Robotics I

February 9, 2012

Exercise 1

Consider the non-spherical wrist of the Comau Smart5 NJ4 170 robot, i.e., the last three revolute joints of this 6R structure (see Fig. 1). The associated Denavit-Hartenberg parameters are given in the three rows of Tab. 1. Note that α, d_4, and d_5 are all positive constants, and d_6 has been set to zero.

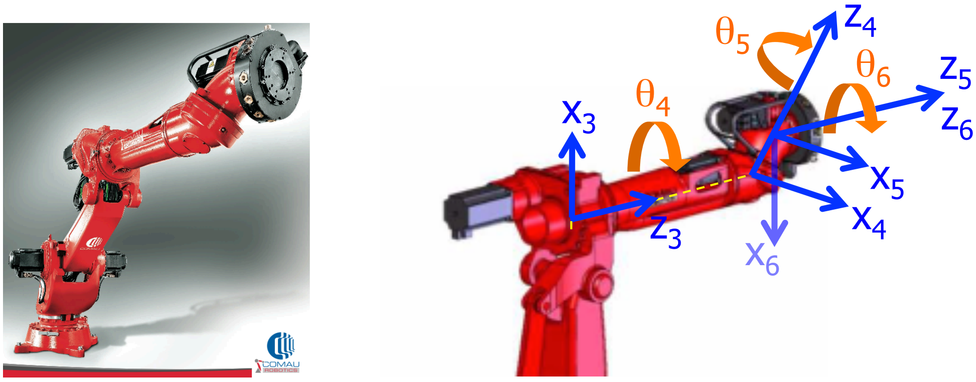

Figure 1: The Comau Smart5 NJ4 170 robot and its last three joints constituting a non-spherical wrist (with DH frames)

i	α_i	a_i	d_i	θ_i
4	α	0	d_4	θ_4
5	$-\alpha$	0	d_5	θ_5
6	0	0	0	θ_6

Table 1: DH parameters of the robot wrist

Provide the explicit relation between the joint velocity $\dot{\boldsymbol{\theta}} = \begin{pmatrix} \dot{\theta}_4 & \dot{\theta}_5 & \dot{\theta}_6 \end{pmatrix}^T$ and the angular velocity ${}^3\boldsymbol{\omega}_e$ of the end-effector frame (labeled as 6 in Fig. 1) expressed in frame 3. Also, analyze the singularities of this differential relation.

Exercise 2

Consider the RRP (polar) robot in Fig. 2, where $d_1 = 1$, and assume that the coordinate q_3 associated to the third (prismatic) joint can only take non-negative values.

- Assign the frames according to the *Denavit-Hartenberg convention* and complete the associated table of parameters. Choose the reference axes so that $\alpha_i \geq 0$, for $i = 1, 2, 3$, and set the origin of the last frame at the end-effector/tip of the robot.

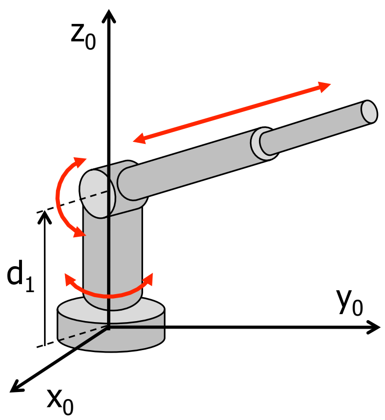

Figure 2: A RRP (polar) robot

- Give the explicit expression of the 3×3 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the linear velocity $\boldsymbol{v}_e$ of the end-effector

$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

  and discuss its singularities.

Exercise 3

Consider the same robot of Exercise 2.

- Define a desired linear path $\boldsymbol{p}_d(\sigma)$, parametrized by its actual length σ, between the initial and final Cartesian points

$$\boldsymbol{p}_{init} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \qquad \boldsymbol{p}_{fin} = \begin{pmatrix} -1 & -1 & 3 \end{pmatrix}^T.$$

  Verify the existence of a value $\sigma = \sigma_s$ at which the desired path encounters a robot singularity. Determine whether or not the desired trajectory $\boldsymbol{p}_d(\sigma(t))$, with $\dot{\sigma} > 0$ at $\sigma = \sigma_s$, can be *perfectly* realized also in that robot configuration. In case it can, provide some reasoning to justify how to execute the desired trajectory; else, explain in detail why this is not possible.

