Robotics I

June 6, 2017

Exercise 1

Consider the planar PRPR manipulator in Fig. 1. The joint variables defined therein are those used by the manufacturer and do not correspond necessarily to a Denavit-Hartenberg (DH) convention. The blue arrow indicates in this case the *positive* increments of the joint variable.
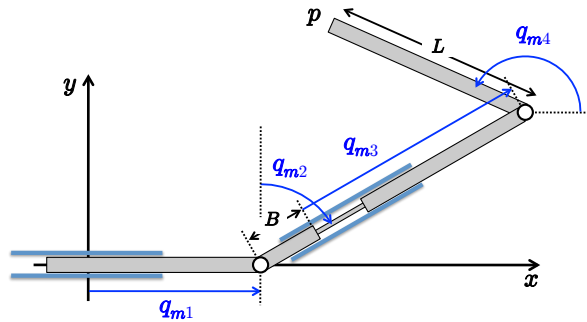

Figure 1: A planar PRPR manipulator with the joint variables $\boldsymbol{q}_m = (q_{m1}, q_{m2}, q_{m3}, q_{m4})$ defined by the manufacturer.

- Determine the direct kinematics $\boldsymbol{f}_m(\boldsymbol{q}_m)$ for the planar position $\boldsymbol{p} \in \mathbb{R}^2$ of the end-effector w.r.t. the given $(\boldsymbol{x}, \boldsymbol{y})$ reference frame, using the joint variables $\boldsymbol{q}_m$ provided by the manufacturer.

- Assign a set of frames according to the DH convention, draw them on the manipulator, and compile the associated table of DH parameters. Choose the origin of frame 0 coincident with that of the manufacturer.

- Determine again the direct kinematics $\boldsymbol{f}(\boldsymbol{q})$ for the planar position $\boldsymbol{p} \in \mathbb{R}^2$ of the end-effector w.r.t. the DH frame 0, using now the joint variables $\boldsymbol{q}$ of your DH notation.

- Draw the manipulator in the two *zero* configurations: the one associated with $\boldsymbol{q}_m = \boldsymbol{0}$ and the one with $\boldsymbol{q} = \boldsymbol{0}$.

- Establish the transformation $\boldsymbol{\phi}(\cdot)$ needed to map one set of variables into the other, say $\boldsymbol{q}_m \to \boldsymbol{q}$, so that for any given $\boldsymbol{q}_m$ the transformed $\boldsymbol{q} = \boldsymbol{\phi}(\boldsymbol{q}_m)$ describes the same physical configuration of the manipulator (in particular, $\boldsymbol{f}_m(\boldsymbol{q}_m) = \boldsymbol{f}(\boldsymbol{\phi}(\boldsymbol{q}_m))$).

- Suppose that a joint velocity command $\dot{\boldsymbol{q}}$ has been determined using routines and control software that operates with the DH variables $\boldsymbol{q}$. What velocity command $\dot{\boldsymbol{q}}_m$ should be transferred to the low-level controller of the robot manufacturer in order to obtain the expected motion behavior?

Exercise 2

With reference to Fig. 2, a laser sensor placed at the base of a 3R robot is monitoring an angular sector of the workspace, scanning it on a horizontal plane at a given height, with minimum and maximum radial sensing distance ρ_{min} and ρ_{max}, respectively. When a human presence is detected by the laser sensor and localized by vector $\boldsymbol{r}_{human}$, the robot end-effector should be controlled for safety, so as to: either $i)$ retract with a velocity $\boldsymbol{v}_e^{[i]}$ along the horizontal and centripetal direction of the workspace, with a scalar speed which is inversely proportional to the distance $\|\boldsymbol{r}_{human}\|$; or $ii)$ be repulsed with a velocity $\boldsymbol{v}_e^{[ii]}$ along an horizontal direction and away from the detected human, with a scalar speed which is inversely proportional to the relative distance between the human and the robot end-effector. The two alternative reactions occur only when $\|\boldsymbol{r}_{human}\| \in [\rho_{min}, \rho_{max}]$. Beyond ρ_{max} the robot continues its original task, as specified by a joint velocity $\dot{\boldsymbol{q}}_{task}$. Finally, when the human is closer than ρ_{min} to the robot base, the robot should simply stop.

