Robotics I

January 9, 2014

Exercise 1

A planar PPR robot is shown in Fig. 1, together with the axes ($\boldsymbol{x}_w$,$\boldsymbol{y}_w$) of a world reference frame RF_w. The third link of the robot has length L. The position of the end-effector in the plane is given by ${}^w\boldsymbol{p} = \begin{pmatrix} {}^wp_x & {}^wp_y \end{pmatrix}^T$.

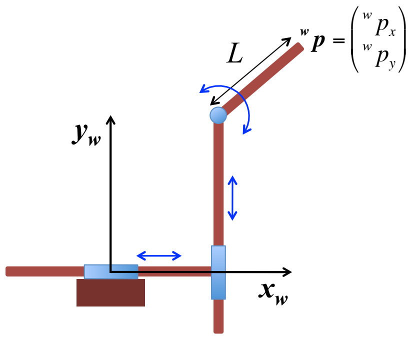

Figure 1: Planar PPR robot

- Assign the frames according to the Denavit-Hartenberg convention and provide the associated table of parameters. Make sure that all constant parameters in the table are *non-negative*.
- Define the homogeneous transformation matrix ${}^w\boldsymbol{T}_0$ between the world reference frame RF_w and the Denavit-Hartenberg frame RF_0 just assigned.
- Assuming that the two prismatic joints have a limited range, $|q_i| < D$ ($i = 1, 2$) with $D > L$:
  - draw the primary and secondary workspaces of the robot, respectively WS_1 and WS_2;
  - for a given end-effector position ${}^w\boldsymbol{p} \in WS_2$, provide all inverse kinematics solutions (q_1, q_2) as parametric functions of q_3.

Exercise 2

Consider a trajectory planning problem for the orientation of the end-effector of a robot. The end-effector should move from an initial orientation, specified by the rotation matrix $\boldsymbol{R}_{in}$, to a final orientation, specified by $\boldsymbol{R}_{fin}$, in time T and with zero initial and final angular velocity ($\boldsymbol{\omega}(0) = \boldsymbol{\omega}(T) = \boldsymbol{0}$). The trajectory has to be designed in terms of the (Y, Z, Y) Euler angles (α, β, γ) of a minimal representation of orientation. The motion time T has to be adjusted so that the norm of the angular velocity $\boldsymbol{\omega}(t)$ does never exceed a constant value $\Omega > 0$, i.e., $\|\boldsymbol{\omega}(t)\| \leq \Omega$ for all $t \in [0, T]$. After sketching the steps of the solution approach, provide a solution to the problem and the associated minimum feasible motion time T^* using as numerical data:

$$\boldsymbol{R}_{in} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{R}_{fin} = \begin{pmatrix} \frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right) & \frac{1}{2} & \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right) \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ -\frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right) & -\frac{1}{2} & -\frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right) \end{pmatrix}, \quad \Omega = \pi \text{ [rad/s]}.$$

Exercise 3

For a 6R manipulator with *spherical* wrist, assume that the origin O_6 of the end-effector frame is placed at the center of the wrist. Let the joint velocity vector be partitioned into base and wrist velocities as $\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{\boldsymbol{q}}_b^T & \dot{\boldsymbol{q}}_w^T \end{pmatrix}^T$, with $\dot{\boldsymbol{q}}_b \in \mathbb{R}^3$ and $\dot{\boldsymbol{q}}_w \in \mathbb{R}^3$.

- Provide the symbolic expressions of $\dot{\boldsymbol{q}}_b$ and $\dot{\boldsymbol{q}}_w$ that assign a desired (zero) angular velocity $\boldsymbol{\omega}_d = \mathbf{0}$ to the end-effector frame and a desired velocity $\boldsymbol{v}_d \neq \mathbf{0}$ to its origin.

