Robotics 1

Inverse kinematics

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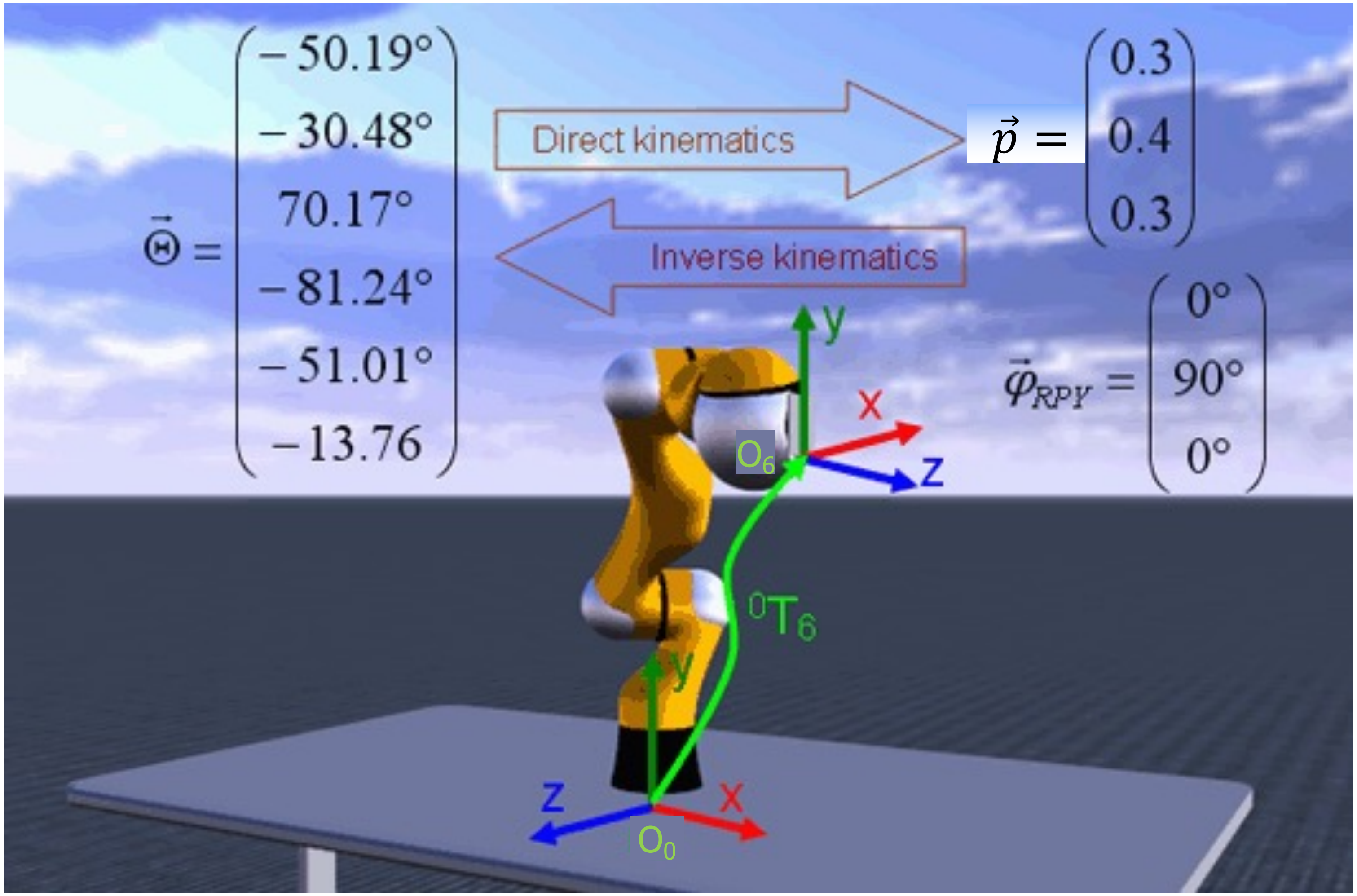
Inverse kinematics
what are we looking for?



direct kinematics is always unique;
how about inverse kinematics for this 6R robot?

Inverse kinematics problem

- given a desired end-effector pose (position + orientation), find the values of the joint variables q that will realize it
- a synthesis problem, with input data in the form
  - $T = \begin{bmatrix} R & p \\ 0^T & 1 \end{bmatrix} = {}^0A_n(q)$
  - $r = f_r(q)$, for a task function

classical formulation:
inverse kinematics for a given end-effector pose T

generalized formulation:
inverse kinematics for a given value r of task variables

- a typical nonlinear problem
  - existence of a solution (workspace definition)
  - uniqueness/multiplicity of solutions ($r \in \mathbb{R}^m$, $q \in \mathbb{R}^n$)
  - solution methods

Solvability and robot workspace

for tasks related to a desired end-effector Cartesian pose

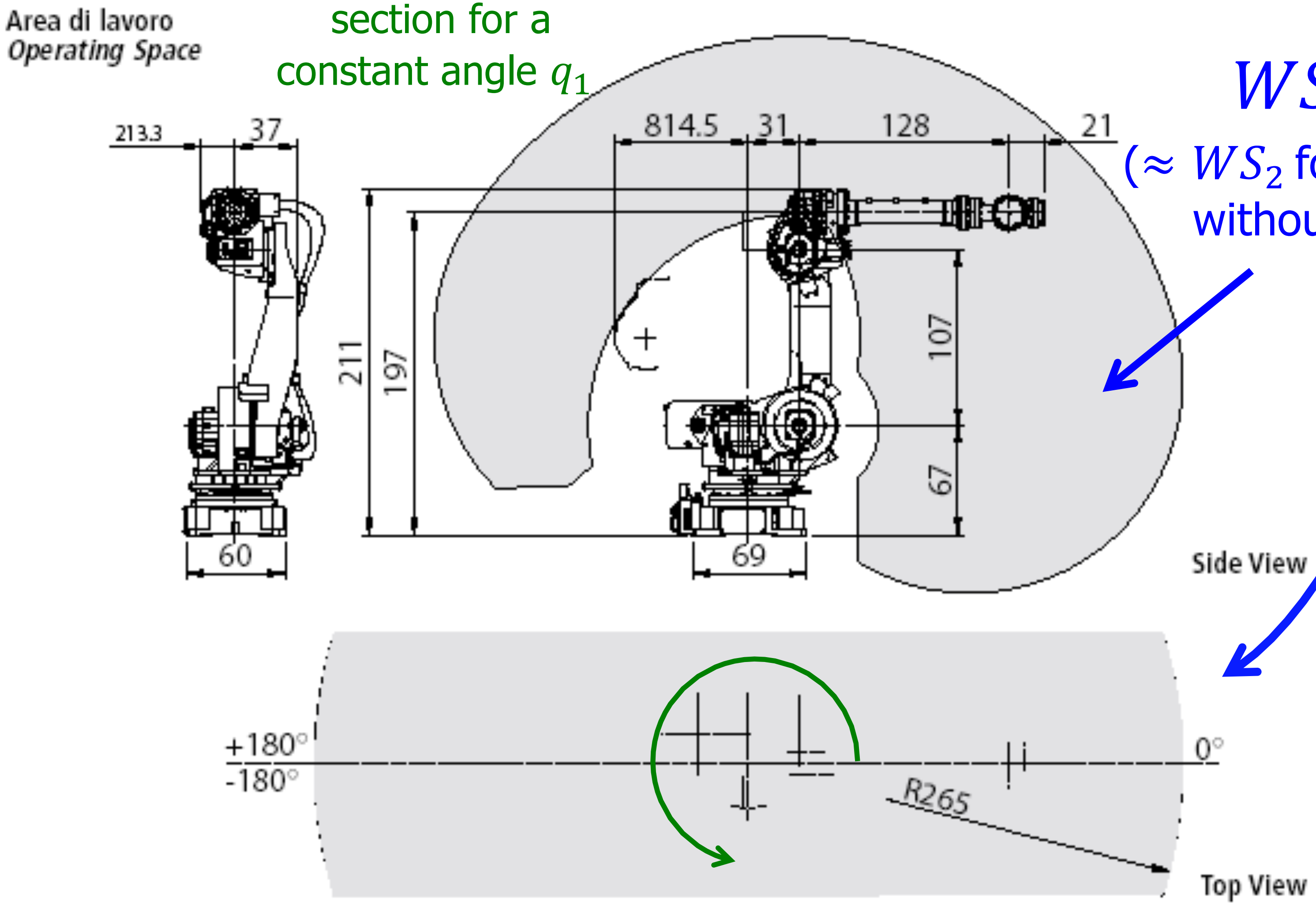
- primary workspace WS_1: set of all positions p that can be reached with at least one orientation (ϕ or R)
  - out of WS_1 there is no solution to the problem
  - if $p \in WS_1$, there is a suitable ϕ (or R) for which a solution exists
- secondary (or dexterous) workspace WS_2: set of positions p that can be reached with any orientation (among those feasible for the robot direct kinematics)
  - if $p \in WS_2$, there exists a solution for any feasible ϕ (or R)
- $WS_2 \subseteq WS_1$

Workspace of Fanuc R-2000i/165F

section for a constant angle q_1

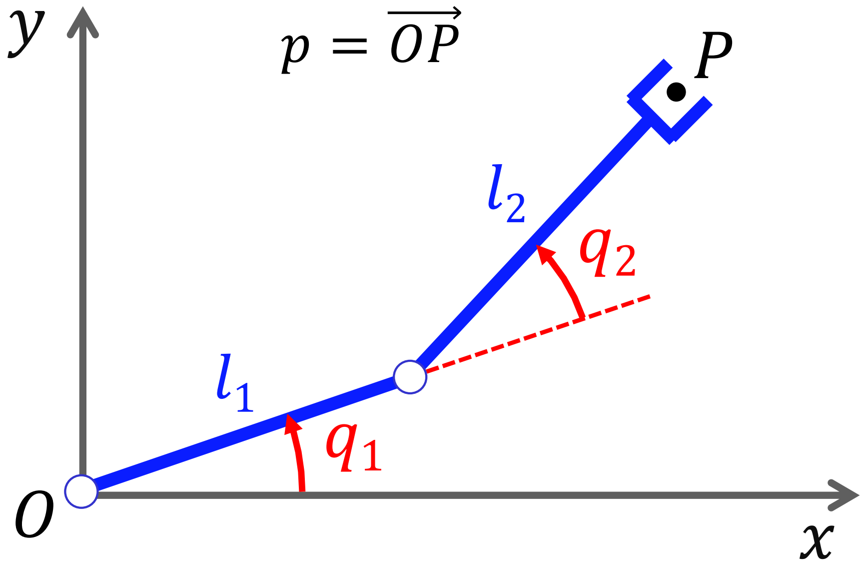
$WS_1 \subset \mathbb{R}^3$

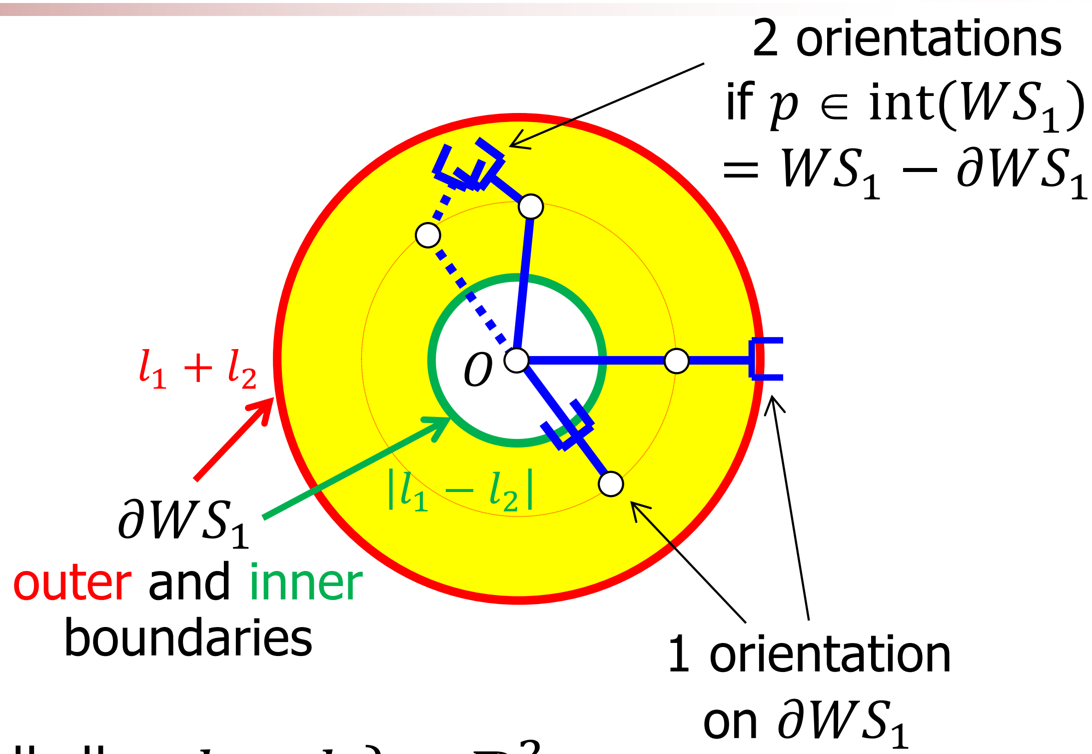
($\approx WS_2$ for spherical wrist without joint limits)



rotating the base joint angle q_1

Workspace of a planar 2R arm





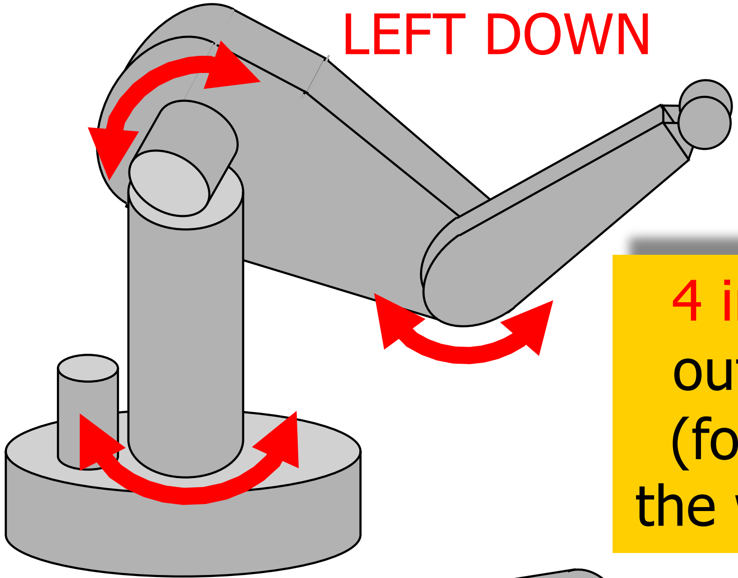
- if $l_1 \neq l_2$
  - $WS_1 = \{p \in \mathbb{R}^2 : |l_1 - l_2| \leq \|p\| \leq l_1 + l_2\} \subset \mathbb{R}^2$
  - $WS_2 = \emptyset$
- if $l_1 = l_2 = l$
  - $WS_1 = \{p \in \mathbb{R}^2 : \|p\| \leq 2l\} \subset \mathbb{R}^2$
  - $WS_2 = \{p = 0\}$ (all feasible orientations at the origin!... an infinite number)

