorem 3 *If* $Axioms \cup \{Sensed[\sigma]\} \models Trans(\Sigma_e(p), end[\sigma], p', s')$*, then*

1. $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(p, end[\sigma], s_f)$
2. $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(\Sigma_e(p), end[\sigma], s_f)$
3. *All online executions from* $(\Sigma_e(p), \sigma)$ *terminate.*

Proof Sketch: (1) and (2) follow immediately from the definition of $Trans$ for Σ_e.

(3) By the definition of $Trans$ for Σ_e, there exists a dp and such that $Axioms \cup \{Sensed[\sigma]\} \models EFDP(dp, end[\sigma]) \land \exists s_f.Trans(dp, end[\sigma], p', s') \land Do(p', s', s_f)$. The conditions of Theorem 2 are satisfied, thus we have that all online executions from (dp, σ) are terminating. Since these include all online executions from (p', σ') with $s' = end[\sigma']$, all online executions from (p', σ') must also be terminating. Hence the thesis follows. ■

Theorem 4 *Let dpt be a tree program, i.e., dpt* $\in$ *TREE. Then, for all histories* σ*,*
if $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dpt, end[\sigma], s_f)$*,*
then $Axioms \cup \{Sensed[\sigma]\} \models EFDP(dpt, end[\sigma])$.

Proof Sketch: By induction on the structure of dpt.

Base cases: for nil, it is known that nil is $Final$, so $Axioms \cup \{Sensed[\sigma]\} \models EFDP(nil, end[\sigma])$ holds; for $False?$, the antecedent is false, so the thesis holds.

Inductive cases: Assume that the thesis holds for dpt_1 and dpt_2. Assume that $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dpt, end[\sigma], s_f)$.

For $dpt = a; dpt_1$: $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(a; dpt_1, end[\sigma], s_f)$ implies that $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dpt_1, do(a, end[\sigma]), s_f)$. Since a is a non-sensing action, $Sensed[\sigma \cdot (a, 1)] = Sensed[\sigma]$, so we also have $Axioms \cup Sensed[\sigma \cdot (a, 1)] \models \exists s_f.Do(dpt_1, end[\sigma \cdot (a, 1)], s_f)$. Thus by the induction hypothesis we have $Axioms \cup \{Sensed[\sigma \cdot (a, 1)]\} \models EFDP(dpt_1, end[\sigma \cdot (a, 1)])$. It follows that $Axioms \cup \{Sensed[\sigma]\} \models EFDP(dpt_1, do(a, end[\sigma])$. The assumption $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(a; dpt_1, end[\sigma], s_f)$ also implies that $Axioms \cup \{Sensed[\sigma]\} \models Poss(a, end[\sigma])$ and this must be known by Theorem 1, i.e., $Axioms \cup \{Sensed[\sigma]\} \models \mathbf{Know}(Poss(a, now), end[\sigma])$. Thus, we have that

$$Axioms \cup \{Sensed[\sigma]\} \models \mathbf{Know}(Trans(a; dpt_1, now, dpt_1, do(a, now)), end[\sigma]).$$

It is also known that this is the only transition possible for $a; dpt_1$, So $Axioms \cup \{Sensed[\sigma]\} \models LEFDP(a; dpt_1, end[\sigma])$. Therefore, $Axioms \cup \{Sensed[\sigma]\} \models EFDP(a; dpt_1, end[\sigma])$.

For $dpt = True?; dpt_1$: the argument is similar, but simpler since the test does not change the situation.