- Consider the same initial point $\boldsymbol{p}_{init}$ and a new final point $\boldsymbol{p}_{new} = \begin{pmatrix} -1 & 1 & 3 \end{pmatrix}^T$, and assume as desired interpolating trajectory $\boldsymbol{p}_d(\sigma(t))$ a linear Cartesian path with constant speed $\dot{\sigma} = 1$. The robot is initially in the configuration $\boldsymbol{q}(0) = \begin{pmatrix} \pi/3 & \pi/2 & 1 \end{pmatrix}^T$. Design *two* different kinematic control laws that exponentially drive the tracking error to zero, either

  - by keeping the Cartesian error along $\boldsymbol{z}_0$ constantly at zero,

  or, respectively,

  - by keeping the joint error on the second coordinate q_2 constantly at zero.

[210 minutes; open books]

Solutions

February 9, 2012

Exercise 1

A first way to solve the problem is to use the 3×3 geometric Jacobian $\boldsymbol{J}_A(\boldsymbol{\theta})$ associated to the angular velocity $\boldsymbol{\omega}_e$ of the end-effector frame,

$$\boldsymbol{\omega}_e = \boldsymbol{J}_A(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}}.$$

Since the considered last three joints of the robot are all revolute and computations have to be performed w.r.t. the robot frame 3, we have

$$\begin{aligned} {}^3\boldsymbol{J}_A(\boldsymbol{\theta}) &= \begin{pmatrix} {}^3\boldsymbol{z}_3 & {}^3\boldsymbol{z}_4(\theta_4) & {}^3\boldsymbol{z}_5(\theta_4,\theta_5) \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & {}^3\boldsymbol{R}_4(\theta_4)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & {}^3\boldsymbol{R}_4(\theta_4)\,{}^4\boldsymbol{R}_5(\theta_5)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \end{aligned} \tag{1}$$

with rotation matrices obtained from the DH table of parameters (using $s_{-\alpha} = -s_\alpha$ and $c_{-\alpha} = c_\alpha$):

$$\begin{aligned} {}^3\boldsymbol{R}_4(\theta_4) &= \begin{pmatrix} c_4 & -s_4c_\alpha & s_4s_\alpha \\ s_4 & c_4c_\alpha & -c_4s_\alpha \\ 0 & s_\alpha & c_\alpha \end{pmatrix} \\ {}^4\boldsymbol{R}_5(\theta_5) &= \begin{pmatrix} c_5 & -s_5c_\alpha & -s_5s_\alpha \\ s_5 & c_5c_\alpha & c_5s_\alpha \\ 0 & -s_\alpha & c_\alpha \end{pmatrix} \\ {}^3\boldsymbol{R}_4(\theta_4)\,{}^4\boldsymbol{R}_5(\theta_5) &= \begin{pmatrix} * & * & (1-c_5)\,s_4s_\alpha c_\alpha - c_4s_5s_\alpha \\ * & * & (c_5-1)\,c_4s_\alpha c_\alpha - s_4s_5s_\alpha \\ * & * & c_5s_\alpha^2 + c_\alpha^2 \end{pmatrix}. \end{aligned}$$

Above, a $*$ denotes quantities that need not to be computed. Note also that the values of d_4 and d_5 (as well as d_6, if present) are irrelevant. Substituting in (1) yields

$${}^3\boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} 0 & s_4s_\alpha & (1-c_5)\,s_4s_\alpha c_\alpha - c_4s_5s_\alpha \\ 0 & -c_4s_\alpha & (c_5-1)\,c_4s_\alpha c_\alpha - s_4s_5s_\alpha \\ 1 & c_\alpha & c_5s_\alpha^2 + c_\alpha^2 \end{pmatrix}.$$