Wrist position and E-E pose

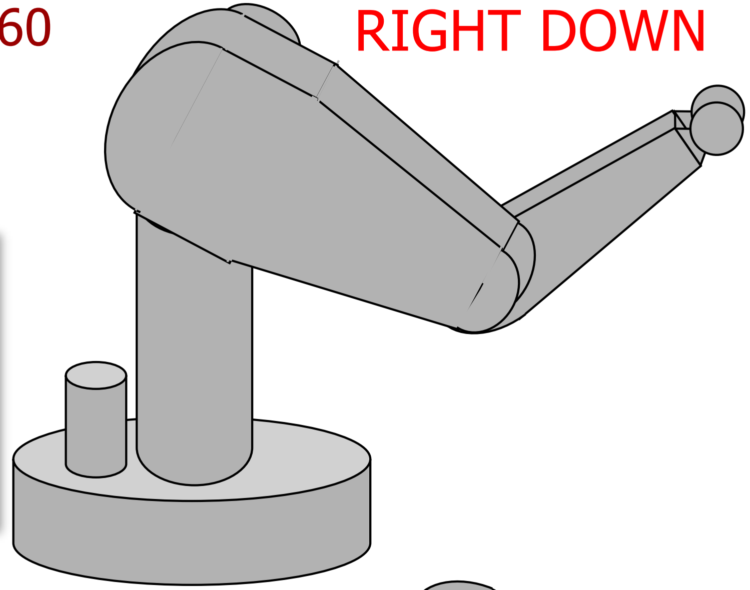
inverse solutions for an articulated 6R robot

Unimation PUMA 560

LEFT DOWN

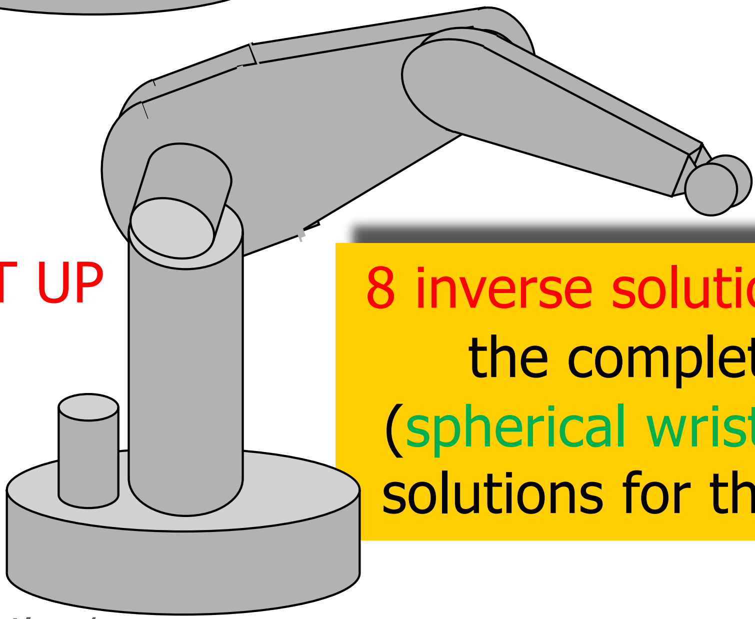


RIGHT DOWN

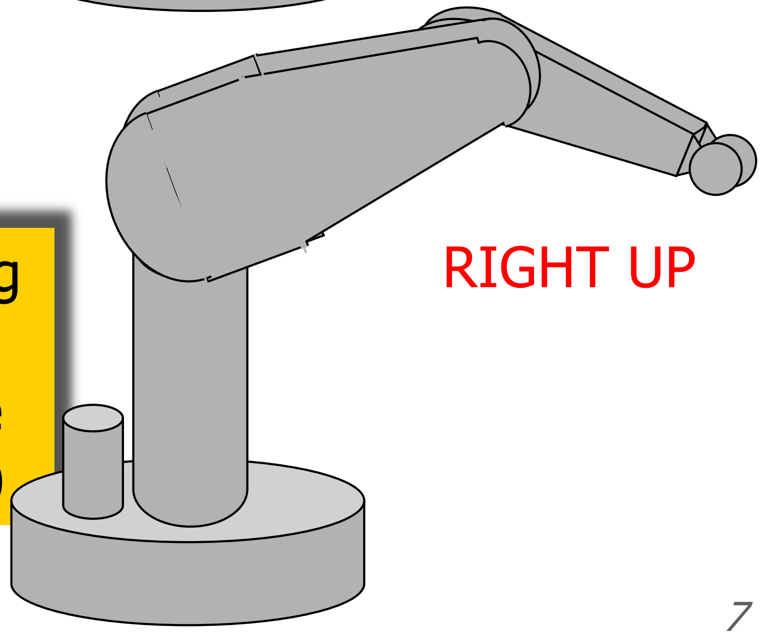


4 inverse solutions
out of singularities
(for the position of
the wrist center only)

LEFT UP



RIGHT UP



8 inverse solutions considering
the complete E-E pose
(spherical wrist: 2 alternative
solutions for the last 3 joints)

Inverse kinematic solutions of UR10

6-dof Universal Robot UR10, with non-spherical wrist

video (slow motion)

desired pose

$$p = \begin{pmatrix} -0.2373 \\ -0.0832 \\ 1.3224 \end{pmatrix} \text{ [m]}$$

$$R = \begin{pmatrix} \sqrt{3}/2 & 0.5 & 0 \\ -0.5 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

home configuration at start

$$q = \begin{pmatrix} 0 & -\pi/2 & 0 & -\pi/2 & 0 & 0 \end{pmatrix}^T$$

[rad]

8 inverse kinematic solutions of UR10

shoulderRight
wristDown
elbowUp

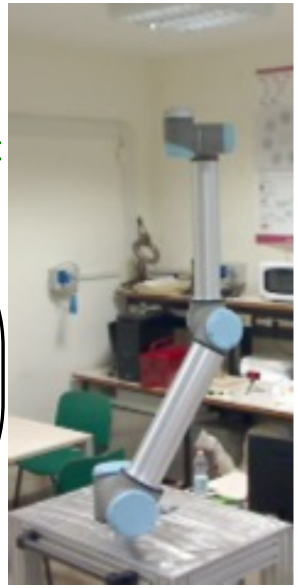
$$q = \begin{pmatrix} 1.0472 \\ -1.2833 \\ -0.7376 \\ -2.6915 \\ -1.5708 \\ 3.1416 \end{pmatrix}$$

shoulderRight
wristDown
elbowDown

$$q = \begin{pmatrix} 1.0472 \\ -1.9941 \\ 0.7376 \\ 2.8273 \\ -1.5708 \\ 3.1416 \end{pmatrix}$$

shoulderRight
wristUp
elbowUp

$$q = \begin{pmatrix} 1.0472 \\ -1.5894 \\ -0.5236 \\ 0.5422 \\ 1.5708 \\ 0 \end{pmatrix}$$

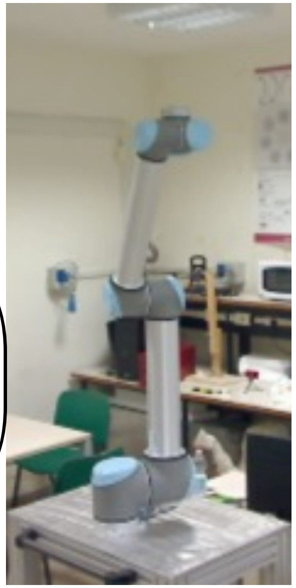


shoulderRight
wristUp
elbowDown

$$q = \begin{pmatrix} 1.0472 \\ -2.0944 \\ 0.5236 \\ 0 \\ 1.5708 \\ 0 \end{pmatrix}$$

shoulderLeft
wristDown
elbowDown

$$q = \begin{pmatrix} 2.7686 \\ -1.0472 \\ -0.5236 \\ 3.1416 \\ -1.5708 \\ 1.4202 \end{pmatrix}$$

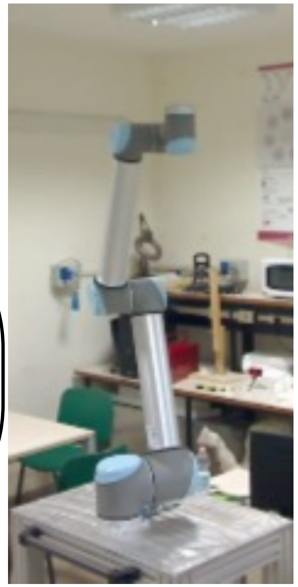


shoulderLeft
wristDown
elbowUp

$$q = \begin{pmatrix} 2.7686 \\ -1.5522 \\ 0.5236 \\ 2.5994 \\ -1.5708 \\ 1.4202 \end{pmatrix}$$

shoulderLeft
wristUp
elbowDown

$$q = \begin{pmatrix} 2.7686 \\ -1.1475 \\ -0.7376 \\ 0.3143 \\ 1.5708 \\ -1.7214 \end{pmatrix}$$



shoulderLeft
wristUp
elbowUp

$$q = \begin{pmatrix} 2.7686 \\ -1.8583 \\ 0.7376 \\ -0.4501 \\ 1.5708 \\ -1.7214 \end{pmatrix}$$

Multiplicity of solutions

few examples

- E-E positioning of planar 2R robot $(m = n = 2)$
  - 2 regular solutions in $\text{int}(WS_1)$
  - 1 solution on ∂WS_1
  - for $l_1 = l_2$: ∞ solutions in WS_2

  } singular solutions
- E-E positioning of elbow-type spatial 3R robot $(m = n = 3)$
  - 4 regular solutions in WS_1 (with singular cases yet to be investigated ...)
- spatial 6R robot arms $(m = n = 6)$
  - $\leq$ 16 distinct solutions, out of singularities: this "upper bound" of solutions was shown to be attained by a particular instance of "orthogonal" robot, i.e., with twist angles $\alpha_i = 0$ or $\pm\pi/2$ $(\forall i)$
  - analysis based on algebraic transformations of robot kinematics
    - transcendental equations are transformed into a single polynomial equation in one variable (number of roots = degree of the polynomial)
    - seek for a transformed polynomial equation of the least possible degree

A 6R robot with 16 IK solutions

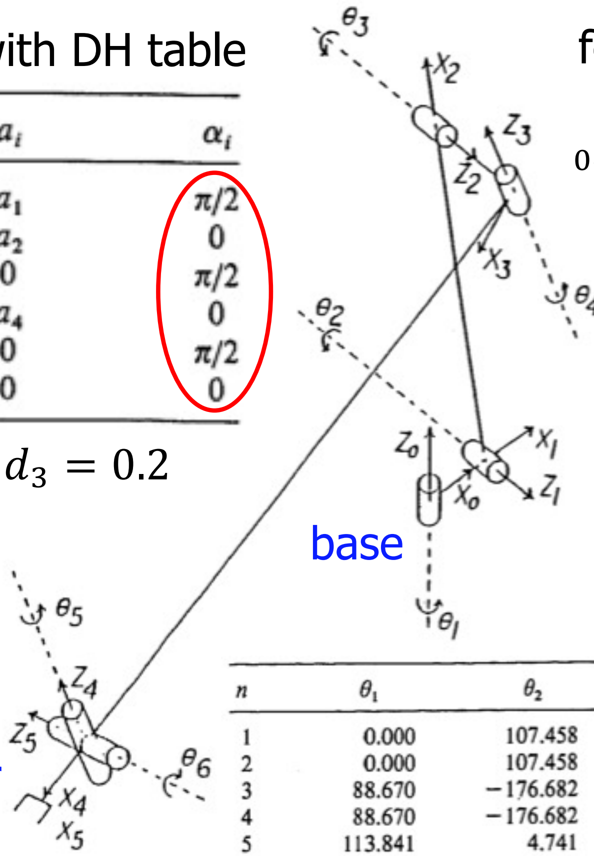
all distinct and non-singular

an orthogonal manipulator with DH table

i	d_i	θ_i	a_i	α_i
1	0	θ_1	a_1	$\pi/2$
2	0	θ_2	a_2	0
3	d_3	θ_3	0	$\pi/2$
4	0	θ_4	a_4	0
5	0	θ_5	0	$\pi/2$
6	0	θ_6	0	0

$$a_1 = 0.3, a_2 = 1, a_4 = 1.5, d_3 = 0.2$$

with non-spherical wrist



end-effector

for the desired end-effector pose

$$^0T_6 = \begin{bmatrix} -0.760117 & -0.641689 & 0.102262 & -1.140175 \\ 0.133333 & 0 & 0.991071 & 0 \\ -0.635959 & 0.766965 & 0.085558 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

there are 16 real solutions of the inverse kinematics!

all non-singular

n	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	det (J^3)
1	0.000	107.458	112.460	−7.662	0.000	0.000	1.310
2	0.000	107.458	−67.540	−172.338	180.000	180.000	1.310
3	88.670	−176.682	−178.394	−63.284	157.829	139.944	−0.800
4	88.670	−176.682	1.606	−116.716	22.171	−40.056	−0.800
5	113.841	4.741	−179.093	−55.954	−63.659	−42.463	−1.256
6	113.841	4.741	0.907	−124.046	−116.341	137.537	−1.256
7	168.703	−104.205	146.556	−16.393	−170.903	98.216	0.803
8	168.703	−104.205	−33.444	−163.607	−9.097	−81.784	0.803
9	180.000	107.458	−147.375	−7.662	−164.675	180.000	0.732
10	180.000	107.458	32.625	−172.338	−15.325	0.000	0.732
11	−120.748	173.066	−178.472	31.328	−146.087	142.605	−0.717
12	−120.748	173.066	1.528	148.672	−33.913	−37.395	−0.717
13	−96.292	−5.766	−179.142	38.477	51.922	−39.631	−1.441
14	−96.292	−5.766	0.858	141.523	128.078	140.369	−1.441
15	−11.768	−105.495	−114.490	1.243	6.408	−79.398	1.318
16	−11.768	−105.495	65.510	178.757	173.592	100.602	1.318

Manseur and Doty:
International Journal of Robotics Research, 1989

solutions found using a fast numerical inversion algorithm …

Algebraic transformations

whiteboard …

start with some trigonometric equation in the joint angle θ to be solved …

$$a\ \sin\theta + b\cos\theta = c \qquad (*)$$

introduce the algebraic transformation (… and the related inverse formulas)

$$u = \tan(\theta/2)$$

$$\Rightarrow \qquad \sin\theta = \frac{2u}{1+u^2} \qquad \cos\theta = \frac{1-u^2}{1+u^2} \qquad (\Rightarrow \ \sin^2\theta + \cos^2\theta = 1)$$

$$\tan\theta = \tan 2(\theta/2) = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{2u}{1-u^2}$$ (using the duplication formula)

substituting in $(*)$

$$a\,\frac{2u}{1+u^2} + b\,\frac{1-u^2}{1+u^2} = c \qquad \Rightarrow$$

polynomial equation of second degree in u

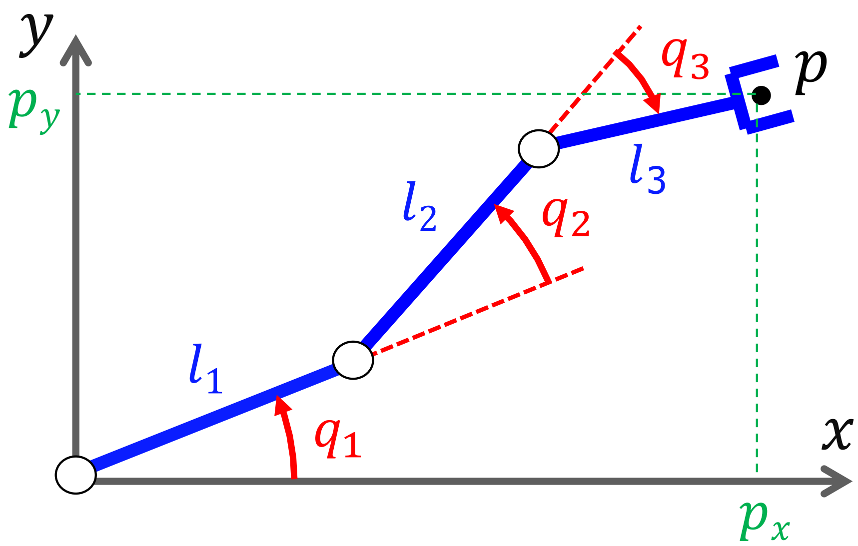
$$(b+c)\,u^2 - 2a\,u - (b-c) = 0$$

$$\Rightarrow \qquad u_{1,2} = \frac{a \pm \sqrt{a^2+b^2-c^2}}{b+c} \qquad \Rightarrow \qquad \theta_{1,2} = 2\arctan\left(u_{1,2}\right)$$

only if argument is real, else no solution

A planar 3R arm

workspace and number/type of inverse solutions



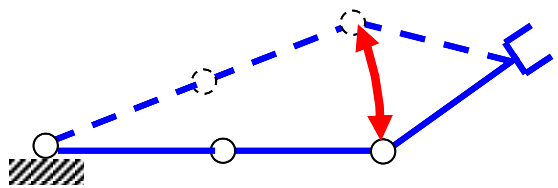
$l_1 = l_2 = l_3 = l \qquad n = 3, m = 2$