- Suppose now that the linear part of the motion task is specified by a desired position trajectory $\boldsymbol{p}_d(t)$, with $\boldsymbol{v}_d(t) = \dot{\boldsymbol{p}}_d(t)$, and that the orientation part is specified by *constant* values of a set of Euler angles $\boldsymbol{\phi}_d = (\alpha_d, \beta_d, \gamma_d)$, with $\boldsymbol{\omega}_d = \boldsymbol{T}(\alpha_d, \beta_d)\dot{\boldsymbol{\phi}}_d = \mathbf{0}$. At time $t = 0$, the robot configuration $\boldsymbol{q}(0)$ is such that the end-effector pose is out of the desired trajectory, i.e., there are initial errors on the task

$$\boldsymbol{e}_p(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) \neq \mathbf{0}, \qquad \boldsymbol{e}_\phi(0) = \boldsymbol{\phi}_d - \boldsymbol{\phi}(0) \neq \mathbf{0}.$$

  Define a kinematic control law for the commands $\dot{\boldsymbol{q}}_b$ and $\dot{\boldsymbol{q}}_w$ such that each component of the error vectors $\boldsymbol{e}_p(t) \in \mathbb{R}^3$ and $\boldsymbol{e}_\phi(t) \in \mathbb{R}^3$ will exponentially converge to zero in an independent and prescribed way as t increases.

In your answer to each problem, specify which relevant matrices are required to be invertible during the entire motion.

[240 minutes; open books]

Solutions

January 9, 2014

Exercise 1

The DH frame assignment is shown in Fig. 2, with the associated Table 1. Note that all constant non-zero parameters are positive, as requested.

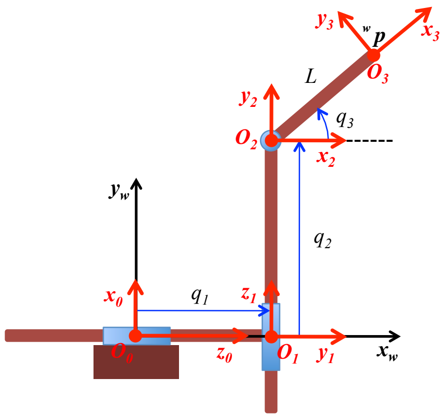

Figure 2: Assignment of Denavit-Hartenberg frames for the planar PPR robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	q_1	$\pi/2$
2	$\pi/2$	0	q_2	$\pi/2$
3	0	L	0	q_3

Table 1: Denavit-Hartenberg parameters for the planar PPR robot

The homogeneous transformation matrix between the (right-handed) frames RF_w and RF_0 is

$$
{}^{w}\boldsymbol{T}_0 = \begin{pmatrix} \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}.
$$

The primary (positional) workspace WS_1 and the secondary (dexterous) workspace WS_2 of the planar PPR robot are depicted in Fig. 3. The primary workspace (displayed in light orange) is the set of points in the plane that can be reached by the robot end-effector, independently from its orientation: it consists of the larger square with side $2(D+L)$ and smoothed corners (rounded as circles of radius L). The external boundary of WS_1 can be generated by sliding the center of a circle of radius L along the borders of the inner square having side $2D$, which is the mobility area of the tip of the second link due to the two prismatic joints. The secondary workspace (displayed in deep orange) is the smaller square of side $2(D-L)>0$: WS_2 contains only those points of WS_1 that can be reached with *all* possible orientations of the end-effector in the plane, i.e., with the third link being able to approach the point from *any* direction (which is obtained by letting q_3 vary in the whole interval $(-\pi,+\pi]$).

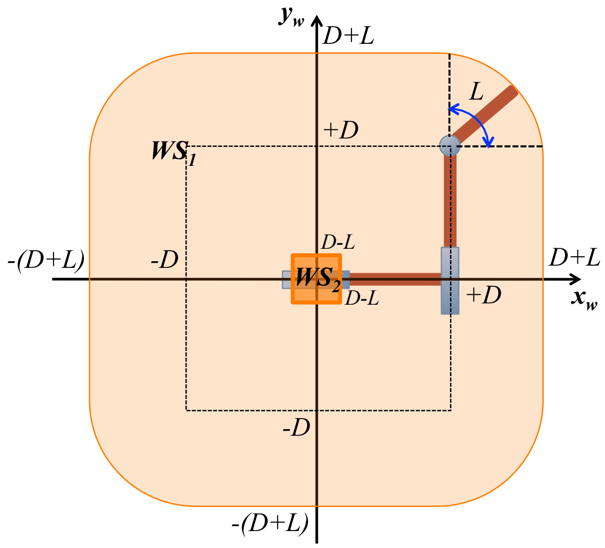

Figure 3: Primary and secondary workspaces of the planar PPR robot

By simple inspection, the direct kinematics of the end-effector position in the plane is

$$
{}^w\boldsymbol{p} = \begin{pmatrix} {}^w p_x \\ {}^w p_y \end{pmatrix} = \begin{pmatrix} q_1 + L\cos q_3 \\ q_2 + L\sin q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).
$$