This matrix has determinant

$$\det\left({}^3\boldsymbol{J}_A(\boldsymbol{\theta})\right) = -s_5\,s_\alpha^2$$

being thus singular for $\theta_5 = 0$ (and $\theta_5 = \pi$, but this is likely to be out the admissible range of this joint). Since the rank of

$${}^3\boldsymbol{J}_A(\boldsymbol{\theta})\big|_{\theta_5=0} = \begin{pmatrix} 0 & s_4s_\alpha & 0 \\ 0 & -c_4s_\alpha & 0 \\ 1 & c_\alpha & 1 \end{pmatrix}$$

is equal to 2, an angular vector ${}^3\boldsymbol{\omega}_e$ of the form

$${}^3\boldsymbol{\omega}_e = \beta\begin{pmatrix} c_4 \\ s_4 \\ * \end{pmatrix} \notin \mathcal{R}\left({}^3\boldsymbol{J}_A(\boldsymbol{\theta})|_{\theta_5=0}\right), \qquad \beta \neq 0$$

cannot be realized in this configuration by any choice of $\dot{\boldsymbol{\theta}}$.

A second way to address the problem would be to use the relation between the derivative of a rotation matrix and the associated angular velocity

$$\boldsymbol{S}(\boldsymbol{\omega}_e) = {}^3\dot{\boldsymbol{R}}_6 \, {}^3\boldsymbol{R}_6^T, \qquad \text{with } {}^3\boldsymbol{R}_6 = {}^3\boldsymbol{R}_4(\theta_4)^4\boldsymbol{R}_5(\theta_4)^5\boldsymbol{R}_6(\theta_6)$$

where $\boldsymbol{S}(\cdot)$ is the skew-symmetric matrix built from the components of $\boldsymbol{\omega}_e$. However, this requires much more computations, in particular the evaluation of all three DH rotation matrices, and their complete product and derivation w.r.t. time (to be performed symbolically).

Exercise 2

A DH frame assignment satisfying the stated requirements is shown in Fig. 3, with the associated parameters given in Tab. 2. Note that the third link is horizontal when $q_2 = \pi/2$.

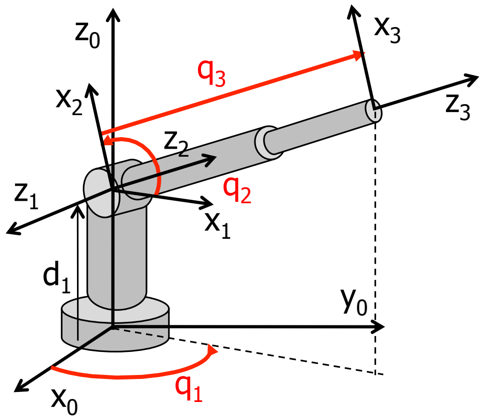

Figure 3: DH frame assignment for the RRP robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	$\pi/2$	0	0	q_2
3	0	0	q_3	0

Table 2: DH parameters for the RRP robot

The end-effector/tip position (i.e, the origin of frame 3) is then obtained as

$$\boldsymbol{p}_e = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} c_1 s_2 \, q_3 \\ s_1 s_2 \, q_3 \\ d_1 - c_2 \, q_3 \end{pmatrix}. \tag{2}$$

The requested Jacobian matrix relating $\dot{\boldsymbol{q}}$ to $\boldsymbol{v}_e = \dot{\boldsymbol{p}}_e$ can be computed either geometrically or by analytic differentiation of (2), yielding the same result in both cases. We have

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1 s_2\, q_3 & c_1 c_2\, q_3 & c_1 s_2 \\ c_1 s_2\, q_3 & s_1 c_2\, q_3 & s_1 s_2 \\ 0 & s_2\, q_3 & -c_2 \end{pmatrix}. \tag{3}$$

For later use, we can write the Jacobian in frame 1, using the rotation matrix (from the DH table)

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

where the expression on the right shows that this is composed by an elementary rotation around the $\boldsymbol{z}_0$ axis and a permutation of axes (preserving the right-hand rule for frames). We compute thus

$${}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 0 & c_2\, q_3 & s_2 \\ 0 & s_2\, q_3 & -c_2 \\ -s_2\, q_3 & 0 & 0 \end{pmatrix}.$$