$WS_1 = \{p \in \mathbb{R}^2 : \|p\| \leq 3l\} \subset \mathbb{R}^2$

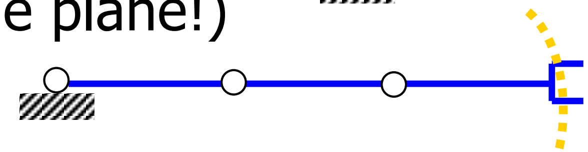
$WS_2 = \{p \in \mathbb{R}^2 : \|p\| \leq l\} \subset \mathbb{R}^2$

any planar orientation is feasible in WS_2

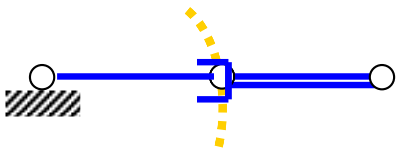
1. in $\mathrm{int}(WS_1)$, except for case 3: ∞^1 regular solutions, at which the E-E can take a continuum of ∞ orientations (but not all orientations in the plane!)

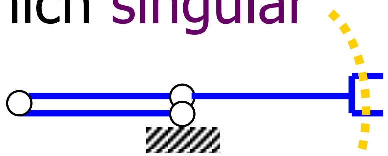


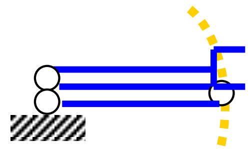
2. if $\|p\| = 3l$: only 1 solution, singular



3. if $\|p\| = l$: ∞^1 solutions, 3 of which singular







4. if $\|p\| < l$: ∞^1 regular solutions (that are never singular)

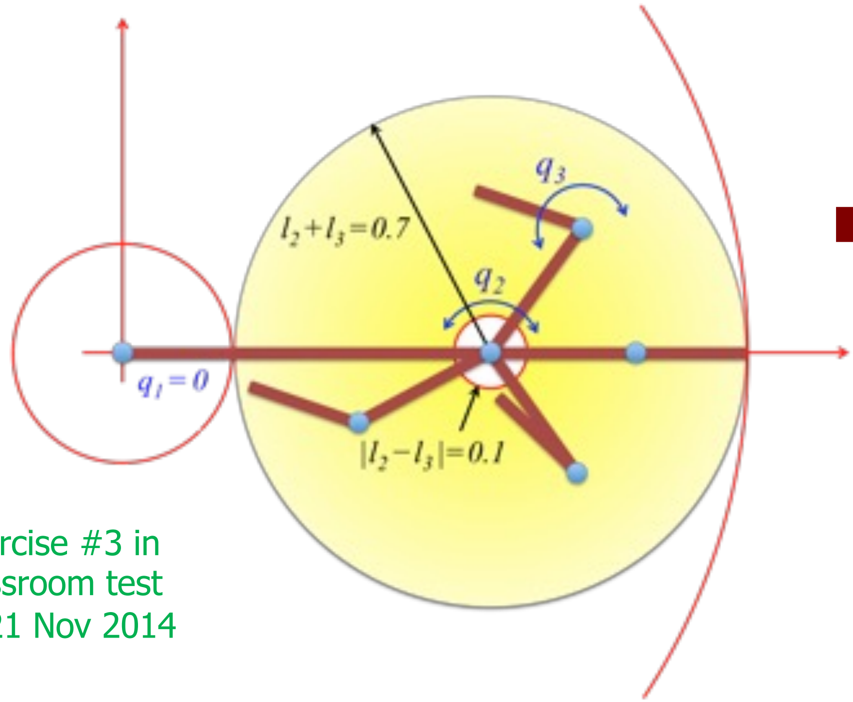
Workspace of a planar 3R arm

with generic link lengths

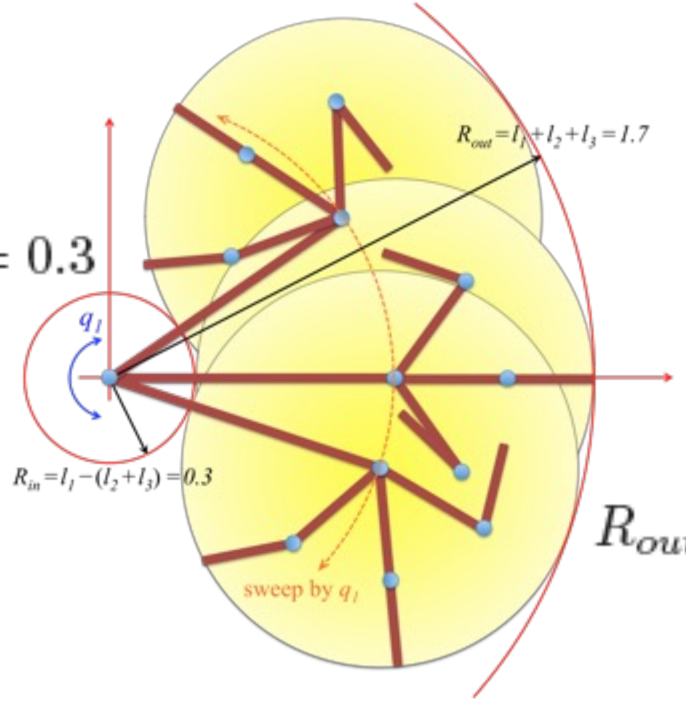
$$l_{max} = \max\{l_i,\ i = 1, 2, 3\} \qquad R_{out} = l_{min} + l_{med} + l_{max} = l_1 + l_2 + l_3$$

$$l_{min} = \min\{l_i,\ i = 1, 2, 3\} \quad \Rightarrow \quad R_{in} = \max\{0, l_{max} - (l_{med} + l_{min})\}$$

a) $l_1 = 1,\ l_2 = 0.4,\ l_3 = 0.3$ [m] $\Rightarrow$ $l_{max} = l_1 = 1,\ l_{med} = l_2 = 0.4,\ l_{min} = l_3 = 0.3$



$\Rightarrow R_{in} = 0.3$



$R_{out} = 1.7$

Exercise #3 in classroom test of 21 Nov 2014

b) $l_1 = 0.5,\ l_2 = 0.7,\ l_3 = 0.5$ [m] $\Rightarrow$ $l_{max} = l_2 = 0.7,\ l_{med} = l_{min} = l_1 (\text{or } l_3) = 0.5$

$\Rightarrow R_{in} = 0,\ R_{out} = 1.7$

Multiplicity of solutions

summary of the general cases

- if $m = n$
  - $\nexists$ solutions
  - a finite number of solutions (regular/generic case)
  - "degenerate" solutions: infinite or finite set, but anyway different in number from the generic case (singularity)
- if $m < n$ (robot is kinematically redundant for the task)
  - $\nexists$ solutions
  - ∞^{n-m} solutions (regular/generic case)
  - a finite or infinite number of singular solutions
- use of the term singularity will become clearer when dealing with differential kinematics
  - instantaneous velocity mapping from joint to task velocity
  - lack of full rank of the associated $m \times n$ Jacobian matrix $J(q)$

Dexter 8R robot arm

- $m = 6$ (position and orientation of E-E)
- $n = 8$ (all revolute joints)
- ∞^2 inverse kinematic solutions (redundancy degree $= n - m = 2$)

video

exploring inverse kinematic solutions by a robot self-motion

Solution methods

ANALYTICAL solution
(in closed form)

$\rightleftharpoons$

NUMERICAL solution
(in iterative form)

- preferred, if it can be found*
- use ad-hoc geometric inspection
- algebraic methods (solution of polynomial equations)
- systematic ways for generating a reduced set of equations to be solved

* sufficient conditions for 6-dof arms

- 3 consecutive rotational joint axes are incident (e.g., spherical wrist), or
- 3 consecutive rotational joint axes are parallel

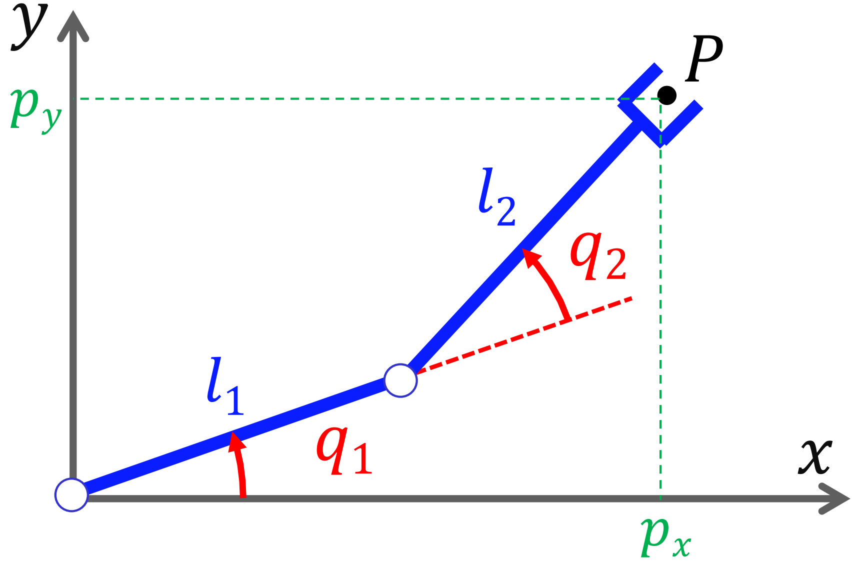
D. Pieper, PhD thesis, Stanford University, 1968

- certainly needed if $n > m$ (redundant case) or at/close to singularities
- slower, but easier to be set up
- in its basic form, it uses the (analytical) Jacobian matrix of the direct kinematics map

$$J_r(q) = \frac{\partial f_r(q)}{\partial q}$$

- Newton method, Gradient method, and so on…

Inverse kinematics of planar 2R arm



direct kinematics

$$p_x = l_1 c_1 + l_2 c_{12}$$

$$\underbrace{p_y}_{\text{data}} = l_1 s_1 + l_2 s_{12}$$

q_1, q_2 unknowns

"squaring and summing" the equations of the direct kinematics

$$p_x^2 + p_y^2 - (l_1^2 + l_2^2) = 2 l_1 l_2 (c_1 c_{12} + s_1 s_{12}) = 2 l_1 l_2 c_2$$

and from this

$$c_2 = \left(p_x^2 + p_y^2 - (l_1^2 + l_2^2)\right)/2 l_1 l_2, \quad s_2 = \pm\sqrt{1 - c_2^2}$$

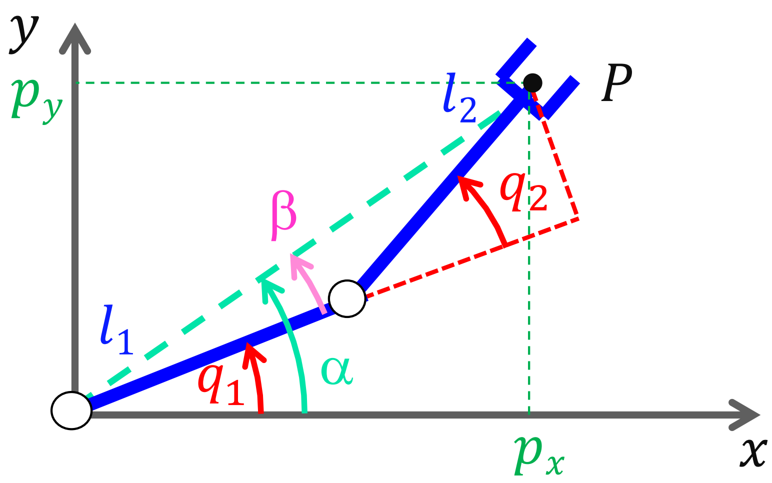
$$\boxed{q_2 = \text{atan2}\{s_2, c_2\}}$$

must be in $[-1,1]$ (else, point P is outside robot workspace!)

2 solutions

in analytical form

Inverse kinematics of 2R arm (cont'd)



by geometric inspection

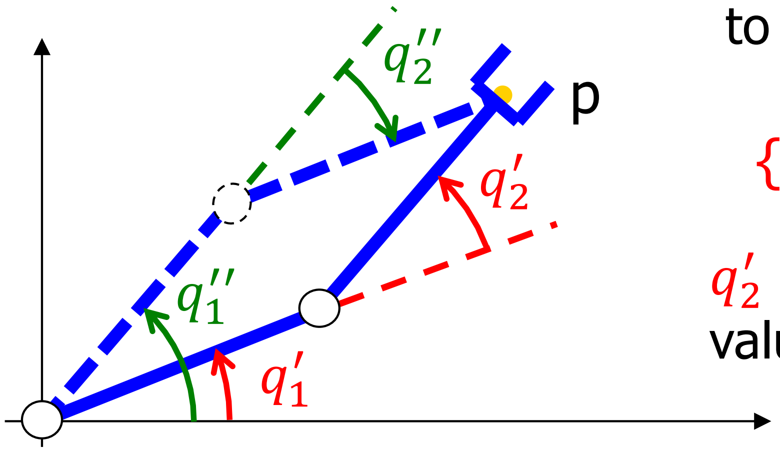
$$q_1 = \alpha - \beta$$

2 solutions
(one for each value of s_2)

$$q_1 = \text{atan2}\{p_y, p_x\} - \text{atan2}\{l_2 s_2, l_1 + l_2 c_2\}$$

note: difference of atan2's needs to be re-expressed in $(-\pi\ , \pi]$!