This result could also be obtained (via lengthy operations) from the first two components of the last column of the matrix product ${}^w\boldsymbol{T}_0\,{}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3)$. Therefore, for a given ${}^w\boldsymbol{p} \in WS_2$, all inverse kinematics solutions can be written in parametric form as

$$
q_1 = {}^w p_x - L\cos q_3, \qquad q_2 = {}^w p_y - L\sin q_3, \qquad \forall q_3 \in (-\pi,+\pi].
$$

Exercise 2

As a preliminary step, we set up the direct and inverse formulas for the (Y,Z,Y) Euler angles $\boldsymbol{\phi}=(\alpha,\beta,\gamma)$ and then the associated differential mapping between $\dot{\boldsymbol{\phi}}$ and the angular velocity $\boldsymbol{\omega}$. Using the elementary rotation matrices

$$
\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad \boldsymbol{R}_Z(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\boldsymbol{R}_Y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix},
$$

the (Y, Z, Y) Euler rotation matrix of the direct problem is obtained as

$$\begin{aligned} \boldsymbol{R}(\alpha, \beta, \gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_Z(\beta)\boldsymbol{R}_Y(\gamma) \\ &= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\beta & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma \\ \sin\beta\cos\gamma & \cos\beta & \sin\beta\sin\gamma \\ -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\beta & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma \end{pmatrix}. \end{aligned} \tag{1}$$

For the inverse problem, let $\boldsymbol{R} = \{R_{ij}\}$ be a given rotation matrix. From the expressions of the elements in the second row of $\boldsymbol{R}(\alpha, \beta, \gamma)$, one has

$$\beta = \text{ATAN2}\left\{\pm\sqrt{R_{21}^2 + R_{23}^2}, R_{22}\right\}, \tag{2}$$

providing two values β_1 and $\beta_2 = -\beta_1$. When $R_{21}^2 + R_{23}^2 \neq 0$ (or, $\sin\beta \neq 0$), the problem is regular and for each $\beta = \beta_i$ $(i = 1, 2)$ in eq. (2) we have an associated solution

$$\alpha = \text{ATAN2}\left\{\frac{R_{32}}{\sin\beta}, \frac{-R_{12}}{\sin\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{23}}{\sin\beta}, \frac{R_{21}}{\sin\beta}\right\}. \tag{3}$$

The singular case occurs when $R_{21} = R_{23} = 0$, or $\sin\beta = 0$ (and thus also $R_{12} = R_{32} = 0$). Being $\cos\beta = \pm 1$, it is

$$\boldsymbol{R}(\alpha, \beta, \gamma)|_{\beta=\{0,\pi\}} = \begin{pmatrix} \pm\cos(\alpha \pm \gamma) & 0 & \sin(\alpha \pm \gamma) \\ 0 & \pm 1 & 0 \\ \mp\sin(\alpha \pm \gamma) & 0 & \cos(\alpha \pm \gamma) \end{pmatrix}.$$

Therefore, we can only determine the *sum* or, respectively, the *difference* of the two angles α and γ, leading to an infinite number of inverse solutions. If $R_{22} = 1$, we have

$$\beta = 0, \qquad \alpha + \gamma = \text{ATAN2}\{R_{13}, R_{33}\}.$$

If $R_{22} = -1$, we have

$$\beta = \pi, \qquad \alpha - \gamma = \text{ATAN2}\{R_{13}, R_{33}\}.$$

The differential relationship between $\dot{\boldsymbol{\phi}}$ and $\boldsymbol{\omega}$ is obtained by adding the contributions to the angular velocity of the time derivatives $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$, respectively along the directions of the rotation axes Y_0, Z_1, and Y_2, once these are expressed in the original reference frame. In particular, since the moving axes Z_1 and Y_2 are

