Robotics I

February 3, 2011

Consider a 3R anthropomorphic robot mounted on the floor and characterized by the Denavit-Hartenberg parameters in Table 1, where D, L_1, L_2, and L_3 are all strictly positive values.

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	D	L_1	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

Table 1: Table of DH parameters

1. Obtain the 3×3 Jacobian matrix ${}^0\boldsymbol{J}_L(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the linear velocity ${}^0\boldsymbol{v}$ of the origin O_3 of frame 3 expressed in frame 0.

2. Characterize the singular configurations $\boldsymbol{q}$ of the Jacobian ${}^3\boldsymbol{J}_L(\boldsymbol{q})$ relating $\dot{\boldsymbol{q}}$ to the linear velocity ${}^3\boldsymbol{v}$ of the origin O_3 of frame 3 expressed in frame 3.

3. Obtain the 3×3 Jacobian matrix ${}^0\boldsymbol{J}_A(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the angular velocity ${}^0\boldsymbol{\omega}$ of frame 3 expressed in frame 0. Show that this matrix is always singular and provide an explanation of this result.

4. Assume that the robot is in the configuration

$$\boldsymbol{q}^* = \begin{pmatrix} 0 & \frac{\pi}{4} & -\frac{\pi}{4} \end{pmatrix}^T \quad \text{[rad]}$$

   with a joint velocity

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{q}_1^* & 0 & 0 \end{pmatrix}^T \quad \text{[rad/s]}, \qquad \text{with } \dot{q}_1^* \neq 0.$$

   Determine the joint acceleration $\ddot{\boldsymbol{q}}$ that should be imposed so that the resulting linear Cartesian acceleration of the origin O_3 is directed along $\boldsymbol{y}_3$ and has an intensity $A \neq 0$. Provide some comment on the structure of the obtained solution. In particular, is there a value A such that only one joint needs to accelerate?

[150 minutes; open books]

Solution

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For item 1, we are interested in the velocity of point O_3, whose position $\boldsymbol{p} = {}^0\boldsymbol{p}$ is given by the direct kinematics map

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos q_1 \, (L_1 + L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \\ \sin q_1 \, (L_1 + L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \\ D + L_2 \sin q_2 + L_3 \sin(q_2 + q_3) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

The Jacobian ${}^0\boldsymbol{J}_L(\boldsymbol{q})$ can be obtained either by analytical differentiation of $\boldsymbol{f}(\boldsymbol{q})$ in (1) w.r.t. $\boldsymbol{q}$ or by using the expression of the first three rows of the geometric Jacobian. Using the usual short notation for trigonometric functions, the result is in both cases

$${}^0\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1(L_1 + L_2c_2 + L_3c_{23}) & -c_1(L_2s_2 + L_3s_{23}) & -L_3c_1s_{23} \\ c_1(L_1 + L_2c_2 + L_3c_{23}) & -s_1(L_2s_2 + L_3s_{23}) & -L_3s_1s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \end{pmatrix}. \tag{2}$$

For item 2, we have that

$$\det {}^3\boldsymbol{J}_L(\boldsymbol{q}) = \det \left({}^2\boldsymbol{R}_3^T(q_3) \, {}^1\boldsymbol{R}_2^T(q_2) \, {}^0\boldsymbol{R}_1^T(q_1) \, {}^0\boldsymbol{J}_L(\boldsymbol{q}) \right) = \det {}^0\boldsymbol{J}_L(\boldsymbol{q}).$$

Nonetheless, it is useful to rewrite the Jacobian in the successive frames 1, 2, and 3, because the resulting expressions will be simplified. From Table 1, we have

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix}, \; {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; {}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From these we obtain

$${}^1\boldsymbol{J}_L(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1) \, {}^0\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & -(L_2s_2 + L_3s_{23}) & -L_3s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix},$$

$${}^2\boldsymbol{J}_L(\boldsymbol{q}) = {}^1\boldsymbol{R}_2^T(q_1) \, {}^1\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & -L_3s_3 & -L_3s_3 \\ 0 & L_2 + L_3c_3 & L_3c_3 \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix},$$

and

$${}^3\boldsymbol{J}_L(\boldsymbol{q}) = {}^2\boldsymbol{R}_3^T(q_1) \, {}^2\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & L_2s_3 & 0 \\ 0 & L_3 + L_2c_3 & L_3 \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix}.$$

In particular from the last expression, it is immediate to see that for any $i \in \{1, 2, 3\}$

$$\det {}^i\boldsymbol{J}_L(\boldsymbol{q}) = -L_2L_3(L_1 + L_2c_2 + L_3c_{23})s_3. \tag{3}$$

Therefore, the singular configurations of $\boldsymbol{J}_L(\boldsymbol{q})$ are:

$$\begin{array}{rcll} s_3 = 0 & \Longleftrightarrow & q_3 = \{0, \pm\pi\} & \text{(third link is stretched or folded)} \\ L_1 + L_2 c_2 + L_3 c_{23} = 0 & \Longleftrightarrow & p_x = p_y = 0 & \text{(}O_3\text{ is on the axis }\boldsymbol{z}_0\text{ of joint 1)} \end{array}$$

For item 3, we compute the expression of the lower three rows of the geometric Jacobian. It is

$$\begin{aligned} {}^0\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & {}^0\boldsymbol{R}_1(q_1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

Matrix ${}^0\boldsymbol{J}_A(\boldsymbol{q})$ is always singular, having constant rank equal to 2. This can be easily explained as follows. The three degrees of freedom of the considered manipulator allow placing the end-effector in any point of the robot primary workspace, and imposing a linear velocity in any direction when the arm is out of singularities. However, the orientation of the end-effector frame can never be changed around the unitary axis $\boldsymbol{n}(q_1) = \begin{pmatrix} c_1 & s_1 & 0 \end{pmatrix}^T$. In fact, $\boldsymbol{\omega} = \alpha\,\boldsymbol{n}(q_1) \notin \mathcal{R}\{{}^0\boldsymbol{J}_A(\boldsymbol{q})\}$, for every $\boldsymbol{q}$ and for any scalar α.