For $dpt = sense_\phi;$ **if** ϕ **then** dpt_1 **else** dpt_2: Suppose that the sensing action returns 1 and let $\sigma_1 = \sigma \cdot (sense_\phi, 1)$. Next we show that $Axioms \cup \{Sensed[\sigma]\} \models LEFDP(dpt, end[\sigma])$. The assumption that $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dpt, end[\sigma], s_f)$ implies that $Axioms \cup \{Sensed[\sigma_1]\} \models \exists s_f. Do(dpt_1, end[\sigma_1], s_f)$. Thus by the induction hypothesis we have $Axioms \cup \{Sensed[\sigma_1)]\} \models EFDP(dpt_1, end[\sigma_1])$. It follows that $Axioms \cup \{Sensed[\sigma]\} \models \phi(do(sense_{phi}, end[\sigma]) \supset EFDP(dpt_1, do(sense_{phi}, end[\sigma])$. By a similar argument, it also follows that we must have that $Axioms \cup \{Sensed[\sigma]\} \models \neg\phi(do(sense_{phi}, end[\sigma]) \supset EFDP(dpt_2, do(sense_{phi}, end[\sigma])$. The assumption $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dpt, end[\sigma], s_f)$ also implies that $Axioms \cup \{Sensed[\sigma]\} \models Poss(sense_\phi, end[\sigma])$ and this must be known by Theorem 1, i.e., $Axioms \cup \{Sensed[\sigma]\} \models \mathbf{Know}(Poss(sense_\phi, now), end[\sigma])$. Thus, we have that

$$\begin{array}{l} Axioms \cup \{Sensed[\sigma]\} \models \mathbf{Know}(\\ \quad Trans(dpt, now, \textbf{if } \phi \textbf{ then } dpt_1 \textbf{ else } dpt_2, do(sense_\phi, now)), end[\sigma]). \end{array}$$

It is also known that this is the only transition possible for dpt, so $Axioms \cup \{Sensed[\sigma]\} \models LEFDP(dpt, end[\sigma])$. Thus, $Axioms \cup \{Sensed[\sigma]\} \models$

$EFDP(dpt, end[\sigma])$. ■

Theorem 5 *For any program dp that is*

1. *an epistemically feasible deterministic program, i.e.,* $Axioms \cup \{Sensed[\sigma]\} \models EFDP(dp, end[\sigma])$ *and*
2. *such that there is a known bound on the number of steps it needs to terminate, i.e., where there is an* n *such that* $Axioms \cup \{Sensed[\sigma]\} \models \exists p', s', k.k \leq n \wedge Trans^k(dp, end[\sigma], p', s') \wedge Final(p', s')$,

there exists a tree program $dpt \in TREE$ *such that* $Axioms \cup \{Sensed[\sigma]\} \models \forall s_f.Do(dp, end[\sigma], s_f) \equiv Do(dpt, end[\sigma], s_f)$.

Proof Sketch: We construct the tree program $dpt = m(dp, \sigma)$ from dp using the following rules:

- $m(dp, \sigma) = False?$ iff $Axioms \cup \{Sensed[\sigma]\}$ is inconsistent, otherwise
- $m(dp, \sigma) = nil$ iff $Axioms \cup \{Sensed[\sigma]\} \models Final(dp, end[\sigma])$, otherwise
- $m(dp, \sigma) = a; m(dp', \sigma \cdot (a, 1))$ iff $Axioms \cup \{Sensed[\sigma]\} \models Trans(dp, end[\sigma], dp', do(a, end[\sigma])$ for some non-sensing action a,
- $m(dp, \sigma) = sense_\phi;$ **if** ϕ **then** $m(dp_1, \sigma \cdot (sense_\phi, 1))$ **else** $m(dp_2, \sigma \cdot (sense_\phi, 0))$ iff $Axioms \cup \{Sensed[\sigma]\} \models Trans(dp, end[\sigma], dp', do(sense_\phi, end[\sigma])$ for some sensing action $sense_\phi$,
- $m(dp, \sigma) = True?; m(dp', \sigma)$ iff $Axioms \cup \{Sensed[\sigma]\} \models Trans(dp, end[\sigma], dp', end[\sigma])$.

Let us show that
$Axioms \cup \{Sensed[\sigma]\} \models Do(dp, end[\sigma], s_f) \equiv Do(m(dp, \sigma), end[\sigma], s_f)$.