Provide the symbolic expression of the commanded joint velocity $\dot{\boldsymbol{q}}$ that realizes these human-robot coexistence control rules in all the considered cases.
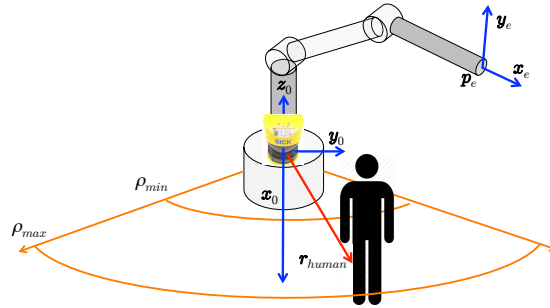

Figure 2: Human-robot coexistence, with a laser scanner measuring the position of a human w.r.t. the base of a spatial 3R robot.

Exercise 3

A generic revolute joint of a robot should move through the sequence of four knots specified by the angular values
$$q_1 = 70°, \quad q_2 = -45°, \quad q_3 = 10°, \quad q_4 = 100°.$$
Define an interpolating path in the form of a cubic spline $q = q(s) \in C^2$, for $s \in [0, 1]$, such that at $s_1 = 0$, $s_2 = 0.25$, $s_3 = 0.5$, and $s_4 = 1$, we have
$$q(0) = q_1, \quad q(0.25) = q_2, \quad q(0.5) = q_3, \quad q(1) = q_4.$$
The path tangent $q'(s)$ at the initial and final knots is given by $(q_2 - q_1)/(s_2 - s_1)$ and $(q_4 - q_3)/(s_4 - s_3)$, respectively.

- Provide the parametric expression and the associated numerical values of the coefficients for the three cubic polynomials in the spline solution.
- Assuming that the path is executed at a constant speed $\dot{s} = 0.4$/s:
  - what is the total traveling time T from q_1 to q_4?
  - which is the value of the joint velocity at the initial and final time, $\dot{q}(0)$ and $\dot{q}(T)$?
  - what is the maximum absolute value of the acceleration $\ddot{q}(t)$ reached during the time interval $[0, T]$, and which is the time instant $t = t_a$ at which this maximum occurs?

[240 minutes, open books but no computer or smartphone]

Solution

June 6, 2017

Exercise 1

A frame assignment for the PRPR manipulator which is consistent with the Denavit-Hartenberg (DH) convention is shown in Fig. 3, and the associated parameters are given in Tab. 1.
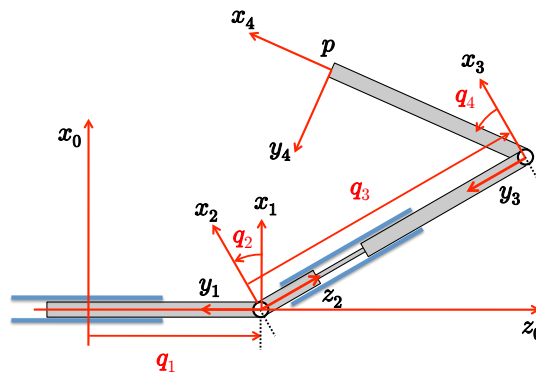

Figure 3: The DH frames assigned to the planar PRPR manipulator, with associated joint variables.

i	α_i	d_i	a_i	θ_i
1	$-\pi/2$	q_1	0	0
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	q_3	0	0
4	0	0	L	q_4

Table 1: Table of DH parameters of the planar PRPR manipulator.

Using the joint variables $\boldsymbol{q}_m = (q_{m1}, q_{m2}, q_{m3}, q_{m4})$ of the manufacturer, by geometric inspection of Fig. 1 the direct kinematics of the PRPR manipulator is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_{m1} + (B + q_{m3})\sin q_{m2} + L\cos q_{m4} \\ (B + q_{m3})\cos q_{m2} + L\sin q_{m4} \end{pmatrix} = \boldsymbol{f}_m(\boldsymbol{q}_m). \tag{1}$$