$$Z_1 = \boldsymbol{R}_Y(\alpha)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}, \qquad Y_2 = \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_Z(\beta)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\alpha\sin\beta \\ \cos\beta \\ \sin\alpha\sin\beta \end{pmatrix},$$

we obtain[1]

$$\begin{aligned} \boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\alpha} + \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}\dot{\beta} + \begin{pmatrix} -\cos\alpha\sin\beta \\ \cos\beta \\ \sin\alpha\sin\beta \end{pmatrix}\dot{\gamma} \\ &= \begin{pmatrix} 0 & \sin\alpha & -\cos\alpha\sin\beta \\ 1 & 0 & \cos\beta \\ 0 & \cos\alpha & \sin\alpha\sin\beta \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\alpha, \beta)\,\dot{\boldsymbol{\phi}}. \end{aligned} \tag{4}$$

[1] An alternative, longer procedure would be to extract $\boldsymbol{\omega}$ from the relation $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}\boldsymbol{R}^T$, with $\boldsymbol{R} = \boldsymbol{R}(\alpha, \beta, \gamma)$ given by eq. (1).

Note that a singularity occurs when $\det \boldsymbol{T} = -\sin\beta = 0$, or $\beta = \{0, \pi\}$. Finally, using the fact that the columns of $\boldsymbol{T}$ are unit vectors (though not necessarily orthogonal to each other), from (4) it follows that

$$\|\boldsymbol{\omega}\|^2 = \boldsymbol{\omega}^T\boldsymbol{\omega} = \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2\,\dot{\alpha}\dot{\gamma}\cos\beta. \tag{5}$$

Note that $\|\boldsymbol{\omega}\|^2 \neq \|\dot{\boldsymbol{\phi}}\|^2 = \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2$.

The first step for determining a solution to the given problem is to compute the initial and final values of the (Y, Z, Y) Euler angles associated to the rotation matrices $\boldsymbol{R}_{in}$ and $\boldsymbol{R}_{fin}$. Since the inverse problem is regular for the initial and final orientation data, from eqs. (2–3) we obtain two sets of possible initial values[2]

$$(\alpha_{in,1}, \beta_{in,1}, \gamma_{in,1}) = \left(0, \frac{\pi}{4}, -\frac{\pi}{2}\right) \quad \text{or} \quad (\alpha_{in,2}, \beta_{in,2}, \gamma_{in,2}) = \left(\pi, -\frac{\pi}{4}, \frac{\pi}{2}\right) \tag{6}$$

and two sets of possible final values

$$(\alpha_{fin,1}, \beta_{fin,1}, \gamma_{fin,1}) = \left(-\frac{3\pi}{4}, \frac{\pi}{4}, -\frac{3\pi}{4}\right) \quad \text{or} \quad (\alpha_{fin,2}, \beta_{fin,2}, \gamma_{fin,2}) = \left(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}\right). \tag{7}$$

Any combination of these boundary conditions (there are four in total) can be chosen to proceed. While the computational steps are formally the same for all cases, it should be noted that the assigned change of orientation may induce larger or smaller variations of the Euler angles, depending on the chosen sets of boundary conditions. Accordingly, for a given motion time T, the angular velocities associated to the four solutions will also be different. Conversely, given the bound on the norm of the angular velocity, each solution will lead in general to a different minimum feasible motion time.

Although not explicitly requested in the text of the problem, we will determine the best among all four possible solutions, namely the one associated to the smallest minimum time T^*. In the following, the four solution trajectories will be labeled as $\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$, where the first index refers to the set of initial values used for the Euler angles and the second to the set of final values from eqs. (6–7).

