Robotics I

June 11, 2018

Exercise 1

Consider the planar 2R robot in Fig. 1, having a L-shaped second link. A frame RF_e is attached to the gripper mounted on the robot end effector.

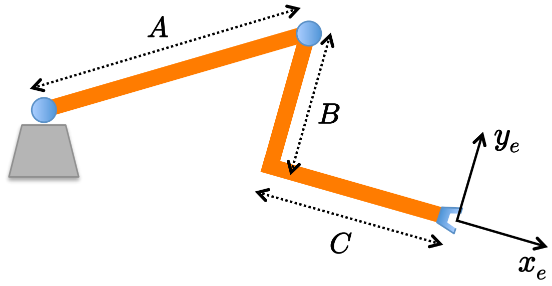

Figure 1: A planar 2R robot with L-shaped second link.

- Assign the link frames and define the joint variables $\boldsymbol{q}$ according to the Denavit-Hartenberg (DH) convention, and complete the associated table of parameters.
- Determine the robot direct kinematics, specifying the position $\boldsymbol{p}_e(\boldsymbol{q}) \in \mathbb{R}^3$ of the origin of frame RF_e and the end-effector orientation as expressed by the rotation matrix ${}^0\boldsymbol{R}_e(\boldsymbol{q}) \in SO(3)$.
- Sketch the robot in the zero configuration ($\boldsymbol{q} = \boldsymbol{0}$); find and sketch also the configuration $\boldsymbol{q}_s$ where the robot gripper is pointing in the direction $\boldsymbol{y}_0$ and is the farthest away from axis $\boldsymbol{x}_0$.
- Draw the primary workspace of the robot using the symbolic values A, B and C for the lengths.
- Provide all closed-form solutions to the inverse kinematics problem, when the end-effector position $\bar{\boldsymbol{p}}_e \in \mathbb{R}^2$ (i.e., reduced to the plane of motion) is given as input.
- Derive the 2×2 Jacobian $\boldsymbol{J}$ in
$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$
where $\boldsymbol{v}_e = \dot{\bar{\boldsymbol{p}}}_e \in \mathbb{R}^2$ is the linear velocity of the end-effector in the plane.
- Determine all singular configurations of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$. In each of these singularities, provide a basis for the null space and the range space of $\boldsymbol{J}$.
- Using the numerical data
$$A = 0.6, \qquad B = 0.3, \qquad C = 0.4 \quad [\text{m}] \tag{1}$$
  - find all joint configurations $\boldsymbol{q}_{sol}$ associated to the end-effector position $\bar{\boldsymbol{p}}_e = (0.6 \;\; -0.5)^T$ [m];
  - for each $\boldsymbol{q}_{sol}$, compute the joint velocity $\dot{\boldsymbol{q}}_{sol}$ that realizes the velocity $\boldsymbol{v}_e = (1 \;\; 0)^T$ [m/s]; are these joint velocities equal or different in norm? why?

Exercise 2

For the robot in Exercise 1, find the minimum time rest-to-rest motion between $\boldsymbol{q}_0 = (1 \;\; -0.5)^T$ and $\boldsymbol{q}_f = (0 \;\; 0.2)^T$ [rad], when the joint velocities and accelerations are subject to the bounds

$$\begin{aligned} |\dot{q}_1| \le V_1 = 0.5 \text{ [rad/s]}, \qquad & |\ddot{q}_1| \le A_1 = 0.8 \text{ [rad/s}^2\text{]}, \\ |\dot{q}_2| \le V_2 = 0.8 \text{ [rad/s]}, \qquad & |\ddot{q}_2| \le A_2 = 0.5 \text{ [rad/s}^2\text{]}. \end{aligned} \tag{2}$$

Draw accurately the minimum time velocity profiles of the two joints, when a coordinated motion is also required.

[180 minutes, open books but no computer or smartphone]

Solution

June 11, 2018

Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1, respectively.