Similarly, using the D-H variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$, by geometric inspection of Fig. 3 (or, computing the sequence of matrix/vector products ${}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\left({}^3\boldsymbol{A}_4(q_4)\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T\right)\right)\right)$ from Tab. 1, and then extracting the $\boldsymbol{z}_0$ and $\boldsymbol{x}_0$ components) we have

$$\boldsymbol{p} = \begin{pmatrix} {}^0p_z \\ {}^0p_x \end{pmatrix} = \begin{pmatrix} q_1 + q_3\cos q_2 - L\sin(q_2 + q_4) \\ q_3\sin q_2 + L\cos(q_2 + q_4) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{2}$$

The zero configurations of the robot arm when $\boldsymbol{q}_m = \boldsymbol{0}$ and when $\boldsymbol{q} = \boldsymbol{0}$ are obviously different

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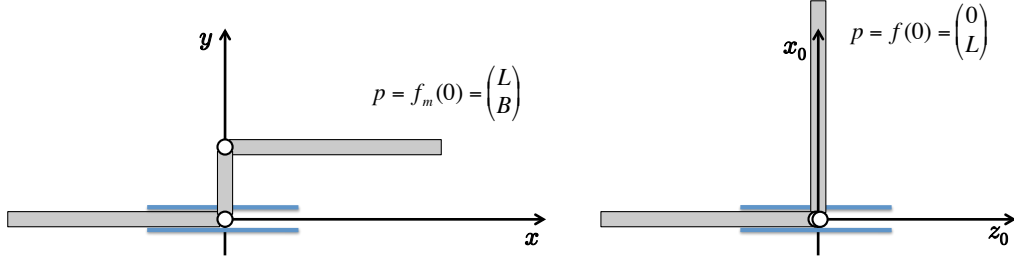
Figure 4: Zero configurations of the robot arm: when $\boldsymbol{q}_m = \boldsymbol{0}$ (left) and when $\boldsymbol{q} = \boldsymbol{0}$ (right).

and are shown in Fig. 4. By comparing eqs. (1) and (2), we find the direct transformation

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} q_{m1} \\ \pi/2 - q_{m2} \\ B + q_{m3} \\ q_{m2} + q_{m4} - \pi \end{pmatrix} = \boldsymbol{\phi}(\boldsymbol{q}_m). \tag{3}$$

From the inverse transformation, we obtain also the linear mapping from the DH velocities $\dot{\boldsymbol{q}}$ to the robot joint velocities $\dot{\boldsymbol{q}}_m$ used by the manufacturer:

$$\boldsymbol{q}_m = \boldsymbol{\phi}^{-1}(\boldsymbol{q}) = \begin{pmatrix} q_1 \\ \pi/2 - q_2 \\ q_3 - B \\ q_2 + q_4 + \pi/2 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}. \tag{4}$$

Exercise 2

Define the projections along the $\boldsymbol{z}$ axis (on an arbitrary plane, say $\boldsymbol{z} = \boldsymbol{0}$) for the two vectors

$$\boldsymbol{p}_e = \begin{pmatrix} p_{e,x} \\ p_{e,y} \\ p_{e,z} \end{pmatrix} \quad \Rightarrow \quad \bar{\boldsymbol{p}}_e = \Pi_z(\boldsymbol{p}_e) = \begin{pmatrix} p_{e,x} \\ p_{e,y} \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{r}_{human} = \begin{pmatrix} r_{human,x} \\ r_{human,y} \\ h \end{pmatrix} \quad \Rightarrow \quad \bar{\boldsymbol{r}}_{human} = \Pi_z(\boldsymbol{r}_{human}) = \begin{pmatrix} r_{human,x} \\ r_{human,y} \\ 0 \end{pmatrix},$$

where h is the (arbitrary) height of the laser scanning plane. The joint velocity command for the requested robot motion is then given by the following cases:

$$\dot{\boldsymbol{q}} = \begin{cases} \boldsymbol{0}, & \text{if } \|\bar{\boldsymbol{r}}_{human}\| < \rho_{min} \\ \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}_e, \text{ with } \boldsymbol{v}_e = \begin{cases} \boldsymbol{v}_e^{[i]} = -\dfrac{\bar{\boldsymbol{p}}_e}{\|\bar{\boldsymbol{p}}_e\|} \cdot \dfrac{1}{\|\bar{\boldsymbol{r}}_{human}\|} \\ \\ \boldsymbol{v}_e^{[ii]} = \dfrac{\bar{\boldsymbol{p}}_e - \bar{\boldsymbol{r}}_{human}}{\|\bar{\boldsymbol{p}}_e - \bar{\boldsymbol{r}}_{human}\|^2} \end{cases}, & \text{if } \|\bar{\boldsymbol{r}}_{human}\| \in [\rho_{min}, \rho_{max}] \\ \dot{\boldsymbol{q}}_{task}, & \text{if } \|\bar{\boldsymbol{r}}_{human}\| > \rho_{max}. \end{cases} \tag{5}$$