For each combination of initial and final conditions, we choose cubic polynomials (with common motion time T) as interpolating trajectories for all three Euler angles. In this way, a zero angular velocity $\boldsymbol{\omega}$ (or, equivalently, a zero time derivative $\dot{\boldsymbol{\phi}}$) can also be imposed at the initial and final instants. For $t \in [0, T]$, we have the general expression

$$a(t) = a_{in} + (a_{fin} - a_{in})\left(3\,(t/T)^2 - 2\,(t/T)^3\right), \qquad \text{where } a = \{\alpha, \beta, \gamma\},$$

with

$$\dot{a}(t) = \frac{6\,(a_{fin} - a_{in})}{T}\left((t/T) - (t/T)^2\right), \qquad \ddot{a}(t) = \frac{6\,(a_{fin} - a_{in})}{T^2}\left(1 - 2\,(t/T)\right).$$

It is easy to see that, at $t = T/2$, the angular variation is the half of the total requested and the absolute value of the velocity reaches its maximum:

$$a(T/2) = a_{in} + \frac{a_{fin} - a_{in}}{2} = \frac{a_{in} + a_{fin}}{2}; \qquad |\dot{a}(T/2)| = \max_{t\in[0,T]} |\dot{a}(t)| = \frac{1.5\,|a_{fin} - a_{in}|}{T}. \tag{8}$$

Note also that, from the boundary conditions (6-7) on β, it follows that either $\beta(t)$ is constant over the whole interval of motion, and so $d = \cos\beta(T/2) = \cos(\pm\pi/4) = \sqrt{2}/2$, or $\beta(t)$ should cross zero at $t = T/2$, and so $d = \cos\beta(T/2) = 1$ is at its maximum.

[2]All angles are assumed to be defined in the interval $(-\pi, \pi]$. Indeed this is only a local representation.

motion solution	$\alpha_{fin} - \alpha_{in}$	$\beta_{fin} - \beta_{in}$	$\gamma_{fin} - \gamma_{in}$	$d = \cos\beta(T/2)$	$\dot{\alpha}(t)\dot{\gamma}(t)$
$\{1,1\}$	$-3\pi/4$	0	$-\pi/4$	$\sqrt{2}/2$	≥ 0
$\{1,2\}$	$\pi/4$	$\pi/2$	$3\pi/4$	1	≥ 0
$\{2,1\}$	$-7\pi/4$	$-\pi/2$	$-5\pi/4$	1	≥ 0
$\{2,2\}$	$-3\pi/4$	0	$-\pi/4$	$\sqrt{2}/2$	≥ 0

Table 2: Quantities used for evaluating the maximum of $\|\boldsymbol{\omega}\|$ in the solution trajectories obtained from the four possible combinations of boundary conditions in (6-7)

Based on eqs. (8) and on the formula (5), Table 2 summarizes the relevant quantities needed for evaluating the maximum of $\|\boldsymbol{\omega}\|$, as associated to the four possible trajectories for the Euler angles. It can be easily verified that in all four combinations of initial and final conditions, one has $\dot{\alpha}(t)\dot{\gamma}(t) \geq 0$ for any $t \in [0, T]$. As a result of this analysis, it can be concluded that the maximum norm of $\boldsymbol{\omega}$ is always attained at the motion midpoint, $t = T/2$, where each of the (positive) terms in the right-hand side of eq. (5) attains its maximum value. Since

$$\|\boldsymbol{\omega}(T/2)\|^2 = \left(\frac{1.5}{T}\right)^2 \left[(\alpha_{fin} - \alpha_{in})^2 + (\beta_{fin} - \beta_{in})^2 + (\gamma_{fin} - \gamma_{in})^2 + 2d\,(\alpha_{fin} - \alpha_{in})\,(\gamma_{fin} - \gamma_{in})\right],$$

the inequality $\|\boldsymbol{\omega}(T/2)\| \leq \Omega$ implies

$$T \geq \frac{1.5}{\Omega}\sqrt{(\alpha_{fin} - \alpha_{in})^2 + (\beta_{fin} - \beta_{in})^2 + (\gamma_{fin} - \gamma_{in})^2 + 2d\,(\alpha_{fin} - \alpha_{in})\,(\gamma_{fin} - \gamma_{in})}. \tag{9}$$

Imposing the equality in (9) and using $\Omega = \pi$ and the values in Tab. 2, we determine and then compare the minimum feasible motion times for all combinations of initial/final conditions. For instance, in the solution trajectory $\{1,2\}$ it is

$$T_{\{1,2\}} = \frac{1.5}{\pi}\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{2}\right)^2 + \left(\frac{3\pi}{4}\right)^2 + 2\cdot 1\cdot\left(\frac{\pi}{4}\right)\left(\frac{3\pi}{4}\right)} = 1.5\sqrt{\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + \frac{6}{16}} = \frac{3\sqrt{5}}{4}.$$