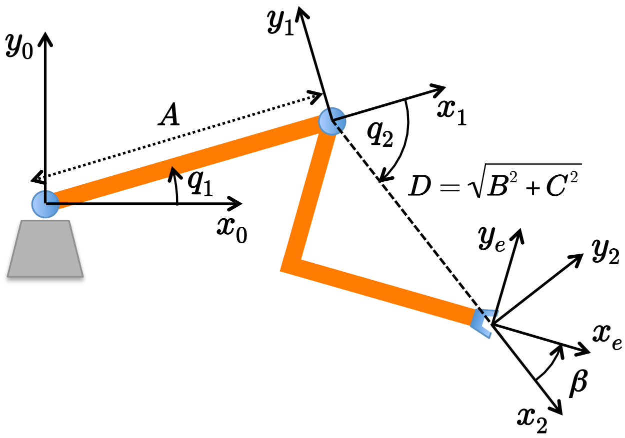

Figure 2: DH frame assignment for the planar 2R robot with a L-shaped second link.

i	α_i	a_i	d_i	θ_i
1	0	A	0	q_1
2	0	D	0	q_2

Table 1: Parameters associated to the DH frames in Fig. 2.

Based on Tab. 1, the DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & A\cos q_1 \\ \sin q_1 & \cos q_1 & 0 & A\sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1(q_1) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & D\cos q_2 \\ \sin q_2 & \cos q_2 & 0 & D\sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2(q_2) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

where $D = \sqrt{B^2 + C^2} > 0$. Moreover, the constant homogeneous transformation between the (last) DH frame RF_2 and the end-effector frame RF_e is given by

$$
{}^2\boldsymbol{T}_e = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_e & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
$$

with $\beta = \arctan(B/C) > 0$. The requested direct kinematics is given by

$$
\boldsymbol{p}_e(\boldsymbol{q}) = {}^0\boldsymbol{p}_e(\boldsymbol{q}) = {}^0\boldsymbol{p}_1(q_1) + {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{p}_2(q_2) = \begin{pmatrix} A\cos q_1 + D\cos(q_1+q_2) \\ A\sin q_1 + D\sin(q_1+q_2) \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{p}}_e(\boldsymbol{q}) \\ 0 \end{pmatrix} \tag{3}
$$

and

$$
{}^0\boldsymbol{R}_e(\boldsymbol{q}) = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_e = \begin{pmatrix} \cos(q_1+q_2+\beta) & -\sin(q_1+q_2+\beta) & 0 \\ \sin(q_1+q_2+\beta) & \cos(q_1+q_2+\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}
$$

It is evident from (3) that the kinematics of this robot is the same as that of a standard planar 2R arm, once we use the value D as length of an equivalent straight second link. On the other hand, the orientation of the end-effector frame is affected by the constant bias angle β, see (4). A sketch of the robot in the zero configuration is given in Fig. 3a, while Fig. 3b shows the requested configuration $\boldsymbol{q}_s = (\pi/2, -\arctan(B/C))$, with the gripper pointing in the $\boldsymbol{y}_0$ direction and placed the farthest away (at a distance $A + C$) from the $\boldsymbol{x}_0$ axis. The primary workspace of the robot is drawn in Fig. 4, with inner radius and outer radius given, respectively, by