$\{q_1, q_2\}_{\text{UP/LEFT}}$



$\{q_1, q_2\}_{\text{DOWN/RIGHT}}$

q_2' and q_2'' have same absolute value, but opposite signs

Algebraic solution for q_1

another solution method…

$$p_x = l_1 c_1 + l_2 c_{12} = l_1 c_1 + l_2 (c_1 c_2 - s_1 s_2)$$

$$p_y = l_1 s_1 + l_2 s_{12} = l_1 s_1 + l_2 (s_1 c_2 + c_1 s_2)$$

linear in s_1 and c_1

$$\underbrace{\begin{bmatrix} l_1 + l_2 c_2 & -l_2 s_2 \\ l_2 s_2 & l_1 + l_2 c_2 \end{bmatrix}} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$\det = l_1^2 + l_2^2 + 2 l_1 l_2 c_2 > 0$$

except if $l_1 = l_2$ and $c_2 = -1$
being then q_1 undefined
(singular case: ∞^1 solutions)

$$\boxed{q_1 =} \ \operatorname{atan2}\{s_1, c_1\}$$

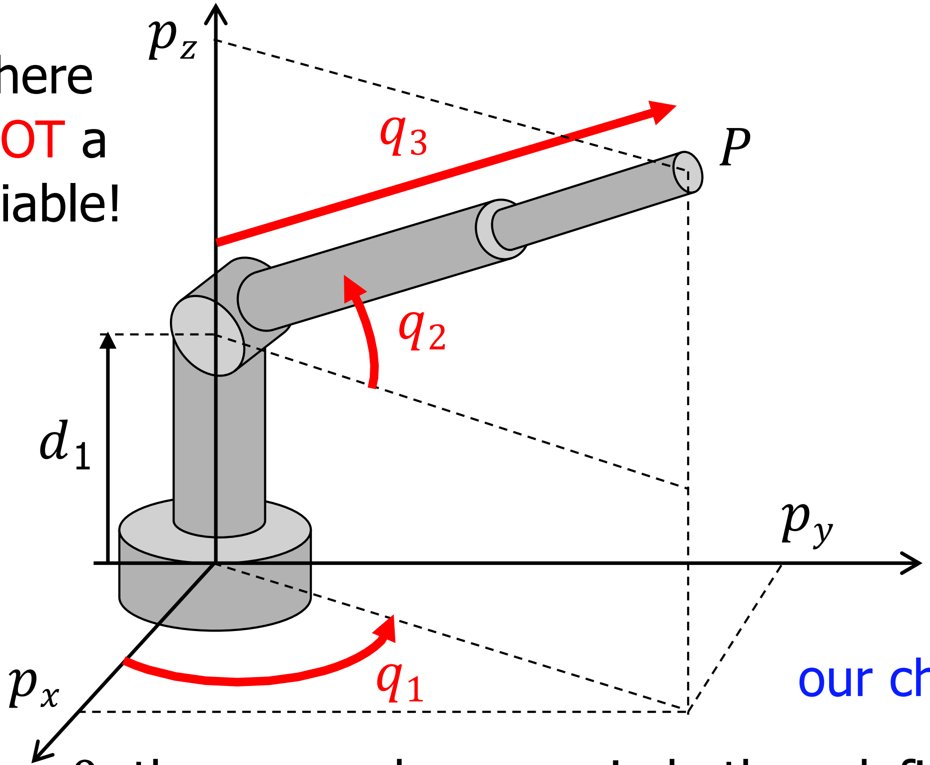
$$= \boxed{\operatorname{atan2}\{(p_y(l_1 + l_2 c_2) - p_x l_2 s_2)/\det, (p_x(l_1 + l_2 c_2) + p_y l_2 s_2)/\det\}}$$

notes: a) this method provides directly the result in $(-\pi, \pi]$

b) when evaluating $\operatorname{atan2}$, $\det > 0$ can be in fact eliminated from the expressions of s_1 and c_1 (not changing the result)

Inverse kinematics of polar (RRP) arm

note: here q_2 is NOT a DH variable!



direct kinematics

$$p_x = q_3 c_2 c_1$$
$$p_y = q_3 c_2 s_1$$
$$p_z = d_1 + q_3 s_2$$

$$p_x^2 + p_y^2 + (p_z - d_1)^2 = q_3^2$$

$$q_3 = +\sqrt{p_x^2 + p_y^2 + (p_z - d_1)^2}$$

our choice: take here only the positive value...

if $q_3 = 0$, then q_1 and q_2 remain both undefined (stop); else

$$q_2 = \text{atan2}\left\{(p_z - d_1)/q_3 \, , \pm\sqrt{p_x^2 + p_y^2}/q_3\right\}$$

(if we stop, it is a singular case: ∞^2 or ∞^1 solutions)

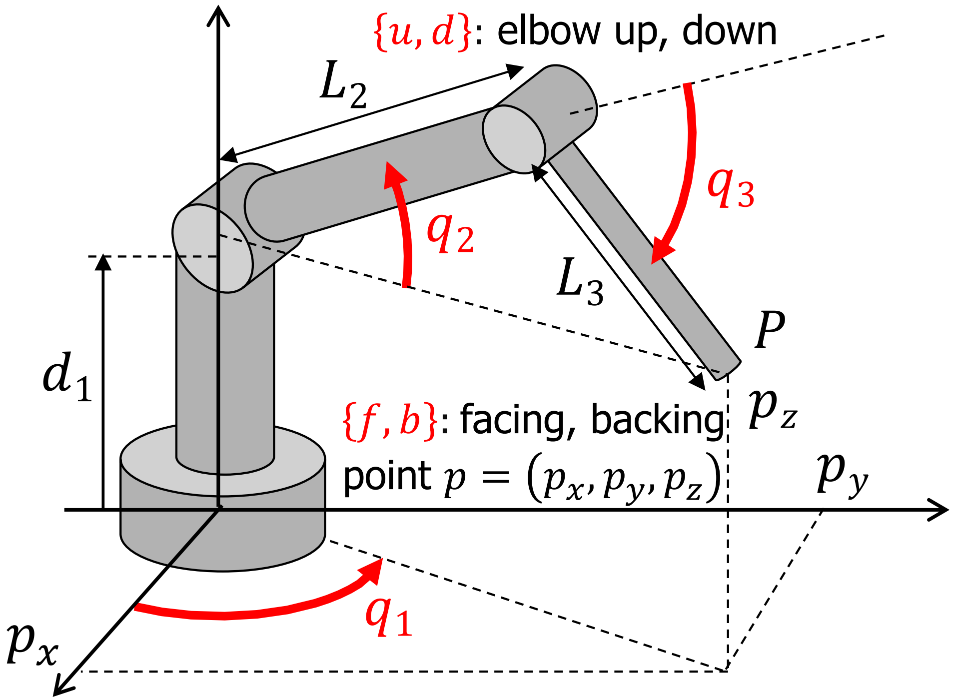
if $p_x^2 + p_y^2 = 0$, then q_1 remains undefined (stop); else

$$q_1 = \text{atan2}\left\{p_y/c_2 \, , p_x/c_2\right\}$$

(2 regular solutions $\{q_1, q_2, q_3\}$)

eliminating $q_3 > 0$ from both arguments

Inverse kinematics of 3R elbow-type arm



symmetric structure without offsets
e.g., first 3 joints of Mitsubishi PA10 robot

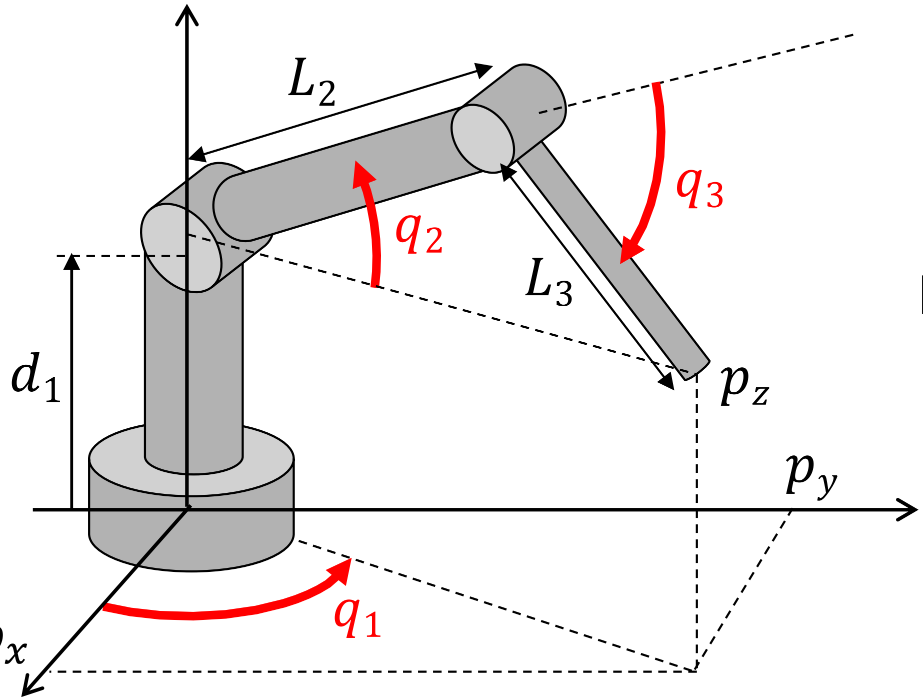
WS_1 = {spherical shell centered at $(0,0,d_1)$,
with outer radius $R_{out} = L_2 + L_3$
and inner radius $R_{in} = |L_2 - L_3|$}

4 regular inverse kinematics solutions in WS_1

more details (e.g., full handling of singular cases) can be found in the solution of Exercise #1 in written exam of 11 Apr 2017

Inverse kinematics of 3R elbow-type arm

step 1



direct kinematics

$$p_x = c_1(L_2c_2 + L_3c_{23})$$
$$p_y = s_1(L_2c_2 + L_3c_{23})$$
$$p_z = d_1 + L_2s_2 + L_3s_{23}$$

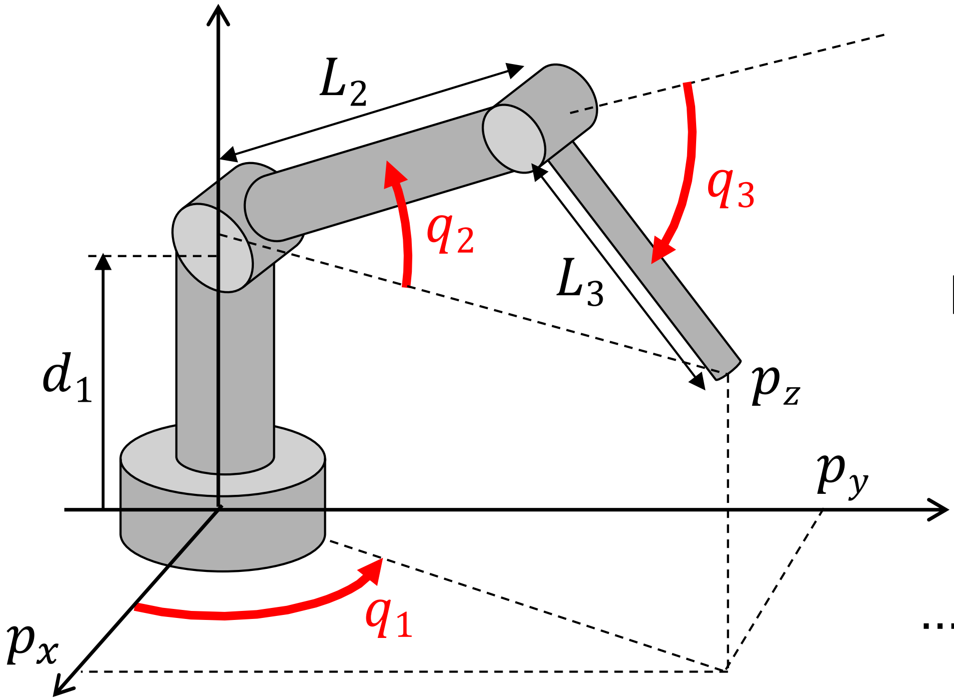
$$p_x^2 + p_y^2 + (p_z - d_1)^2 = c_1^2(L_2c_2 + L_3c_{23})^2 + s_1^2(L_2c_2 + L_3c_{23})^2 + (L_2s_2 + L_3s_{23})^2$$
$$= \cdots = L_2^2 + L_3^2 + 2L_2L_3(c_2c_{23} + s_2s_{23}) = L_2^2 + L_3^2 + 2L_2L_3c_3$$

$c_3 = \left(p_x^2 + p_y^2 + (p_z - d_1)^2 - L_2^2 - L_3^2\right)/2L_2L_3 \in [-1, +1]$ (else, p is out of workspace!)

$\pm s_3 = \pm\sqrt{1 - c_3^2}$ ⇒ two solutions

$$\begin{cases} q_3^{\{+\}} = \text{atan2}\{s_3, c_3\} \\ q_3^{\{-\}} = \text{atan2}\{-s_3, c_3\} = -q_3^{\{+\}} \end{cases}$$

Inverse kinematics of 3R elbow-type arm
step 2



direct kinematics

$$p_x = c_1(L_2c_2 + L_3c_{23})$$
$$p_y = s_1(L_2c_2 + L_3c_{23})$$
$$p_z = d_1 + L_2s_2 + L_3s_{23}$$

... being $p_x^2 + p_y^2 = (L_2c_2 + L_3c_{23})^2 > 0$

only when $p_x^2 + p_y^2 > 0$ ...

(else q_1 is undefined —infinite solutions!)

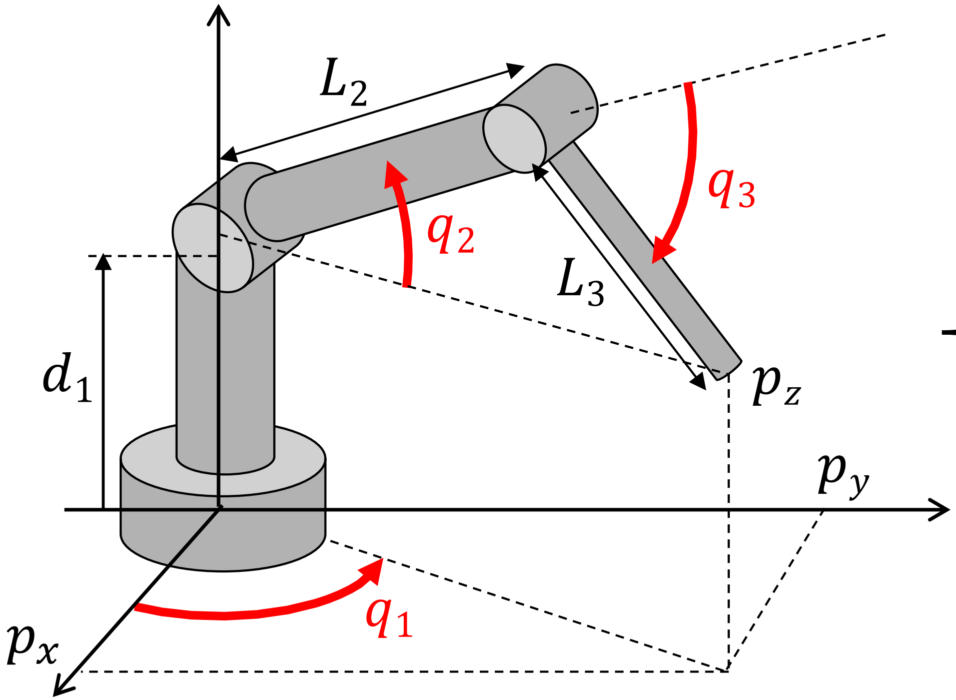
$$\begin{cases} c_1 = p_x / \pm\sqrt{p_x^2 + p_y^2} \\ s_1 = p_y / \pm\sqrt{p_x^2 + p_y^2} \end{cases}$$

again, two solutions

$$\begin{cases} q_1^{\{+\}} = \text{atan2}\{p_y, p_x\} \\ q_1^{\{-\}} = \text{atan2}\{-p_y, -p_x\} \end{cases}$$