Therefore, the smallest minimum time is

$$\begin{aligned} T^* &= \min\left\{T_{\{1,1\}}, T_{\{1,2\}}, T_{\{2,1\}}, T_{\{2,2\}}\right\} \\ &= \min\left\{\frac{1.5\sqrt{10+3\sqrt{2}}}{4}, \frac{3\sqrt{5}}{4}, \frac{3\sqrt{37}}{4}, \frac{1.5\sqrt{10+3\sqrt{2}}}{4}\right\} \\ &\approx \min\{1.4152, 1.6771, 4.5621, 1.4152\} = 1.4152 \text{ [s]}, \end{aligned}$$

which is attained with the solution trajectory $\{1,1\}$ as well as with $\{2,2\}$.

Choosing for instance the solution $\{1,1\}$ leads to the following trajectories for the Euler angles, with $t \in [0, T^*]$:

$$\begin{aligned} \alpha(t) &= \alpha_{in,1} + (\alpha_{fin,1} - \alpha_{in,1})\left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right) \\ &= -\frac{3\pi}{4}\left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right) \end{aligned}$$

$$\begin{aligned}
\beta(t) &= \beta_{in,1} + (\beta_{fin,1} - \beta_{in,1}) \left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right) \\
&= \frac{\pi}{4} \\
\gamma(t) &= \gamma_{in,1} + (\gamma_{fin,1} - \gamma_{in,1}) \left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right) \\
&= -\frac{\pi}{2} - \frac{\pi}{4} \left(3\,(t/T^*)^2 - 2\,(t/T^*)^3\right).
\end{aligned}$$

Figure 4 shows the evolution of the Euler angles in the minimum time solution $\{1,1\}$. Note that in this case $\beta(t)$ is kept constant at the value $\pi/4$ [rad]. The maximum absolute velocity is attained by the angle $\alpha(t)$ at the trajectory midpoint ($\dot{\alpha}(T^*/2) = -2.5$ [rad/s]). In Fig. 5, the plot of the associated $\|\boldsymbol{\omega}\|$, computed from eq. (5), shows that the given bound $\Omega = \pi$ [rad] is never violated and reached only at the trajectory midpoint $t = T^*/2$, as predicted by our analysis. For comparison, the evolution of the Euler angles in the alternative solution $\{1,2\}$ are shown in Fig. 6. All angles will move in this case, while the minimum feasible motion time T_{12} is about 18% longer than the optimal T^*. The maximum absolute velocity is attained here by the angle $\gamma(t)$ ($\dot{\gamma}(T_{12}/2) \approx 2.15$ [rad/s]). Indeed, also in this case the associated $\|\boldsymbol{\omega}\|$ (not reported) remains always feasible and reaches the bound $\Omega = \pi$ only at the trajectory midpoint.

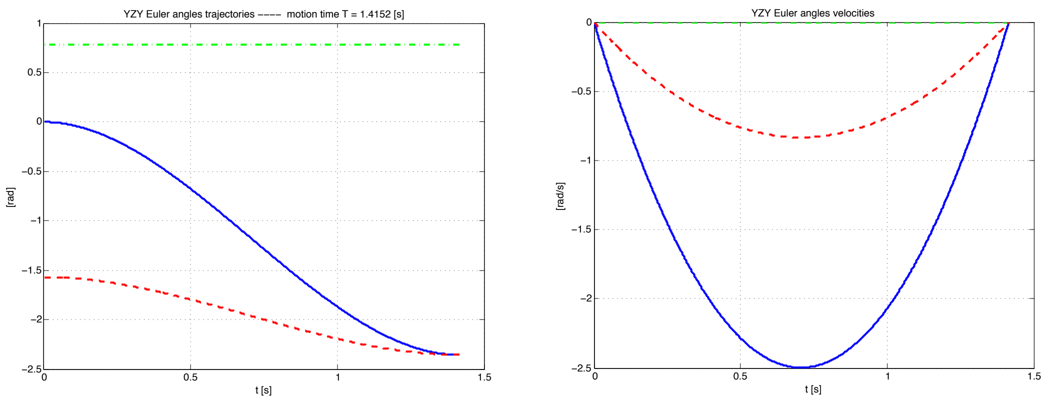

Figure 4: Trajectories (left) and velocities (right) of the Euler angles in the minimum time solution $\{1,1\}$: $\alpha(t)$ and $\dot{\alpha}(t)$ (blue, solid), $\beta(t)$ and $\dot{\beta}(t)$ (green, dashdot), $\gamma(t)$ and $\dot{\gamma}(t)$ (red, dashed)