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Exercise 3

Using path parametrization, the three cubic tracts of the interpolating spline are conveniently defined as

$$q_A(\sigma_A) = q_1 + a_1\sigma_A + a_2\sigma_A^2 + a_3\sigma_A^3, \qquad \sigma_A = \frac{s - s_1}{s_2 - s_1} \in [0,1], \quad s \in [s_1, s_2] \qquad (6)$$

$$q_B(\sigma_B) = q_2 + b_1\sigma_B + b_2\sigma_B^2 + b_3\sigma_B^3, \qquad \sigma_B = \frac{s - s_2}{s_3 - s_2} \in [0,1], \quad s \in [s_2, s_3] \qquad (7)$$

$$q_C(\sigma_C) = q_3 + c_1\sigma_C + c_2\sigma_C^2 + c_3\sigma_C^3, \qquad \sigma_C = \frac{s - s_3}{s_4 - s_3} \in [0,1], \quad s \in [s_3, s_4], \qquad (8)$$

with the nine coefficients $a_1, \ldots, c_3$ determined by satisfying the nine boundary conditions

$$
\begin{aligned}
q_A(1) &= q_2, \\
q_B(1) &= q_3, \\
q_C(1) &= q_4,
\end{aligned}
\qquad
\begin{aligned}
q_A'(0) &= \frac{q_2 - q_1}{s_2 - s_1}, \\
q_C'(1) &= \frac{q_4 - q_3}{s_4 - s_3},
\end{aligned}
\qquad
\begin{aligned}
q_A'(1) &= q_B'(0)\,[=v_2], \\
q_B'(1) &= q_C'(0)\,[=v_3],
\end{aligned}
\qquad
\begin{aligned}
q_A''(1) &= q_B''(0), \\
q_B''(1) &= q_C''(0).
\end{aligned}
\qquad (9)
$$

However, the *divide et impera* approach is more convenient if we assume, for the time being, that we know the value of the first derivatives v_2 and v_3 at the two intermediate knots (at $s = s_2$ and $s = s_3$, respectively). The coefficients of each of the three cubic polynomials would then be completely defined by the four local boundary conditions at the two extremes of their interval of definition. Performing computations for the cubic A yields the coefficients

$$a_1 = q_2 - q_1, \qquad a_2 = (q_2 - q_1) - v_2(s_2 - s_1), \qquad a_3 = v_2(s_2 - s_1) - (q_2 - q_1), \qquad (10)$$

and thus

$$q_A''(1) = \frac{2a_2 + 6a_3}{(s_2 - s_1)^2} = \frac{4v_2}{s_2 - s_1} - \frac{4(q_2 - q_1)}{(s_2 - s_1)^2}. \qquad (11)$$

Similarly, for the cubic B

$$b_1 = v_2(s_3 - s_2), \quad b_2 = 3(q_3 - q_2) - (2v_2 + v_3)(s_3 - s_2), \quad b_3 = -2(q_3 - q_2) + (v_2 + v_3)(s_3 - s_2), \quad (12)$$

and thus

$$q_B''(0) = \frac{2b_2}{(s_3 - s_2)^2} = \frac{6(q_3 - q_2)}{(s_3 - s_2)^2} - \frac{4v_2 + 2v_3}{s_3 - s_2} \qquad (13)$$

and

$$q_B''(1) = \frac{2b_2 + 6b_3}{(s_3 - s_2)^2} = \frac{2v_2 + 4v_3}{s_3 - s_2} - \frac{6(q_3 - q_2)}{(s_3 - s_2)^2}. \qquad (14)$$

Finally, for the cubic C

$$c_1 = v_3(s_4 - s_3), \qquad c_2 = 2(q_4 - q_3) - 2v_3(s_4 - s_3), \qquad c_3 = v_3(s_4 - s_3) - (q_4 - q_3), \qquad (15)$$

and thus

$$q_C''(0) = \frac{2c_2}{(s_4 - s_3)^2} = \frac{4(q_4 - q_3)}{(s_4 - s_3)^2} - \frac{4v_3}{s_4 - s_3}. \qquad (16)$$