It turns out that, under the hypothesis of the theorem, for all dp and all σ, (dp, σ) is bisimilar to $(m(dp, \sigma), \sigma)$ with respect to online executions. Indeed, it is easy to check that the relation $[(dp, \sigma), (m(dp, \sigma), \sigma)]$ is a bisimulation, i.e., for all dp and σ, $[(dp, \sigma), (m(dp, \sigma), \sigma)]$ implies that

- $Axioms \cup \{Sensed[\sigma]\} \models Final(dp, end[\sigma])$ iff $Axioms \cup \{Sensed[\sigma]\} \models Final(m(dp, \sigma), end[\sigma])$,
- for all dp', σ' if $Axioms \cup \{Sensed[\sigma]\} \models Trans(dp, end[\sigma], dp', end[sigma'])$ with $Axioms \cup \{Sensed[sigma']\}$ consistent, then $Axioms \cup \{Sensed[\sigma]\} \models Trans(m(dp, \sigma), end[\sigma], m(dp', \sigma'), end[\sigma'])$ and $[(dp', \sigma'), (m(dp', \sigma'), \sigma')]$,

- for all dp', σ' if $Axioms \cup \{Sensed[\sigma]\} \models Trans(m(dp,\sigma), end[\sigma], m(dp',\sigma'), end[\sigma'])$ with $Axioms \cup \{Sensed[sigma']\}$ consistent, then $Axioms \cup \{Sensed[\sigma]\} \models Trans(dp, end[\sigma], dp', end[sigma'])$ and $[(dp',\sigma'), (m(dp',\sigma'),\sigma')]$.

Now, assume that $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dp, end[\sigma], s_f)$, then since dp is an $EFDP$, by Theorem 2 all online execution from (dp,σ) terminate. Hence since $(dp,\sigma$ and $(m(dp,\sigma),\sigma)$ are bisimilar, $(m(dp,\sigma),\sigma)$ has the same online execution (apart from the program appearing in the configurations).

Next, observe that given an online execution of (dp,σ) terminating in (dp_f,σ_f), in all models of $Axioms \cup \{Sensed[\sigma]\}$ with sensing outcomes as in σ_f both the program dp and $m(dp,\sigma)$ reach the same situation $end[\sigma_f]$. Since there are terminating online executions for all possible sensing outcomes, the thesis follows. ■

Theorem 6 *Let dpl be a linear program, i.e., dpl $\in LINE$. Then, for all histories σ, if $Axioms \cup \{Sensed[\sigma]\} \models \exists s_f.Do(dpl, end[\sigma], s_f)$, then $Axioms \cup \{Sensed[\sigma]\} \models EFDP(dpl, end[\sigma])$.*

Proof Sketch: This is a corollary of Theorem 4 for tree programs. Since linear programs are tree programs, the thesis follows immediately from this theorem. ■

Theorem 7 *For any dp that does not include sensing actions, such that*

$$Axioms \cup \{Sensed[\sigma]\} \models EFDP(dp, end[\sigma]),$$

there exists a linear program dpl such that

$$Axioms \cup \{Sensed[\sigma]\} \models \forall s_f.Do(dp, end[\sigma], s_f) \equiv Do(dpl, end[\sigma], s_f).$$

Proof Sketch: We show this using the same approach as for Theorem 5 for tree programs. Since dp cannot contain sensing actions, the construction method used in the proof of Theorem 5 produces a tree program that contains no branching and is in fact a linear program. Then, by the same argument as used there, the thesis follows. ■

Theorem 8 $Axioms \cup \{Sensed[\sigma]\} \models Trans(\Sigma_l(p), end[\sigma], dpl, s')$ *if and only if there exists a situation s_f such that $Axioms \cup \{Sensed[\sigma]\} \models Do(p, end[\sigma], s_f)$.*

Proof Sketch: $\Leftarrow$ If for same s_f we have $Axioms \cup \{Sensed[\sigma]\} \models Do(p, end[\sigma], s_f)$ then the sequence of actions from $end[\sigma]$ to s_f is an $LINE$ program, which trivially satisfies the left-hand-side of the axiom for Σ_l. Observe that if $s' = end[\sigma]$ then the linear program can be simply $True?$.

$\Rightarrow$ By hypothesis there exists a dpl that is a $LINE$. If $s' = s$ and then $dpl = true?; dpl'$ and if $s' = do(a, s)$, for same action a, and then $dpl = a; dpl'$. In both cases dpl' must be an $LINE$. In every model dpl' reaches from s' a final situation of the original program p. Observe that such situation will be the same in every model since the sequence of actions α starting from s' is fixed by dpl'. It follows that the sequence of action done by dpl starting from s reaches a situation s_f such that $Axioms \cup \{Sensed[\sigma]\} \models Do(p, end[\sigma], s_f)$. ■