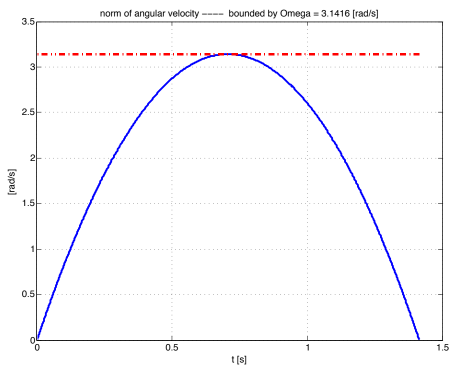

Figure 5: Norm of the angular velocity $\boldsymbol{\omega}$ in the minimum time solution $\{1,1\}$

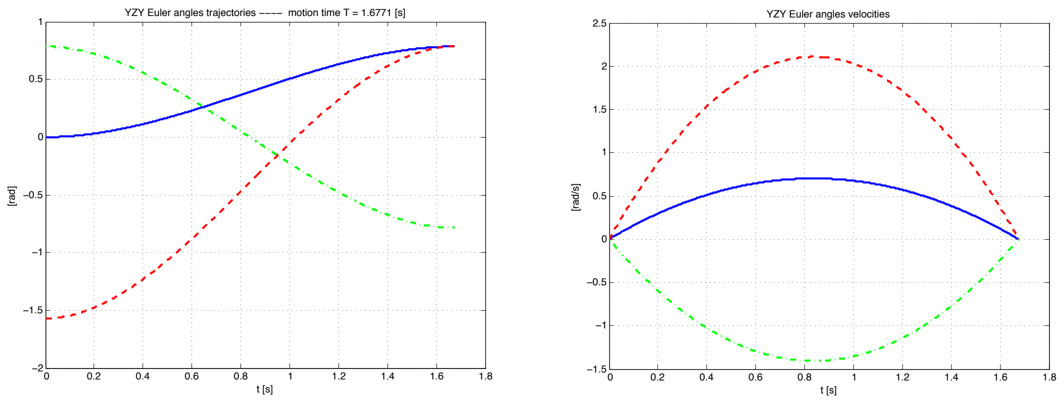

Figure 6: Trajectories (left) and velocities (right) of the Euler angles in the alternative solution $\{1,2\}$: $\alpha(t)$ and $\dot\alpha(t)$ (blue, solid), $\beta(t)$ and $\dot\beta(t)$ (green, dashdot), $\gamma(t)$ and $\dot\gamma(t)$ (red, dashed)

Exercise 3

For a 6R manipulator with a spherical wrist, the (6×6) geometric Jacobian that relates the joint velocity vector $\dot{\boldsymbol{q}}$ to the linear and angular velocity of the end-effector at a configuration $\boldsymbol{q}$ can be written, under the given assumptions, in the partitioned way

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{11}(\boldsymbol{q}) & \boldsymbol{O} \\ \boldsymbol{J}_{21}(\boldsymbol{q}) & \boldsymbol{J}_{22}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}}_b \\ \dot{\boldsymbol{q}}_w \end{pmatrix}.$$

Assuming that the Jacobian is non-singular at $\boldsymbol{q}$ (i.e., that both (3×3) diagonal blocks $\boldsymbol{J}_{11}$ and $\boldsymbol{J}_{22}$ are invertible) and dropping dependencies, we have

$$\begin{pmatrix} \dot{\boldsymbol{q}}_b \\ \dot{\boldsymbol{q}}_w \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{11}^{-1} & \boldsymbol{O} \\ -\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\boldsymbol{J}_{11}^{-1} & \boldsymbol{J}_{22}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix}.$$