The determinant of $\boldsymbol{J}(\boldsymbol{q})$, equal to that of ${}^1\boldsymbol{J}(\boldsymbol{q})$, is

$$\det \boldsymbol{J}(\boldsymbol{q}) = s_2\, q_3^2.$$

Therefore, the robot is in a singular configuration whenever $q_3 = 0$ or $q_2 \in \{0, \pi\}$. In the first case, the rank of the Jacobian drops to 1, whereas in the second case the rank is 2. When both singularity conditions hold true, the rank of the Jacobian is still equal to 1. In particular, when $s_2 = 0$ the end-effector position $\boldsymbol{p}_e$ is placed on the axis $\boldsymbol{z}_0$ of joint 1 and the Jacobian ${}^1\boldsymbol{J}(\boldsymbol{q})$ becomes

$${}^1\boldsymbol{J}(\boldsymbol{q})\big|_{s_2=0} = \begin{pmatrix} 0 & \pm q_3 & 0 \\ 0 & 0 & \mp 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This clearly shows that, when the end-effector is on the $\boldsymbol{z}_0$ axis, any desired Cartesian velocity vector lying only in the plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$, i.e., with zero component along the $\boldsymbol{z}_1$ axis, belongs to the range space of the Jacobian, and thus can be perfectly realized by the robot. Velocity vectors $\boldsymbol{v}_e \in \mathcal{R}\left(\boldsymbol{J}(\boldsymbol{q})|_{s_2=0}\right)$ have the form

$${}^1\boldsymbol{v}_e = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix} \quad \Rightarrow \quad {}^0\boldsymbol{v}_e = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{v}_e = \begin{pmatrix} \beta_1 c_1 \\ \beta_1 s_1 \\ \beta_2 \end{pmatrix} \tag{4}$$

when expressed, respectively, in frame 1 or in frame 0, and for arbitrary β_1 and β_2. Finally, note that also the (double) singularity $q_3 = 0$ corresponds to a situation in which the end-effector is on the $\boldsymbol{z}_0$ axis.

Exercise 3

The linear path is parametrized as follows:

$$\boldsymbol{p}_d(\sigma) = \boldsymbol{p}_{init} + \frac{\sigma}{L}\left(\boldsymbol{p}_{fin} - \boldsymbol{p}_{init}\right), \qquad L = \left\|\boldsymbol{p}_{fin} - \boldsymbol{p}_{init}\right\| = \sqrt{12}$$

or

$$\boldsymbol{p}_d(\sigma) = \begin{pmatrix} 1 - \dfrac{2\sigma}{\sqrt{12}} \\ 1 - \dfrac{2\sigma}{\sqrt{12}} \\ 1 + \dfrac{2\sigma}{\sqrt{12}} \end{pmatrix}, \qquad \sigma \in [0, L].$$

It is easy to see that this path will cross the $\boldsymbol{z}_0$ axis for

$$\sigma = \sigma_s = \frac{\sqrt{12}}{2} \qquad \Rightarrow \qquad p_{d,x}(\sigma_s) = p_{d,y}(\sigma_s) = 0$$

so that a singularity of the RRP robot is encountered. The desired velocity along the path has indeed a constant direction

$$\dot{\boldsymbol{p}}_d(t) = \frac{\dot{\sigma}(t)}{L}\left(\boldsymbol{p}_{fin} - \boldsymbol{p}_{init}\right) = \frac{\dot{\sigma}(t)}{\sqrt{12}} \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}. \tag{5}$$

In particular, at $\sigma = \sigma_s$, this Cartesian velocity is still a feasible one for the RRP robot, since (5) can be written in the form (4) of a vector $\boldsymbol{v}_e \in \mathcal{R}(\boldsymbol{J})$ by setting, e.g.,

$$q_1 = \frac{\pi}{4} \quad \rightarrow \quad c_1 = s_1 = \frac{\sqrt{2}}{2}, \qquad \beta_1 = \frac{-2\,\dot{\sigma}}{\sqrt{6}}, \qquad \beta_2 = \frac{2\,\dot{\sigma}}{\sqrt{12}}.$$