Inverse kinematics of 3R elbow-type arm

step 3



combine first the two equations of direct kinematics and rearrange the last one

$$\begin{cases} c_1 p_x + s_1 p_y = L_2 c_2 + L_3 c_{23} \\ \qquad = (L_2 + L_3 c_3) c_2 - L_3 s_3 s_2 \\ p_z - d_1 = L_2 s_2 + L_3 s_{23} \\ \qquad = L_3 s_3 c_2 + (L_2 + L_3 c_3) s_2 \end{cases}$$

define and solve a linear system $Ax = b$ in the algebraic unknowns $x = (c_2, s_2)$

$$\begin{bmatrix} L_2 + L_3 c_3 & -L_3 s_3^{\{+,-\}} \\ L_3 s_3^{\{+,-\}} & L_2 + L_3 c_3 \end{bmatrix} \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} c_1^{\{+,-\}} p_x + s_1^{\{+,-\}} p_y \\ p_z - d_1 \end{bmatrix}$$

coefficient matrix A known vector b

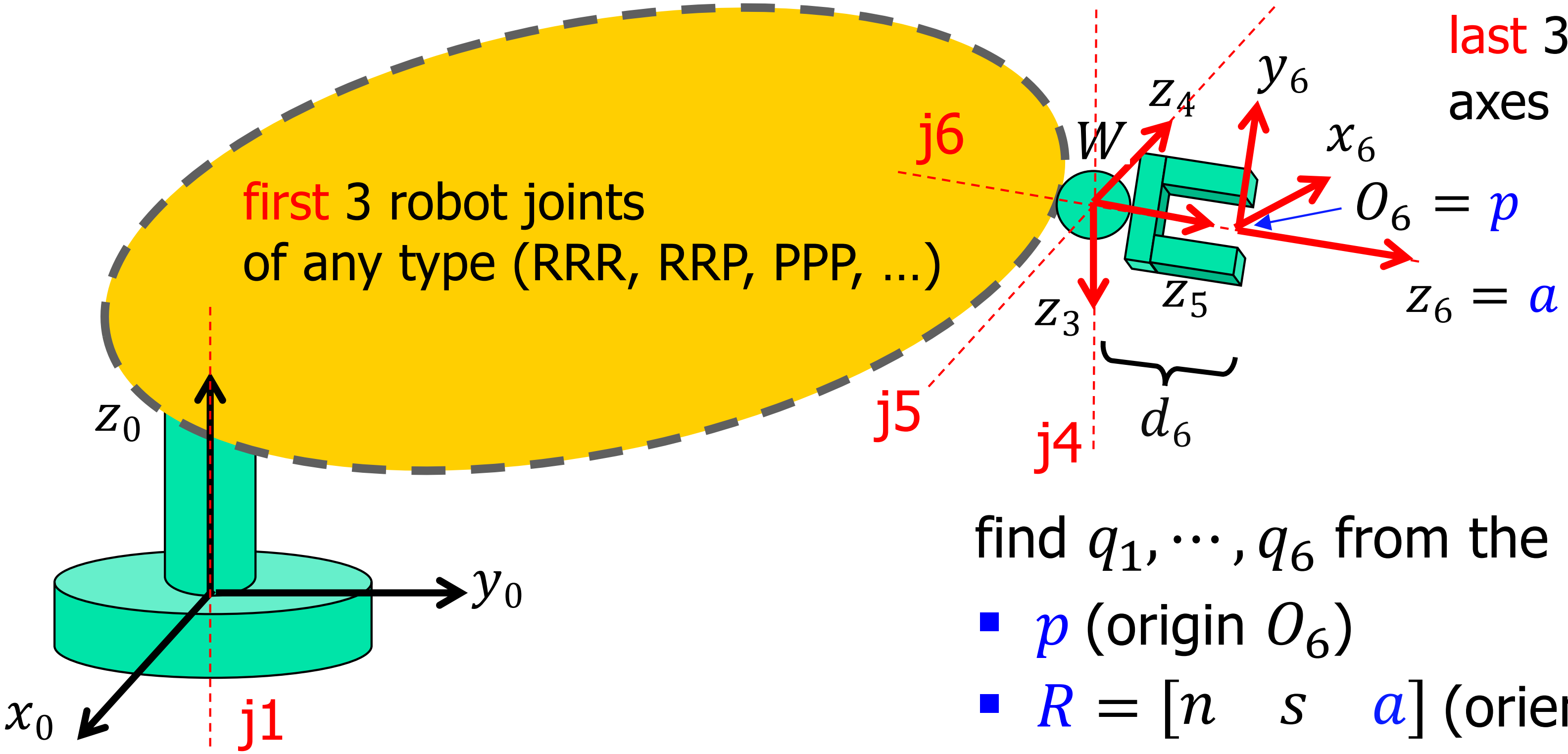
4 regular solutions for q_2, depending on the combinations of $\{+,-\}$ from q_1 and q_3

$$q_2^{\{\{f,b\},\{u,d\}\}} = \text{atan2}\left\{ s_2^{\{\{f,b\},\{u,d\}\}}, c_2^{\{\{f,b\},\{u,d\}\}} \right\}$$

provided $\det A = p_x^2 + p_y^2 + (p_z - d_1)^2 \neq 0$

(else q_2 is undefined —infinite solutions!)

Inverse kinematics for robots with spherical wrist



find $q_1, \cdots, q_6$ from the input data

- p (origin O_6)
- $R = [n \quad s \quad a]$ (orientation of RF_6)

1. $W = p - d_6 a \Rightarrow q_1, q_2, q_3$ (inverse "position" kinematics for main axes)
2. $R = {}^0R_3(q_1,q_2,q_3)\ \underbrace{{}^3R_6(q_4,q_5,q_6)} \Rightarrow {}^3R_6(q_4,q_5,q_6) = {}^0R_3^T R \Rightarrow q_4, q_5, q_6$

given

known, after step 1

Euler ZYZ or ZXZ rotation matrix with q_4,q_5,q_6 $(\theta_4,\theta_5,\theta_6)$ → two regular solutions

(inverse "orientation" kinematics for the wrist)

6R robot Unimation PUMA 600

spherical wrist

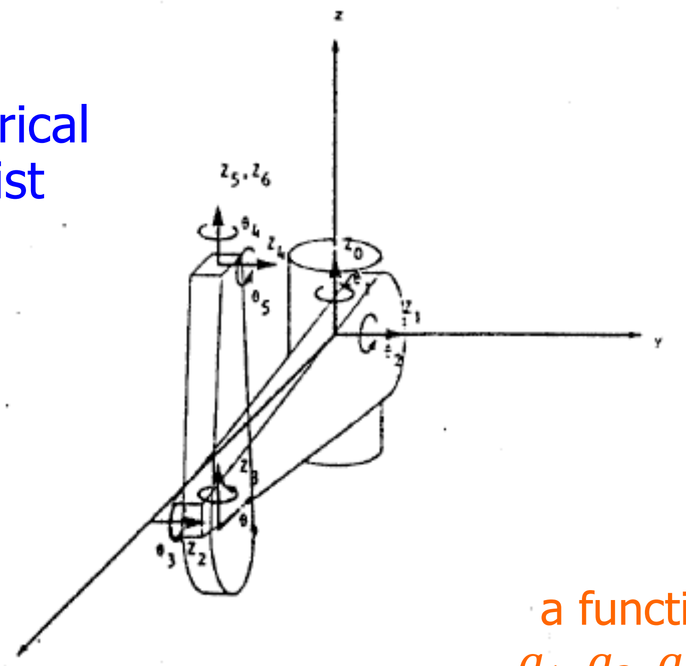


TABLE I
LINK PARAMETERS FOR PUMA ARM

Joint	$\alpha°$	$\theta°$	d	a	Range
1	$-90°$	θ_1	0	0	θ_1: +/− 160°
2	0	θ_2	0	a_2	θ_2: +45 → −225°
3	90°	θ_3	d_3	a_3	θ_3: 225° → −45°
4	$-90°$	θ_4	d_4	0	θ_4: +/− 170°
5	90°	θ_5	0	0	θ_5: +/− 135°
6	0	θ_6	0	0	θ_6: +/− 170°
$a_2 = 17.000$	$a_3 = 0.75$				
$d_3 = 4.937$	$d_4 = 17.000$				

here $d_6 = 0$,
so that $O_6 = W$ directly

a function of
q_1, q_2, q_3 only!

$$\left.\begin{aligned}
n_x &= C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] \\
&\quad - S_1[S_4C_5C_6 + C_4S_6] \\
n_y &= S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] \\
&\quad + C_1[S_4C_5C_6 + C_4S_6] \\
n_z &= -S_{23}(C_4C_5C_6 - S_4S_6) - C_{23}S_5C_6
\end{aligned}\right\} \; n = {}^0x_6(q)$$

$$\left.\begin{aligned}
o_x &= C_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] \\
&\quad - S_1[-S_4C_5S_6 + C_4C_6] \\
o_y &= S_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] \\
&\quad + C_1[-S_4C_5S_6 + C_4C_6] \\
o_z &= S_{23}(C_4C_5S_6 + S_4C_6) + C_{23}S_5S_6
\end{aligned}\right\} \; s = {}^0y_6(q)$$

$$\left.\begin{aligned}
a_x &= C_1(C_{23}C_4S_5 + S_{23}C_5) - S_1S_4S_5 \\
a_y &= S_1(C_{23}C_4S_5 + S_{23}C_5) + C_1S_4S_5 \\
a_z &= -S_{23}C_4S_5 + C_{23}C_5
\end{aligned}\right\} \; a = 0z_6(q)$$

$$\left.\boxed{\begin{aligned}
p_x &= C_1(d_4S_{23} + a_3C_{23} + a_2C_2) - S_1d_3 \\
p_y &= S_1(d_4S_{23} + a_3C_{23} + a_2C_2) + C_1d_3 \\
p_z &= -(-d_4C_{23} + a_3S_{23} + a_2S_2).
\end{aligned}}\right\} \; p = O_6(q)$$

8 different (regular) inverse solutions
that can be found in closed form

Finding nice kinematic relations

whiteboard …

- the most complex inverse kinematics that can be solved in principle in closed form (i.e., analytically) is that of a 6R serial manipulator, with arbitrary DH table
  - ways to systematically generate equations from the direct kinematics that could be easier to solve ⇒ some scalar equations may contain perhaps a single unknown variable!

$$^0T_6 = {}^0A_1(\theta_1)\, {}^1A_2(\theta_2) \cdots {}^5A_6(\theta_6) = U_0$$

method used for the Unimation PUMA 600 in (*)

$$^0A_1^{-1}\, {}^0T_6 = U_1 \; (= {}^1A_2 \cdots {}^5A_6)$$
$$^1A_2^{-1}\, {}^0A_1^{-1}\, {}^0T_6 = U_2 \left(= {}^2A_3 \cdots {}^5A_6\right)$$
$$\cdots$$
$$^4A_5^{-1} \cdots {}^1A_2^{-1}\, {}^0A_1^{-1}\, {}^0T_6 = U_5 \; (= {}^5A_6)$$

or also …

$$^0T_6\, {}^5A_6^{-1} = V_5 \; (= {}^0A_1 \cdots {}^4A_5)$$
$$^0T_6\, {}^5A_6^{-1}\, {}^4A_5^{-1} = V_4 \left(= {}^0A_1 \cdots {}^3A_4\right)$$
$$\cdots$$
$$^0T_6\, {}^5A_6^{-1}\, {}^4A_5^{-1} \cdots {}^1A_2^{-1} = V_1 \; (= {}^0A_1)$$

(*) Paul, Shimano, and Mayer: IEEE Transactions on Systems, Man, and Cybernetics, 1981

  - generating from the direct kinematics a reduced set of equations to be solved (setting w.l.o.g. $d_1 = d_6 = 0$) ⇒ 4 compact scalar equations in the 4 unknowns $\theta_2, \dots, \theta_5$

$$^0T_6 = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0A_6(\theta) \longrightarrow$$

$$a_z = a^T(\theta)\, z \qquad \|p\|^2 = p^T(\theta)\, p(\theta)$$
$$p_z = p^T(\theta)\, z \qquad p^T a = p^T(\theta)\, a(\theta)$$

solved analytically or numerically …

$$z = [0 \quad 0 \quad 1]^T$$

… then solve easily for the remaining θ_1 and θ_6

Manseur and Doty: International Journal of Robotics Research, 1988

Numerical solution of inverse kinematics problems

- use when a closed-form solution q to $r_d = f_r(q)$ does not exist or is "too hard" to be found
- all methods are iterative and need the matrix $J_r(q) = \dfrac{\partial f_r(q)}{\partial q}$ (analytical Jacobian)
- Newton method (here only for $m = n$, at the kth iteration)
  - $r_d = f_r(q) = f_r(q^k) + J_r(q^k)(q - q^k) + o(\|q - q^k\|)$ ← neglected in Taylor expansion

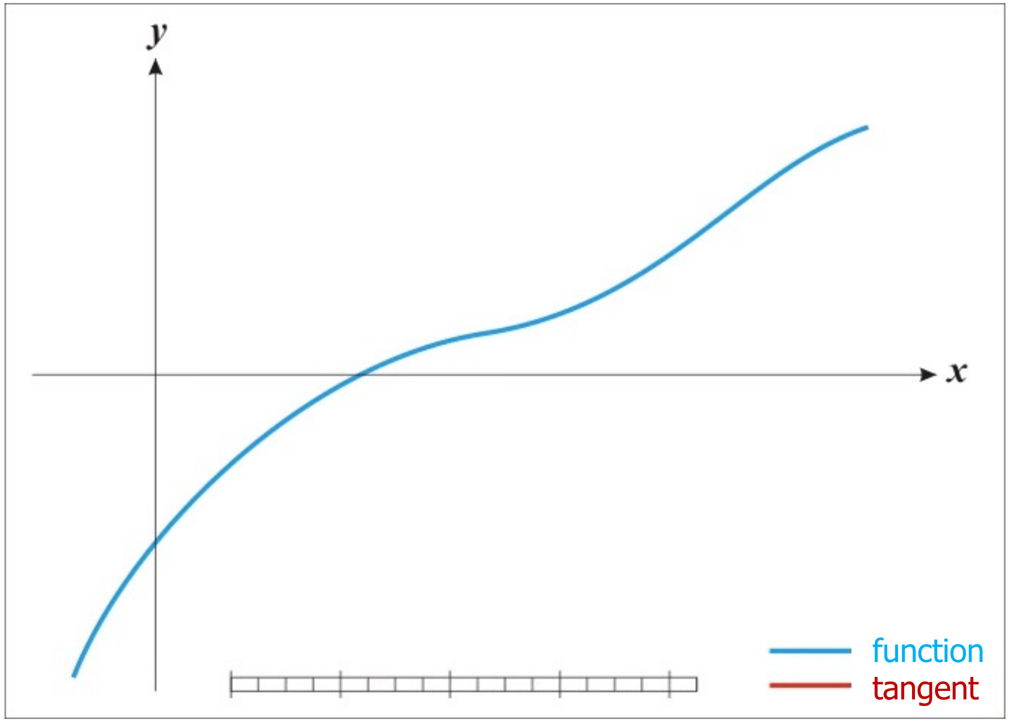
$$\boxed{q^{k+1} = q^k + J_r^{-1}(q^k)\left[r_d - f_r(q^k)\right]}$$

  - convergence for q^0 (initial guess) close enough to some $q^*: f_r(q^*) = r_d$
  - problems near singularities of the Jacobian matrix $J_r(q)$
  - in case of robot redundancy ($m < n$), use the pseudoinverse $J_r^{\#}(q)$
  - has quadratic convergence rate when near to a solution (fast!)