Finally, for item 4 we use the second-order differential map

$${}^0\ddot{\boldsymbol{p}} = {}^0\boldsymbol{J}_L(\boldsymbol{q})\ddot{\boldsymbol{q}} + {}^0\dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{5}$$

evaluated at $\boldsymbol{q} = \boldsymbol{q}^*$, $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^*$. The Cartesian acceleration is specified as

$${}^0\ddot{\boldsymbol{p}} = {}^0\boldsymbol{R}_3(\boldsymbol{q})\,{}^3\ddot{\boldsymbol{p}} = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3)\begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix} = \begin{pmatrix} -A\,c_1 s_{23} \\ -A\,s_1 c_{23} \\ A\,c_{23} \end{pmatrix},$$

which, when evaluated at $\boldsymbol{q} = \boldsymbol{q}^*$, yields the desired value

$${}^0\ddot{\boldsymbol{p}}_d = {}^0\ddot{\boldsymbol{p}}\big|_{\boldsymbol{q}=\boldsymbol{q}^*} = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix}, \tag{6}$$

i.e., the acceleration of the end-effector should be directed along $\boldsymbol{z}_0$, the vertical direction. Since the determinant (3) of the associated Jacobian is nonzero at the given configuration, the solution for the joint acceleration is obtained from (5) as

$$\ddot{\boldsymbol{q}} = {}^0\boldsymbol{J}_L^{-1}(\boldsymbol{q}^*)\left({}^0\ddot{\boldsymbol{p}}_d - {}^0\dot{\boldsymbol{J}}_L(\boldsymbol{q}^*)\dot{\boldsymbol{q}}^*\right),$$

where

$${}^0\boldsymbol{J}_L^{-1}(\boldsymbol{q}^*) = \begin{pmatrix} 0 & -L_2\dfrac{\sqrt{2}}{2} & 0 \\ L_1 + L_2\dfrac{\sqrt{2}}{2} + L_3 & 0 & 0 \\ 0 & L_2\dfrac{\sqrt{2}}{2} + L_3 & L_3 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \dfrac{1}{L_1 + L_2\frac{\sqrt{2}}{2} + L_3} & 0 \\ -\dfrac{\sqrt{2}}{L_2} & 0 & 0 \\ \dfrac{1}{L_3} + \dfrac{\sqrt{2}}{L_2} & 0 & \dfrac{1}{L_3} \end{pmatrix}. \tag{7}$$

Let ${}^0\boldsymbol{J}_1$ be the first column of the Jacobian ${}^0\boldsymbol{J}_L$. Thanks to the simple structure of $\dot{\boldsymbol{q}}^*$, for the term involving the time derivative of the Jacobian we need only to compute

$$\left({}^0\dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\Big|_{\boldsymbol{q}=\boldsymbol{q}^*,\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}^*} = \left({}^0\dot{\boldsymbol{J}}_1(\boldsymbol{q})\right)\Big|_{\boldsymbol{q}=\boldsymbol{q}^*,\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}^*} \dot{q}_1^* = \left(\frac{\partial\,{}^0\boldsymbol{J}_1(\boldsymbol{q})}{\partial q_1}\,\dot{q}_1^*\right)\Big|_{\boldsymbol{q}=\boldsymbol{q}^*} \dot{q}_1^*$$

$$= \begin{pmatrix} -c_1(L_1+L_2c_2+L_3c_{23}) \\ -s_1(L_1+L_2c_2+L_3c_{23}) \\ 0 \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}^*} (\dot{q}_1^*)^2 = \begin{pmatrix} -(L_1+L_2\frac{\sqrt{2}}{2}+L_3) \\ 0 \\ 0 \end{pmatrix} (\dot{q}_1^*)^2 . \tag{8}$$

As a result, from (6–8) the final solution is

$$\ddot{\boldsymbol{q}} = A \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{L_3} \end{pmatrix} + (L_1 + L_2\,\frac{\sqrt{2}}{2} + L_3)\,(\dot{q}_1^*)^2 \begin{pmatrix} 0 \\ -\dfrac{\sqrt{2}}{L_2} \\ \dfrac{1}{L_3} + \dfrac{\sqrt{2}}{L_2} \end{pmatrix} .$$

We note that no acceleration should be applied to the first joint ($\ddot{q}_1 = 0$), as could be argued already from (6). In fact, any angular acceleration imposed to joint 1 (along the vertical joint axis $\boldsymbol{z}_0$) would produce a centrifugal acceleration on the end-effector, which is in contrast with the requested zero acceleration along the $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ axes in (6). Moreover, if

$$A = -\left(1 + \frac{L_3}{L_2}\sqrt{2}\right)(L_1 + L_2\,\frac{\sqrt{2}}{2} + L_3)\,(\dot{q}_1^*)^2$$

then $\ddot{q}_1 = \ddot{q}_3 = 0$ in the solution.

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