Imposing the continuity of the second derivative at the internal knots

$$q_A''(1) = q_B''(0) \qquad q_B''(1) = q_C''(0),$$

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and using eqs. (11), (13–14) and (16), leads to the linear system of equations

$$A \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = b$$

with

$$A = \begin{pmatrix} 4(s_3 - s_1) & 2(s_2 - s_1) \\ 2(s_4 - s_3) & 4(s_4 - s_2) \end{pmatrix}, \qquad b = \begin{pmatrix} 6(q_3 - q_2)\dfrac{s_2 - s_1}{s_3 - s_2} + 4(q_2 - q_1)\dfrac{s_3 - s_2}{s_2 - s_1} \\[2mm] 4(q_4 - q_3)\dfrac{s_3 - s_2}{s_4 - s_3} + 6(q_3 - q_2)\dfrac{s_4 - s_3}{s_3 - s_2} \end{pmatrix}.$$

Replacing the numerical data, the system is solved as

$$\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = A^{-1}b = \begin{pmatrix} -147.2727 \\ 329.0909 \end{pmatrix},$$

and the coefficients (10), (12), and (15) of the three cubic polynomials take then the numerical values

$$a_0 = q_1 = 70, \quad a_1 = -115, \quad a_2 = -78.1818, \quad a_3 = 78.1818,$$

$$b_0 = q_2 = -45, \quad b_1 = -36.8182, \quad b_2 = 156.3636, \quad b_3 = -64.5455,$$

$$c_0 = q_3 = 10, \quad c_1 = 164.5455, \quad c_2 = -149.0909, \quad c_3 = 74.5455.$$

The plots of the interpolating geometric spline $q(s)$, for $s \in [0, 1]$, and of its first and second derivatives $q'(s)$ and $q''(s)$ are shown in Fig. 5.

If the motion on the geometric path $q(s)$ is at constant speed $\dot{s} = k = 0.4$ s^{-1} ($\ddot{s} = 0$), then the total time for tracing the interval between $s = 0$ and $s = 1$ is simply $T = 1/k = 1/0.4 = 2.5$ s. Moreover, velocity and acceleration of the joint moving at constant speed are $\dot{q} = q' \dot{s} = q' k$ and $\ddot{q} = q' \ddot{s} + q'' \dot{s}^2 = q'' k^2$, respectively. Therefore, the initial and final velocity will be

$$\dot{q}(0) = q'_A(0)\,\dot{s}(0) = \frac{q_2 - q_1}{s_2 - s_1} k = -184°/\text{s}, \qquad \dot{q}(T) = q'_C(1)\,\dot{s}(T) = \frac{q_4 - q_3}{s_4 - s_3} k = 72°/\text{s}.$$

On the other hand, since the second derivative of a geometric cubic spline is made by linear segments between the knots, its maximum necessarily occurs in one of the knots. Being the joint acceleration simply a scaled version of the second spatial derivative, the maximum (absolute) value of the acceleration is found as

$$\max_{t \in [0,T]} |\ddot{q}(t)| = k^2 \max_{s \in [0,1]} |q''(s)| = k^2 \max \{|q''_A(0)|, |q''_B(0)|, |q''_C(0)|, |q''_C(1)|\}$$

$$= 0.16 \max \{|-2501.8|, |5003.6|, |-1192.7|, |596.3|\}$$

$$= \max \{|-400.2909|, |800.5818|, |-190.8364|, |95.4182|\} = 800.5818°/\text{s}^2,$$