Operation of Newton method

- in the scalar case, also known as "method of the tangent"
- for a differentiable function $f(x)$, find a root x^* of $f(x^*) = 0$ by iterating as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



an approximating sequence

$$\{x_1, x_2, x_3, x_4, x_5, \cdots\} \longrightarrow x^*$$

animation from
http://en.wikipedia.org/wiki/File:NewtonIteration_Ani.gif

Numerical solution of inverse kinematics problems (cont'd)

- Gradient method (max descent)
  - minimize the error function

$$H(q) = \frac{1}{2}\|r_d - f_r(q)\|^2 = \frac{1}{2}(r_d - f_r(q))^T(r_d - f_r(q))$$

$$q^{k+1} = q^k - \alpha \, \nabla_q H(q^k)$$

from

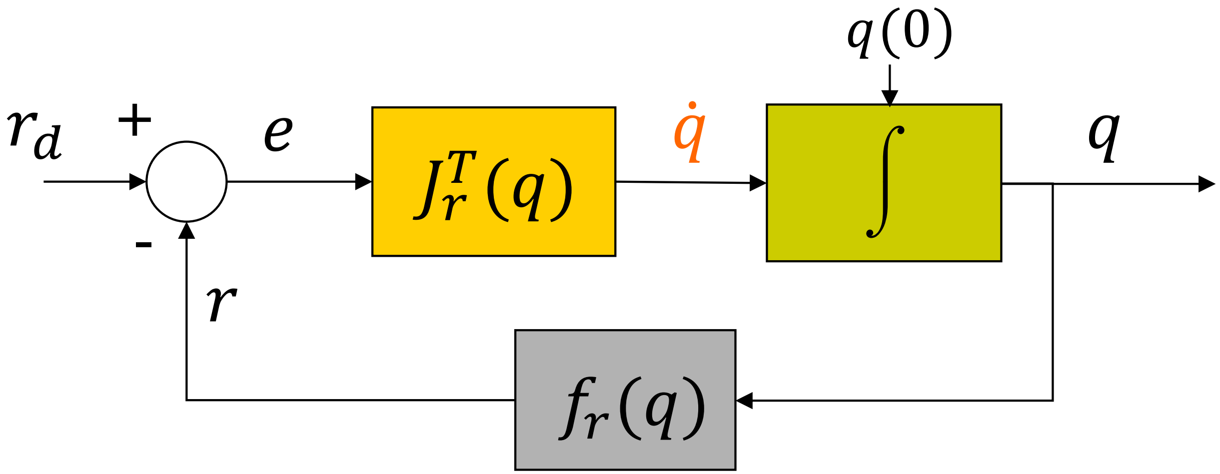
$$\nabla_q H(q) = (\partial H(q)/\partial q)^T = -\left(\left(r_d - f_r(q)\right)^T (\partial f_r(q)/\partial q)\right)^T = -J_r^T(q)(r_d - f_r(q))$$

we get

$$\boxed{q^{k+1} = q^k + \alpha \, J_r^T(q^k)\left(r_d - f_r(q^k)\right)}$$

  - the scalar step size $\alpha > 0$ should be chosen so as to guarantee a decrease of the error function at each iteration: too large values for α may lead the method to "miss" the minimum
  - when the step size is too small, convergence is extremely slow

Revisited as a feedback scheme



r_d = cost

$(\alpha = 1)$

$e = r_d - f_r(q) \rightarrow 0 \quad \Leftrightarrow$ closed-loop equilibrium $e = 0$ is asymptotically stable

$V = \frac{1}{2} e^T e \geq 0$ is a Lyapunov candidate function

$$\dot{V} = e^T \dot{e} = e^T \frac{d}{dt}(r_d - f_r(q)) = -e^T J_r(q) \dot{q} = -e^T J_r(q) J_r^T(q) e \leq 0$$

$\dot{V} = 0 \quad \Leftrightarrow \quad e \in \mathcal{N}(J_r^T(q))$ in particular, $\boxed{e = 0}$

null space

asymptotic stability

Properties of Gradient method

- computationally simpler: use the Jacobian transpose, rather than its (pseudo)inverse
- same use also for robots that are redundant ($n > m$) for the task
- may not converge to a solution, but it never diverges
- the discrete-time evolution of the continuous scheme

$$q^{k+1} = q^k + \Delta T\, J_r^T(q^k)\big(r_d - f_r(q^k)\big), \qquad \alpha = \Delta T$$

  is equivalent to an iteration of the Gradient method
- the scheme can be accelerated by using a gain matrix $K > 0$

$$\dot{q} = J_r^T(q)\, Ke = J_r^T(q)\, K(r_d - f_r(q))$$

note: $K \longrightarrow K + K_s$, with K_s skew-symmetric, can be used also to "escape" from being stuck in a stationary point of $V = \frac{1}{2}\, e^T K e$, by rotating the error Ke out of the null space of J_r^T (when a singularity is encountered)

A case study

analytic expressions of Newton and gradient iterations

- 2R robot with $l_1 = l_2 = 1$, desired end-effector position $r_d = p_d = (1,1)$
- direct kinematic function and error

$$f_r(q) = \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix} \qquad e = p_d - f_r(q) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - f_r(q)$$

- Jacobian matrix

$$J_r(q) = \frac{\partial f_r(q)}{\partial q} = \begin{pmatrix} -(s_1 + s_{12}) & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix}$$

- Newton versus Gradient iteration

$$\boxed{J_r^{-1}(q^k)} \qquad \boxed{e_k}$$

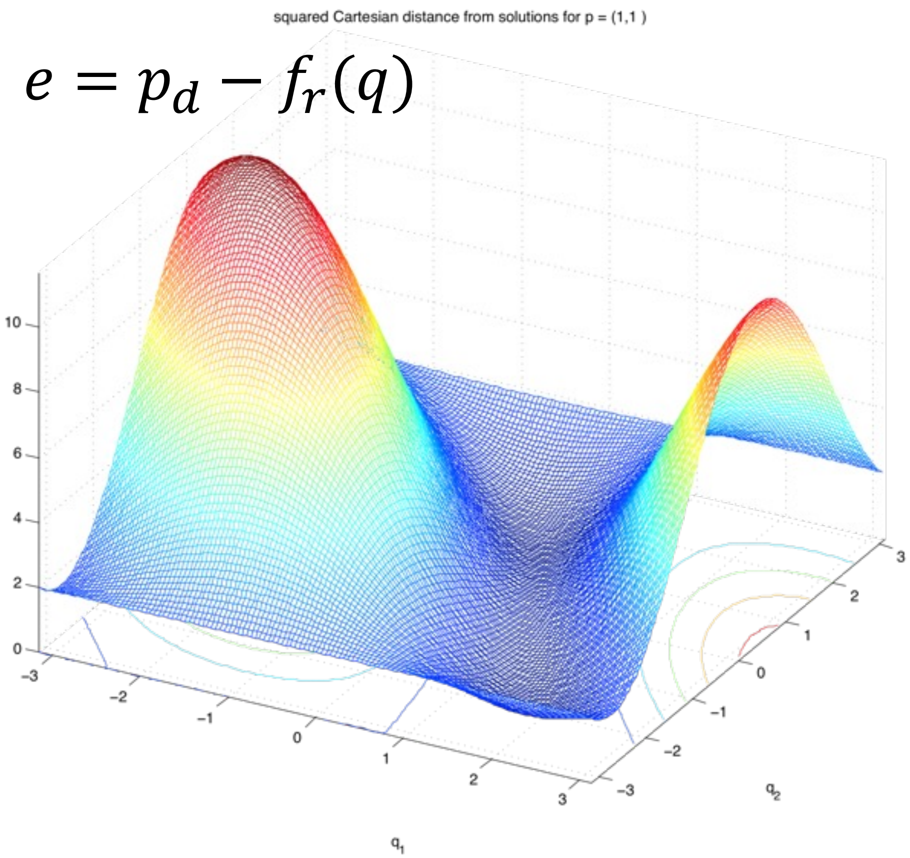
$$q^{k+1} = q^k + \begin{cases} \dfrac{1}{\underbrace{s_2}_{\det J_r(q)}} \begin{pmatrix} c_{12} & s_{12} \\ -(c_1 + c_{12}) & -(s_1 + s_{12}) \end{pmatrix}_{|q=q^k} \\ \alpha \begin{pmatrix} -(s_1 + s_{12}) & c_1 + c_{12} \\ -s_{12} & c_{12} \end{pmatrix}_{|q=q^k} \end{cases} \times \begin{pmatrix} 1 - (c_1 + c_{12}) \\ 1 - (s_1 + s_{12}) \end{pmatrix}_{|q=q^k}$$

$$\boxed{J_r^T(q^k)}$$

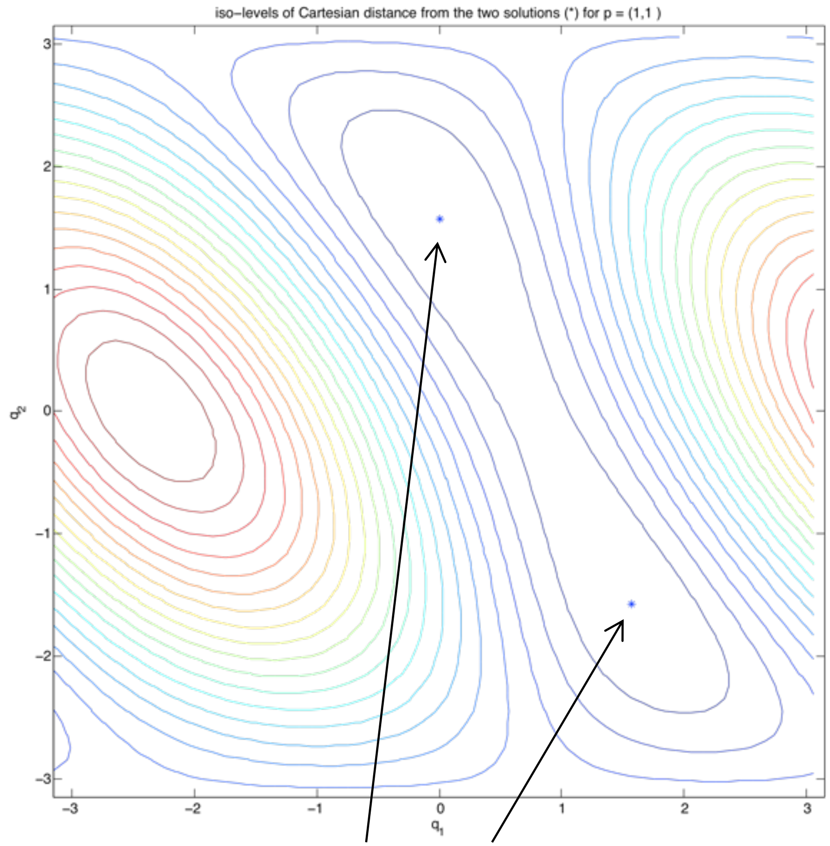
Error function

- 2R robot with $l_1 = l_2 = 1$ and desired end-effector position $p_d = (1,1)$

$$e = p_d - f_r(q)$$



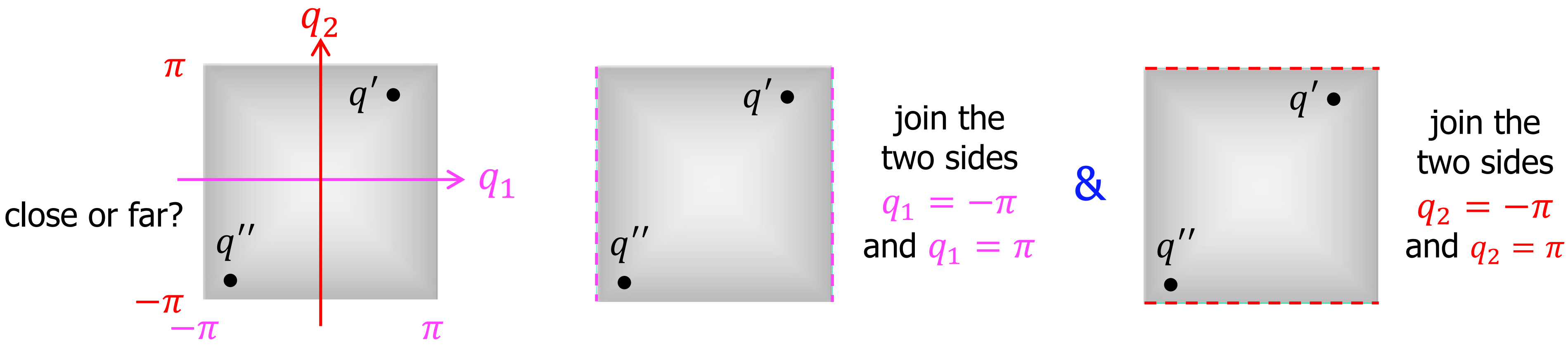
plot of $\|e\|^2$ as a function of $q = (q_1, q_2)$



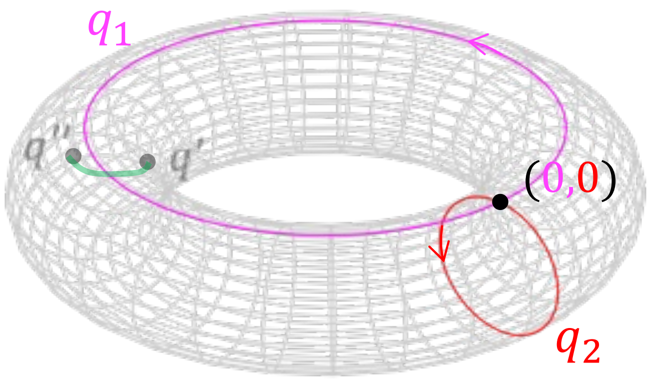
two local minima
(inverse kinematic solutions)

Configuration space of 2R robot
whiteboard …

- can we represent the correct "distance" between two configurations q' and q'' of this robot on a (square) region in $\mathbb{R}^2$?