This implies that the first joint should be rotated so that the whole linear path belongs to the plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$. If the robot initial configuration is set at $q_1 = \pi/4$, then the entire desired trajectory can be realized by using only joint 2 and 3 of the RRP robot (namely, by the planar RP robot obtained by freezing the first joint) while the first joint does not need to move. This reasoning suggests also that the inversion of the 3×3 Jacobian in (3), which would run into problems close to or crossing the singularity $s_2 = 0$, can be completely avoided.

In fact, consider the 2×2 top right sub-matrix of the Jacobian ${}^1\boldsymbol{J}(\boldsymbol{q})$, i.e.,

$${}^1\bar{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} c_2\, q_3 & s_2 \\ s_2\, q_3 & -c_2 \end{pmatrix}$$

and note that this matrix is never singular, provided that $q_3 \neq 0$ (so, it is independent from the value of q_2). By defining $\dot{\bar{\boldsymbol{q}}} = \begin{pmatrix} \dot{q}_2 & \dot{q}_3 \end{pmatrix}^T$ and ${}^1\dot{\bar{\boldsymbol{p}}} = \begin{pmatrix} {}^1\dot{p}_x & {}^1\dot{p}_y \end{pmatrix}^T$, the following differential relation holds

$${}^1\dot{\bar{\boldsymbol{p}}} = {}^1\bar{\boldsymbol{J}}(\boldsymbol{q})\dot{\bar{\boldsymbol{q}}}.$$

Express now the desired Cartesian velocity in frame 1, for a constant $q_1 = \pi/4$

$${}^1\dot{\boldsymbol{p}}_d = {}^0\boldsymbol{R}_1^T(q_1)\Big|_{q_1 = \frac{\pi}{4}} \cdot \dot{\boldsymbol{p}}_d = \frac{\dot{\sigma}}{\sqrt{12}} \begin{pmatrix} \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \dfrac{\sqrt{2}}{2} & -\dfrac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \frac{\dot{\sigma}}{\sqrt{12}} \begin{pmatrix} -2\sqrt{2} \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} {}^1\dot{\bar{\boldsymbol{p}}}_d \\ 0 \end{pmatrix}.$$

Then, the entire desired Cartesian trajectory will be *perfectly* executed if the robot starts at $t = 0$ in the configuration

$$\boldsymbol{q}(0) = \begin{pmatrix} q_1(0) \\ \bar{\boldsymbol{q}}(0) \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4} \\ \frac{\pi}{2} \\ \sqrt{2} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{f}(\boldsymbol{q}(0)) = \boldsymbol{p}_d(0) = \boldsymbol{p}_{init} \tag{6}$$

(which is one of the two solutions found by solving the inverse kinematics problem, taking into account that only $q_3 \geq 0$ is allowed) by setting, for all $t \geq 0$,

$$\begin{aligned} q_1(t) &= \frac{\pi}{4} \\ \bar{\boldsymbol{q}}(t) &= \bar{\boldsymbol{q}}(0) + \int_0^t {}^1\bar{\boldsymbol{J}}^{-1}(\boldsymbol{q}(\tau))\, {}^1\dot{\bar{\boldsymbol{p}}}_d(\sigma(\tau)) d\tau\,. \end{aligned} \tag{7}$$

The total motion time T will depend from the time profile of $\sigma(t)$, under the minimal necessary boundary conditions $\sigma(0) = 0$ and $\sigma(T) = L$. Accordingly, we will obtain from (7) a final value $\boldsymbol{q}(T)$ such that $\boldsymbol{f}(\boldsymbol{q}(T)) = \boldsymbol{p}_{fin}$.