Thus, by setting $\boldsymbol{v} = \boldsymbol{v}_d \neq \boldsymbol{0}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_d = \boldsymbol{0}$, we have

$$\begin{aligned} \dot{\boldsymbol{q}}_b &= \boldsymbol{J}_{11}^{-1}\boldsymbol{v}_d \\ \dot{\boldsymbol{q}}_w &= -\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b = -\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\boldsymbol{J}_{11}^{-1}\boldsymbol{v}_d. \end{aligned}$$

In the presence of errors on both the positional (linear) and orientation (angular) task quantities, a suitable kinematic control law can be defined as

$$\begin{aligned} \dot{\boldsymbol{q}}_b &= \boldsymbol{J}_{11}^{-1}(\boldsymbol{q})\left(\boldsymbol{v}_d + \boldsymbol{K}_p\left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\right)\right) \\ \dot{\boldsymbol{q}}_w &= -\boldsymbol{J}_{22}^{-1}(\boldsymbol{q})\boldsymbol{J}_{21}(\boldsymbol{q})\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}^{-1}(\boldsymbol{q})\boldsymbol{T}(\alpha(\boldsymbol{q}),\beta(\boldsymbol{q}))\boldsymbol{K}_\phi\left(\boldsymbol{\phi}_d - \boldsymbol{\phi}(\boldsymbol{q})\right), \end{aligned} \tag{10}$$

where the two gain matrices $\boldsymbol{K}_p$ and $\boldsymbol{K}_\phi$ are positive definite and diagonal, matrix $\boldsymbol{T}$ relates the time derivative $\dot{\boldsymbol{\phi}}$ of the chosen set of Euler angles to the angular velocity $\boldsymbol{\omega}$, and the functional expressions $\boldsymbol{p}(\boldsymbol{q})$ and $\boldsymbol{\phi}(\boldsymbol{q}) = (\alpha(\boldsymbol{q}),\beta(\boldsymbol{q}),\gamma(\boldsymbol{q}))$ are given by the appropriate direct kinematics mappings.

It is easy to verify that the evolutions of the errors

$$\dot{\boldsymbol{e}}_p = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}} = \boldsymbol{v}_d - \boldsymbol{J}_{11}\dot{\boldsymbol{q}}_b = \boldsymbol{v}_d - \boldsymbol{J}_{11}\boldsymbol{J}_{11}^{-1}\left(\boldsymbol{v}_d + \boldsymbol{K}_p\boldsymbol{e}_p\right) = -\boldsymbol{K}_p\boldsymbol{e}_p$$

and

$$
\begin{aligned}
\dot{\boldsymbol{e}}_\phi = \dot{\boldsymbol{\phi}}_d - \dot{\boldsymbol{\phi}} = -\boldsymbol{T}^{-1}\boldsymbol{\omega} &= -\boldsymbol{T}^{-1}\left(\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}\dot{\boldsymbol{q}}_w\right) \\
&= -\boldsymbol{T}^{-1}\left(\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}\left(-\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}\dot{\boldsymbol{q}}_b + \boldsymbol{J}_{22}^{-1}\boldsymbol{T}\boldsymbol{K}_\phi\boldsymbol{e}_\phi\right)\right) = -\boldsymbol{K}_\phi\boldsymbol{e}_\phi
\end{aligned}
$$

are exponentially converging to zero with a rate prescribed by the elements of the gain matrices $\boldsymbol{K}_p$ and $\boldsymbol{K}_\phi$, as desired. The independent behavior of the error components is enforced by the choice of diagonal gain matrices. The non-singularity of the blocks $\boldsymbol{J}_{11}$ and $\boldsymbol{J}_{22}$ in the Jacobian matrix is again required for the feasibility of the kinematic control law (10). On the other hand, since matrix $\boldsymbol{T}$ needs not to be inverted in the law (10), its possible rank deficiencies will not lead the control command to grow unbounded. Nonetheless, $\boldsymbol{T}$ should remain invertible (during the transient behavior and along the nominal desired trajectory) in order to guarantee a trajectory tracking behavior with all the features requested in the problem formulation.

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