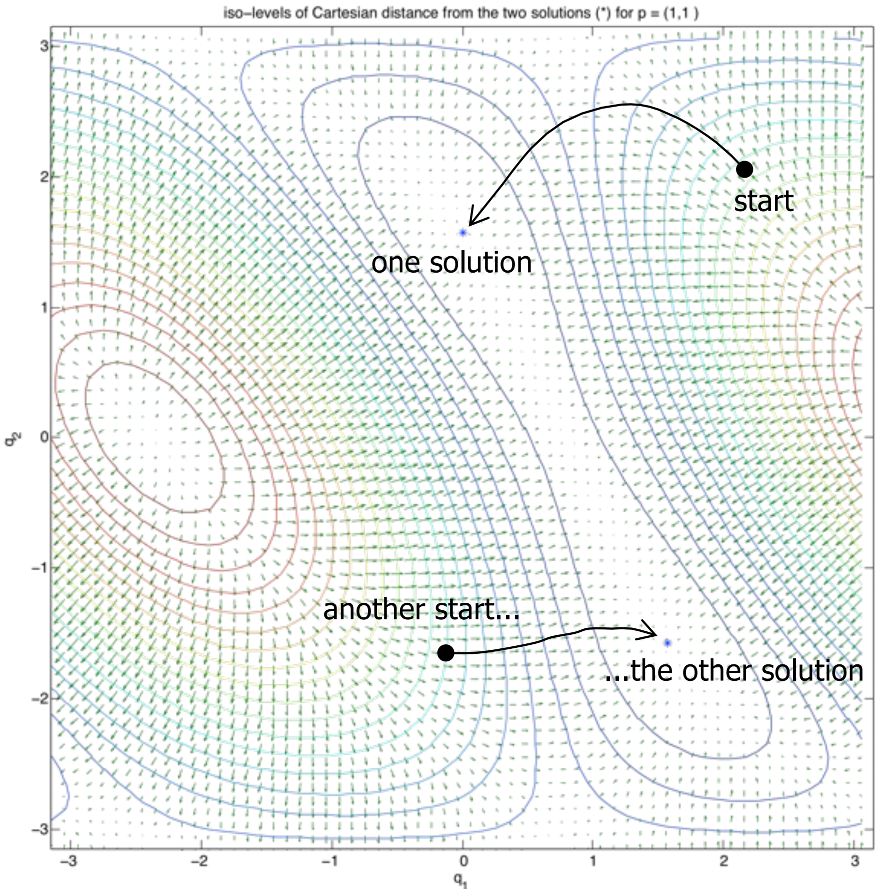
- configuration space is a torus $SO(1) \times SO(1)$, i.e., the surface of a "donut"

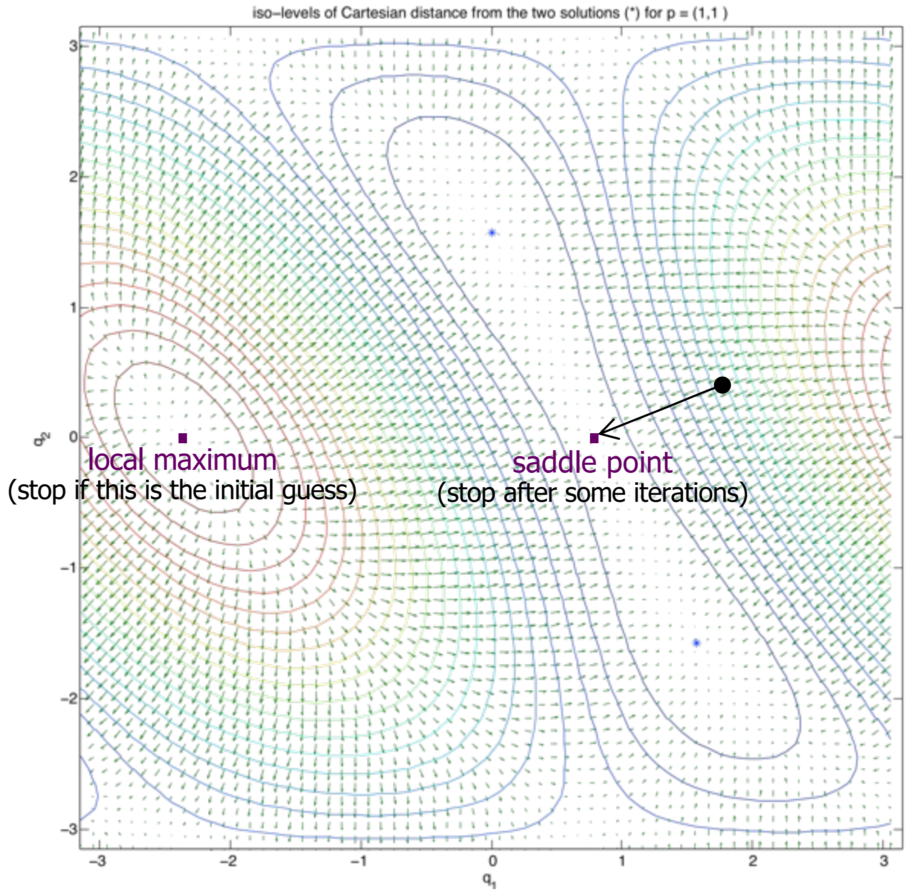


- the right metric is a geodesic on the torus …

Error reduction by Gradient method

- flow of iterations along the negative (or anti-) gradient
- two possible cases: convergence or stuck (at zero gradient)





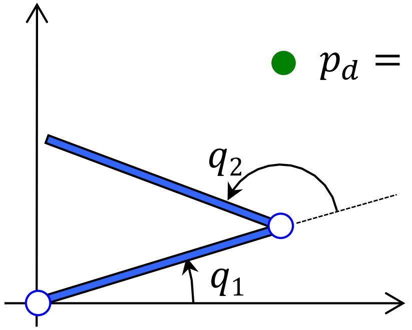
$(q_1, q_2)' = (0, \pi/2)$ $\quad$ $(q_1, q_2)'' = (\pi/2, -\pi/2)$

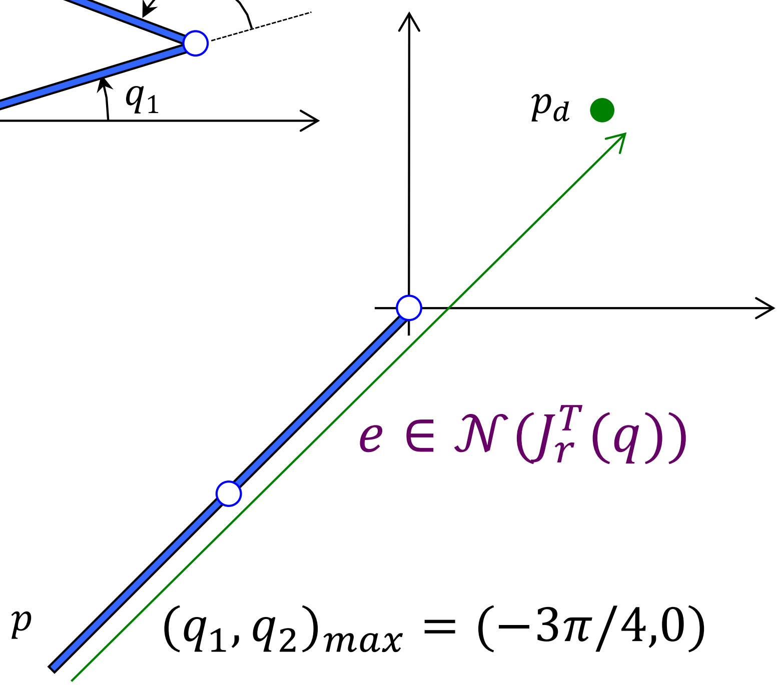
$$\underbrace{(q_1, q_2)_{max} = (-3\pi/4, 0) \quad (q_1, q_2)_{saddle} = (\pi/4, 0)}_{e \in \mathcal{N}(J_r^T(q))\,!}$$

Convergence analysis

when does the gradient method get stuck?

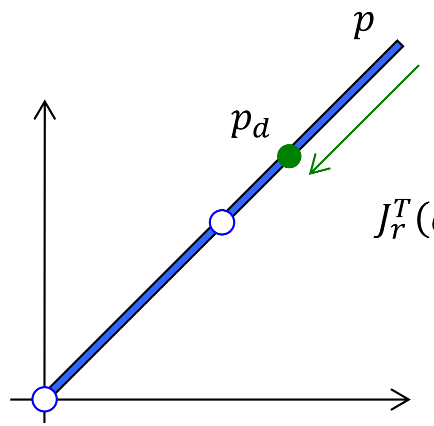
- lack of convergence occurs when
  - the Jacobian matrix $J_r(q)$ is not full rank (the robot is in a "singular configuration")
  - **AND** the error e is in the null space of $J_r^T(q)$





$e \in \mathcal{N}(J_r^T(q))$

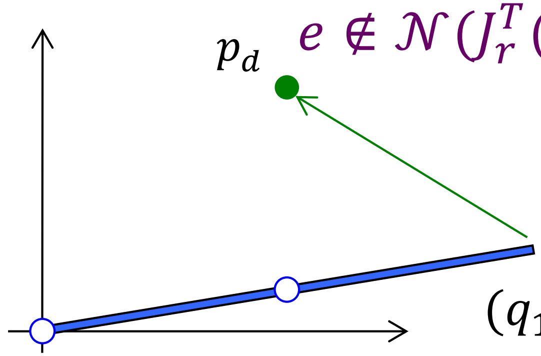
$(q_1, q_2)_{max} = (-3\pi/4, 0)$



$$e = p_d - p = \begin{pmatrix} 1-\sqrt{2} \\ 1-\sqrt{2} \end{pmatrix}$$

$$J_r^T(q) = \begin{pmatrix} -(s_1 + s_{12}) & c_1 + c_{12} \\ -s_{12} & c_{12} \end{pmatrix}_{|q=q_{saddle}} = \begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$$

$(q_1, q_2)_{saddle} = (\pi/4, 0)$ $\quad e \in \mathcal{N}(J_r^T(q))$



$e \notin \mathcal{N}(J_r^T(q))$!!

the algorithm will proceed in this case, moving out of the singularity

$(q_1, q_2) = (\pi/9, 0)$

Issues in implementation

- initial guess q^0
  - only one inverse solution is generated for each guess
  - multiple initializations for obtaining other solutions
- optimal step size $\alpha > 0$ in Gradient method
  - a constant step may work good initially, but not close to the solution (or vice versa)
  - an adaptive one-dimensional line search (e.g., Armijo's rule) could be used to choose the best α at each iteration
- stopping criteria

Cartesian error (possibly, separate for position and orientation) $\|r_d - f_r(q^k)\| \leq \varepsilon$

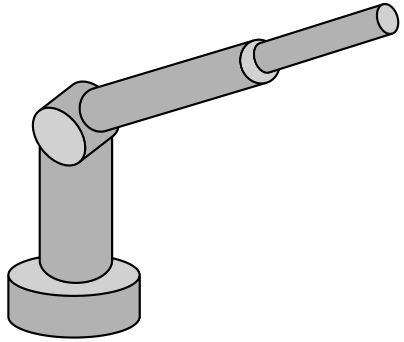
algorithm increment $\|q^{k+1} - q^k\| \leq \varepsilon_q$

- understanding closeness to singularities

$$\sigma_{min}\{J_r(q^k)\} \geq \sigma_0$$

good numerical conditioning of Jacobian matrix (SVD)
(or a simpler test on its determinant, for $m = n$)

Numerical tests on RRP robot



- RRP/polar robot: desired E-E position $r_d = p_d = (1,1,1)$
  —see slide #22, with $d_1 = 0.5$
- the two (known) analytical solutions, with $q_3 \geq 0$, are

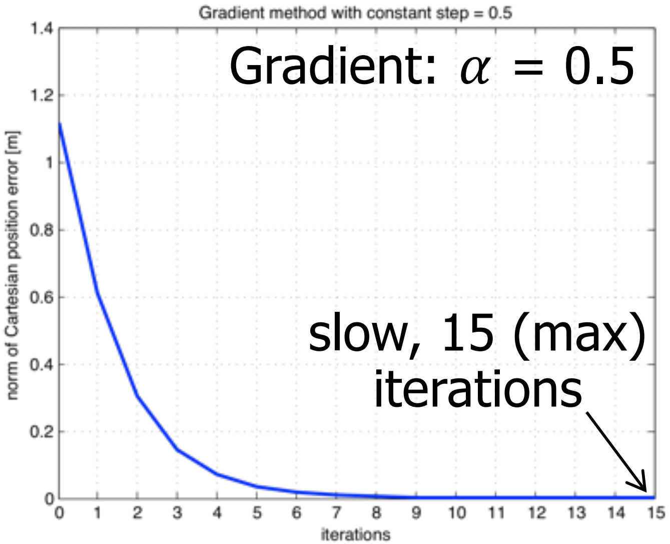
$$q^* = (0.7854, 0.3398, 1.5)$$

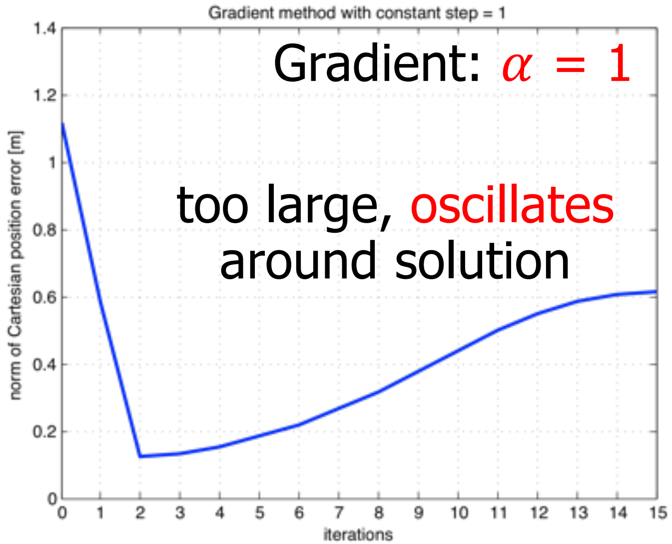
$$q^{**} = (q_1^* - \pi, \pi - q_2^*, q_3^*) = (-2.3562, 2.8018, 1.5)$$

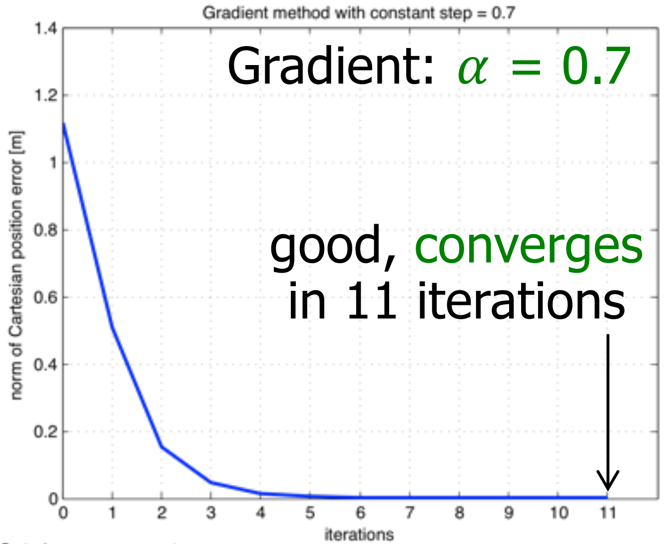
- norms $\varepsilon = 10^{-5}$ (max Cartesian error), $\varepsilon_q = 10^{-6}$ (min joint increment)
- $k_{max} = 15$ (max # iterations), $|\det J_r(q)| \leq 10^{-4}$ (singularity closeness)
- numerical performance of Gradient (with different steps α) vs. Newton
  - test 1: $q^0 = (0,0,1)$ as initial guess
  - test 2: $q^0 = (-\pi/4, \pi/2, 1)$ — "singular" start, since $c_2 = 0$ (see slide #22)
  - test 3: $q^0 = (0, \pi/2, 0)$ — "doubly singular" start, since also $q_3 = 0$
  - solution and plots with MATLAB code

Numerical test - 1

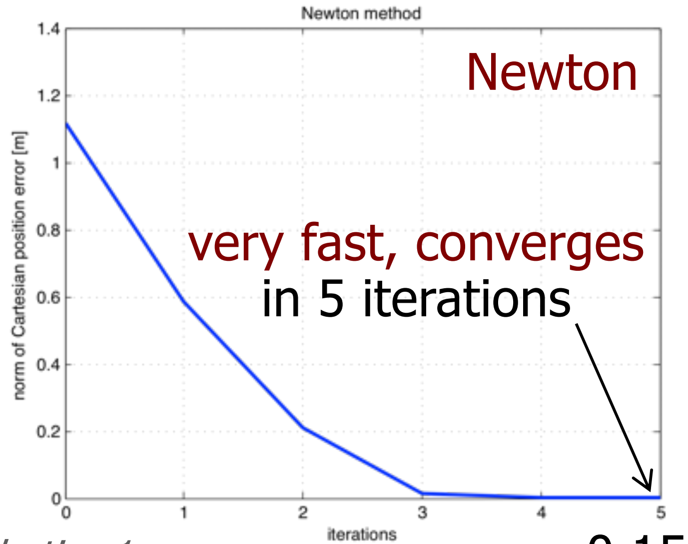
- test 1: $q^0 = (0, 0, 1)$ as initial guess; evolution of the error norm