For the second part of this Exercise, we have $L = \|\boldsymbol{p}_{new} - \boldsymbol{p}_{init}\| = \sqrt{8}$ and the new desired trajectory $\boldsymbol{p}_d(\sigma(t))$ is given by the linear geometric path

$$\boldsymbol{p}_d(\sigma) = \boldsymbol{p}_{init} + \frac{\sigma}{L}\left(\boldsymbol{p}_{new} - \boldsymbol{p}_{init}\right) = \begin{pmatrix} 1 - \frac{2\sigma}{\sqrt{8}} \\ 1 \\ 1 + \frac{2\sigma}{\sqrt{8}} \end{pmatrix}, \qquad \sigma \in [0, \sqrt{8}],$$

with the timing law $\sigma = \sigma(t)$. Since $\dot{\sigma}(t) = 1$ (constant) is assigned, we have $\sigma(t) = t$, with $t \in [0, T]$ and $T = L = \sqrt{8}$. At the given initial configuration $\boldsymbol{q}(0) = \begin{pmatrix} \pi/3 & \pi/2 & 1 \end{pmatrix}^T$, it follows from (2)

$$\boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boldsymbol{p}_{init} = \boldsymbol{p}_d(0)$$

so that there is an initial position error w.r.t. the desired Cartesian trajectory. However, the initial error along the $\boldsymbol{z}_0$ axis is zero. In order to be matched with $\boldsymbol{p}_d(0)$, the initial configuration of the RRP robot should be instead, e.g.,

$$\boldsymbol{q}_d(0) = \begin{pmatrix} \frac{\pi}{4} & \frac{\pi}{2} & \sqrt{2} \end{pmatrix}^T \tag{8}$$

as verified in (6). Indeed, also the initial joint position error is different from zero, but note that we have now $q_2(0) = q_{d,2}(0)$. Moreover, if the robot starts from the desired initial configuration (8), we could generate an entire desired joint trajectory $\boldsymbol{q}_d(t)$ associated to $\boldsymbol{p}_d(t)$ as

$$\boldsymbol{q}_d(t) = \boldsymbol{q}_d(0) + \int_0^t \dot{\boldsymbol{q}}_d(\tau)\, d\tau = \boldsymbol{q}_d(0) + \int_0^t \boldsymbol{J}^{-1}(\boldsymbol{q}_d(\tau))\, \dot{\boldsymbol{p}}_d(\tau)\, d\tau \tag{9}$$

since the Jacobian never encounters a singularity in this case.

With the above in mind, in order to recover the initial error and asymptotically track the desired Cartesian trajectory, a feedback/feedforward control law has to be designed at the kinematic level (i.e., considering $\dot{\boldsymbol{q}}$ as the control input). Depending on the additional requirements, the first solution is a law driven by the *Cartesian error* $\boldsymbol{e}_c = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})$, namely

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\big(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_c\,(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}))\big), \qquad \text{with } \boldsymbol{K}_c > 0 \text{ and diagonal.}$$

This will force each component of the Cartesian error $\boldsymbol{e}_c$ to converge exponentially to zero with a rate prescribed by the associated diagonal element of $\boldsymbol{K}_c$, or $e_{c,i}(t) = \exp(-K_{c,i}\,t)\,e_{c,i}(0)$. As a consequence, the error along the $\boldsymbol{z}_0$ component (i.e, for $i = 3$) will remain constantly zero also during the transient phase of the trajectory tracking task ($e_{c,3}(0) = 0 \rightarrow e_{c,3}(t) \equiv 0$), so that the first requested behavior is obtained.

On the other hand, the second solution is a law driven by the *joint error* $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, with $\boldsymbol{q}_d(t)$ given by (9), namely

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_d + \boldsymbol{K}\,(\boldsymbol{q}_d - \boldsymbol{q})\,, \qquad \text{with } \boldsymbol{K} > 0 \text{ and diagonal.}$$

This will force each component of the joint error $\boldsymbol{e}$ to converge exponentially to zero with a rate prescribed by the associated diagonal element of $\boldsymbol{K}$, or $e_i(t) = \exp(-K_i\,t)\,e_i(0)$. Thus, the error on the second component of the joint configuration vector (i.e., on q_2) will remain constantly zero ($e_2(0) = 0 \rightarrow e_2(t) \equiv 0$) and the second requirement is satisfied.

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