occurring at the second knot. The plots of the interpolating trajectory $q(t)$, for $t \in [0, T]$, and of its velocity $\dot{q}(t)$ and acceleration $\ddot{q}(t)$ are shown in Fig. 6.
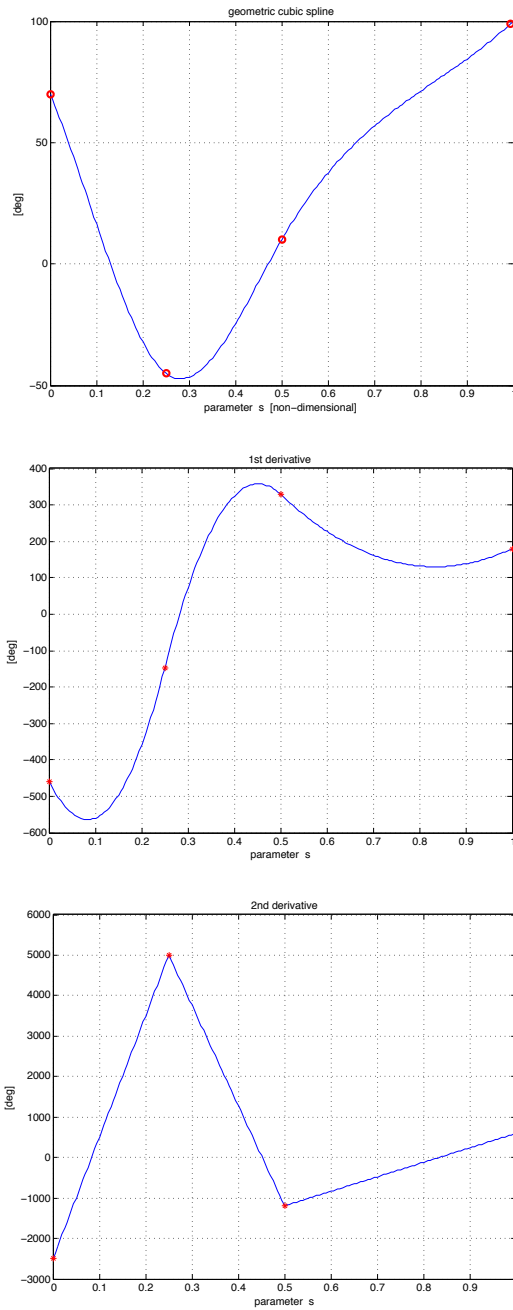
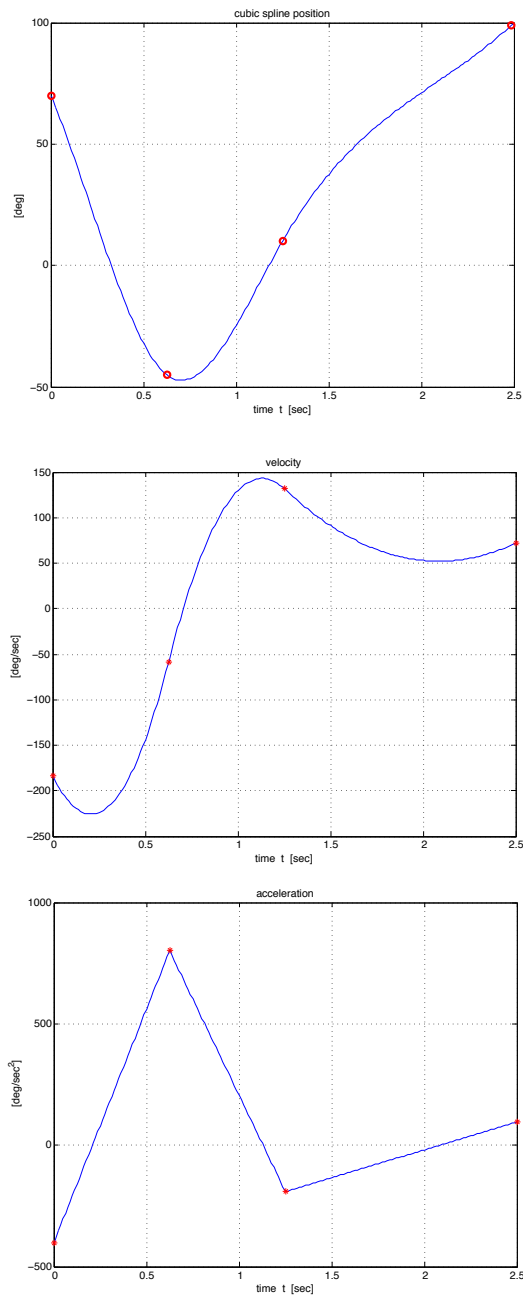Figure 5: Geometric spline $q(s)$, with first and second derivative.

Figure 6: Spline trajectory $q(t)$, with velocity and acceleration.

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