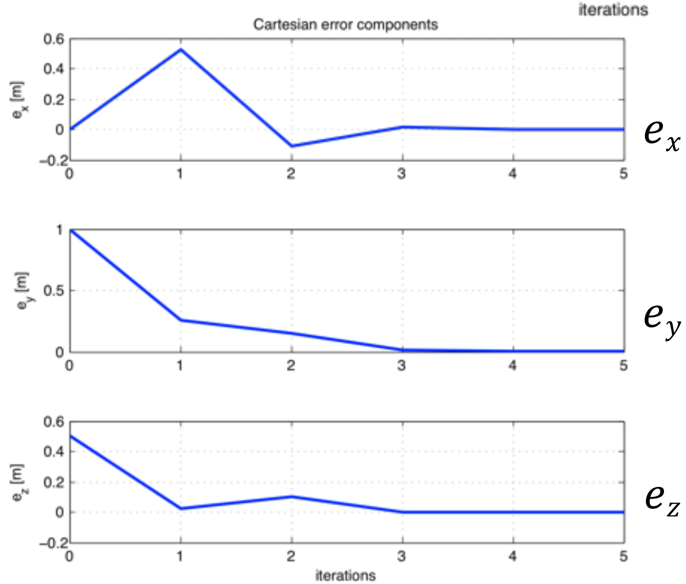


$0.57 \cdot 10^{-5}$



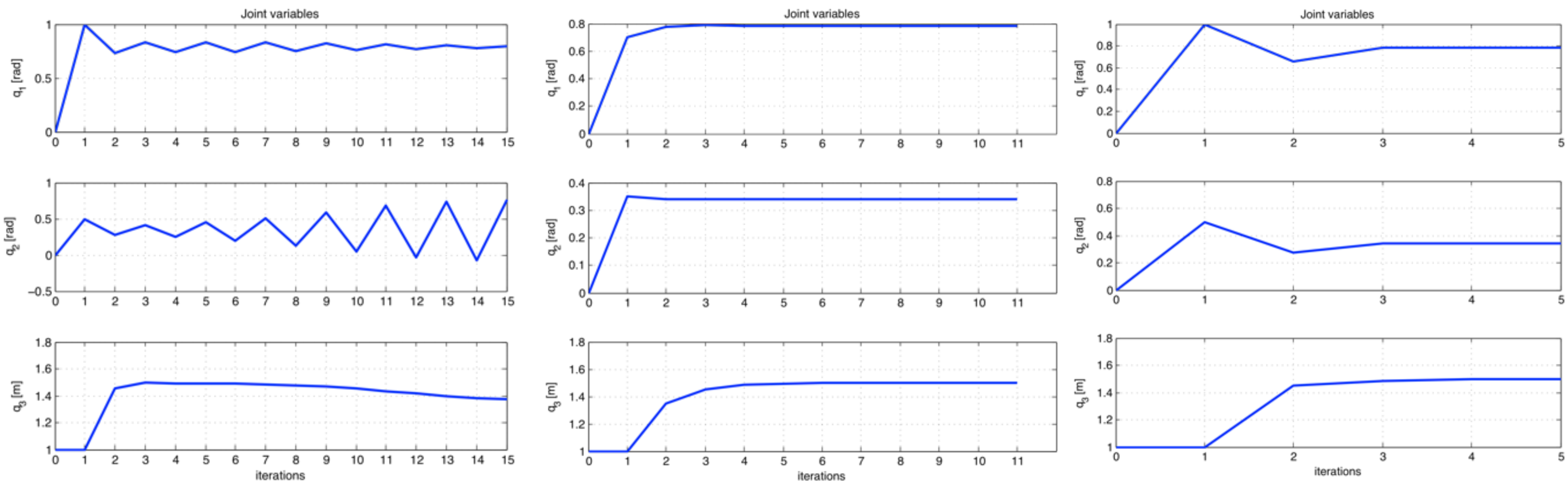
$0.15 \cdot 10^{-8}$

Cartesian errors component-wise



Numerical test - 1

- test 1: $q^0 = (0, 0, 1)$ as initial guess; evolution of joint variables



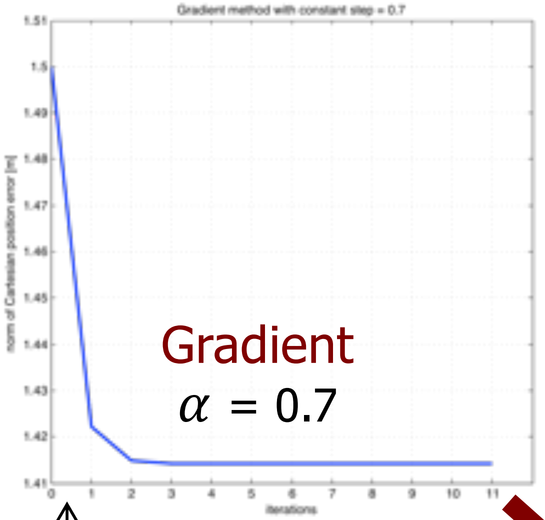
Gradient: $\alpha = 1$	Gradient: $\alpha = 0.7$	Newton
not converging to a solution	converges in 11 iterations	converges in 5 iterations

both to the same solution $q^* = (0.7854, 0.3398, 1.5)$

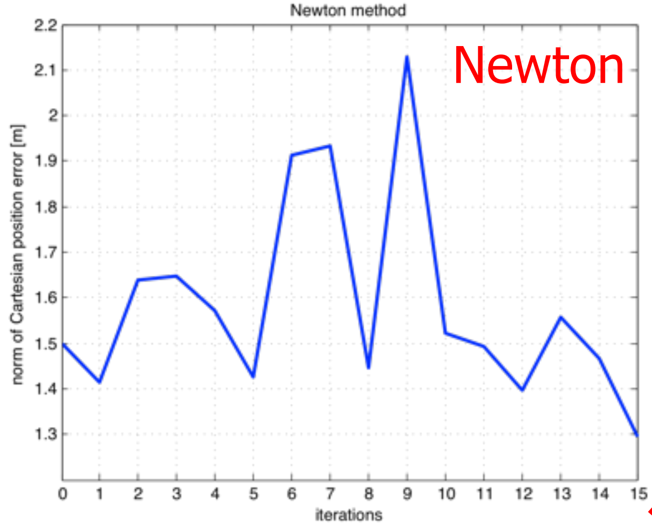
Numerical test - 2

- test 2: $q^0 = (-\pi/4, \pi/2, 1)$: singular start

error norms



Gradient
α = 0.7



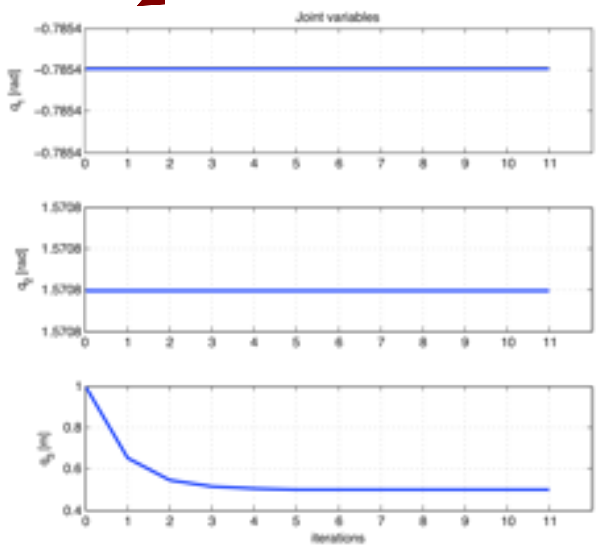
Newton $\leftrightarrow$ with check of singularity: blocked at start

$\leftrightarrow$ without check: it diverges!

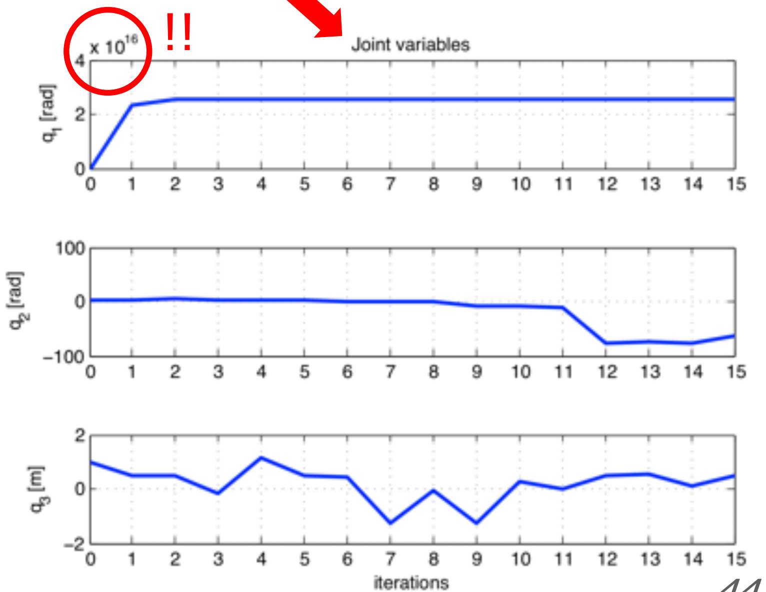
$\updownarrow$

starts toward solution, but slowly stops (in singularity): when Cartesian error vector $e \in \mathcal{N}(J_r^T(q))$

joint variables



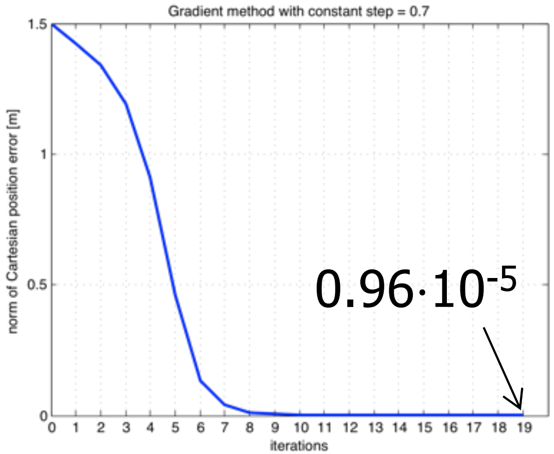
!!



Numerical test - 3

- test 3: $q^0 = (-\pi/4, \pi/2, 1)$: doubly singular start

error norm



Gradient (with α = 0.7)

① starts toward solution

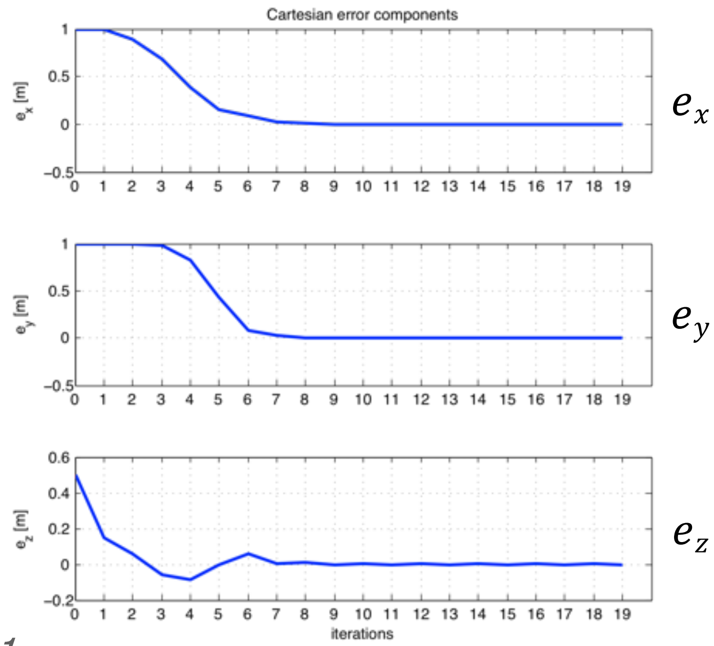
② exits the double singularity

③ slowly converges in 19 iterations to the solution $q^* = (0.7854, 0.3398, 1.5)$

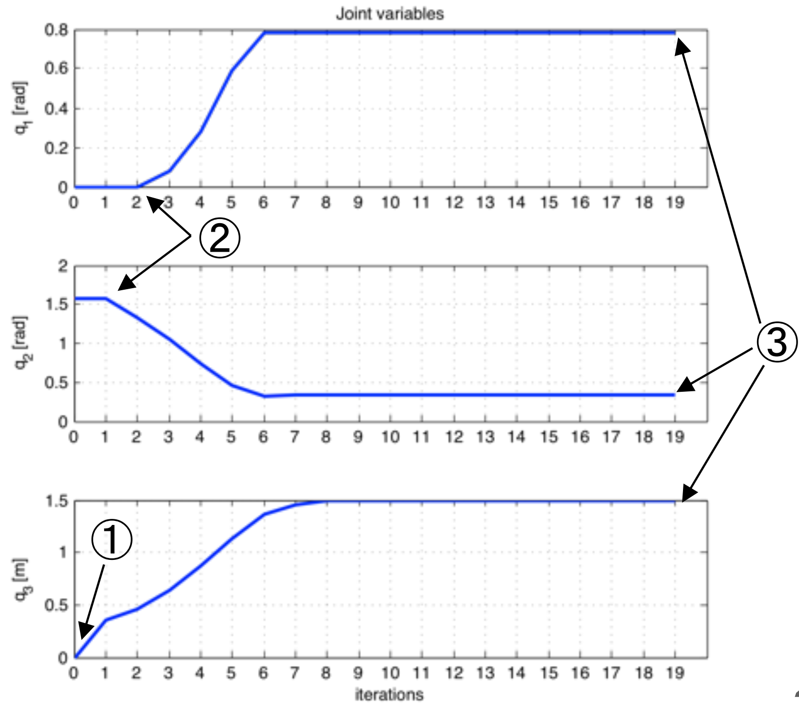
Newton is either blocked at start or (w/o check) explodes!

⇒ "NaN" in MATLAB

Cartesian errors



joint variables



Final remarks

- an efficient iterative scheme can be devised by combining
  - initial iterations using Gradient ("sure but slow", linear convergence rate)
  - switch then to Newton method (quadratic terminal convergence rate)
- joint range limits are considered only at the end
  - check if the solution found is feasible, as for analytical methods
- or, an optimization criterion and/or constraints included in the search
  - drive iterations toward an inverse kinematic solution with nicer properties
- if the problem has to be solved on-line
  - execute iterations and associate an actual robot motion: repeat steps at times t_0, $t_1 = t_0 + T$, ..., $t_k = t_{k-1} + T$ (e.g., every $T = 40$ ms)
  - a "good" choice for the initial guess q^0 at t_k is the solution of the previous problem at t_{k-1} (provides continuity, requires only 1-2 Newton iterations)
  - crossing of singularities and handling of joint range limits need special care
- Jacobian-based inversion schemes are used also for kinematic control, moving along/tracking a continuous task trajectory $r_d(t)$