$$
r = |A - D|\,, \qquad R = A + D.
$$

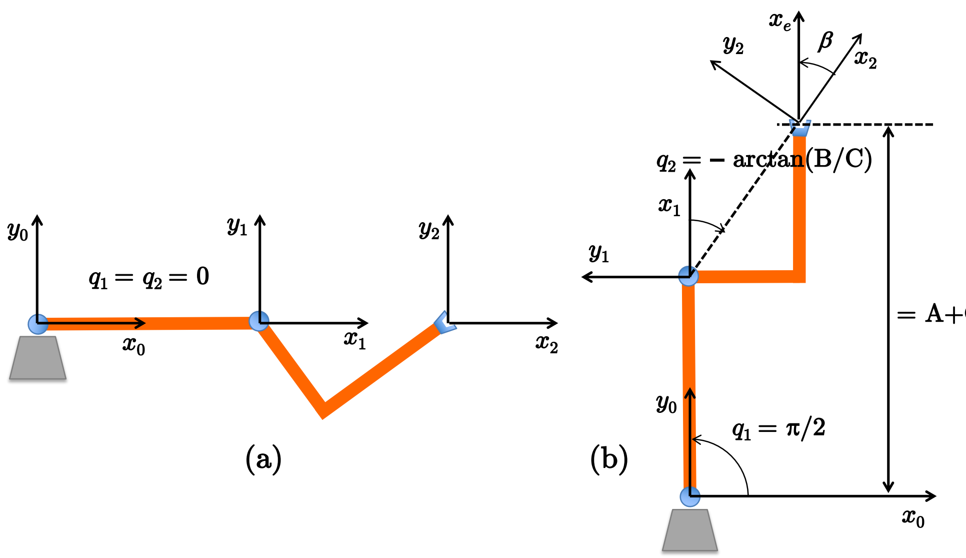

Figure 3: Two special robot postures: (a) the configuration $\boldsymbol{q} = \boldsymbol{0}$, and (b) the configuration $\boldsymbol{q}_s = (\pi/2, -\arctan(B/C))$, in which the robot gripper points in the direction $\boldsymbol{y}_0$ while being the farthest away from the base axis $\boldsymbol{x}_0$.

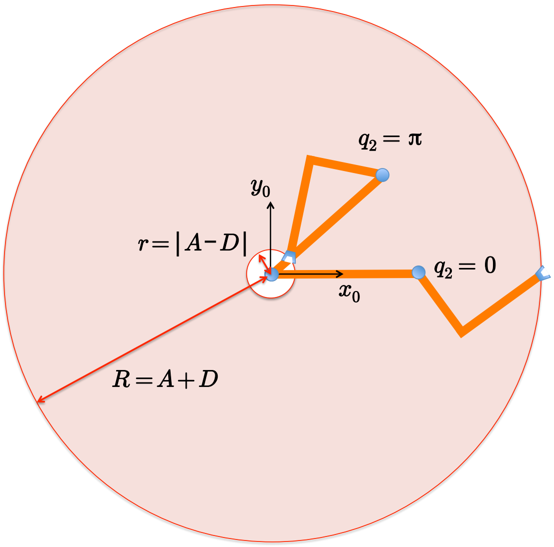

Figure 4: Primary workspace of the planar 2R robot with a L-shaped second link and generic lengths A, B, and C (with $D = \sqrt{B^2 + C^2}$).

From the previous argument, the inverse kinematics problem for this robot has two solutions (out of singularities). Given an end-effector position $\bar{\boldsymbol{p}}_e = (p_x, p_y)$ inside the primary workspace, we have

$$c_2 = \frac{p_x^2 + p_y^2 - A^2 - D^2}{2AD} \in (-1, 1), \qquad s_2 = \pm\sqrt{1 - c_2^2}, \tag{5}$$

and then (for each of the two possible signs of s_2)

$$q_1 = \text{ATAN2}\left\{p_y(A + Dc_2) - p_x D s_2,\, p_x(A + Dc_2) + p_y D s_2\right\}, \qquad q_2 = \text{ATAN2}\left\{s_2,\, c_2\right\}. \tag{6}$$

The requested (analytic) Jacobian is computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \bar{\boldsymbol{p}}_e(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(A \sin q_1 + D \sin(q_1 + q_2)) & -D \sin(q_1 + q_2) \\ A \cos q_1 + D \cos(q_1 + q_2) & D \cos(q_1 + q_2) \end{pmatrix},$$

with $\det \boldsymbol{J}(\boldsymbol{q}) = AD \sin q_2$. Singularities occur at $q_2 = \{0, \pi\}$ (with arbitrary q_1). For $q_2 = 0$ (and a generic q_1), we have

$$\boldsymbol{J}(q_1, 0) = \begin{pmatrix} -(A + D) \sin q_1 & -D \sin q_1 \\ (A + D) \cos q_1 & D \cos q_1 \end{pmatrix} \quad \Rightarrow \quad \mathcal{N}\{\boldsymbol{J}\} = \nu \begin{pmatrix} -D \\ A + D \end{pmatrix}, \ \mathcal{R}\{\boldsymbol{J}\} = \rho \begin{pmatrix} \sin q_1 \\ -\cos q_1 \end{pmatrix},$$

while at $q_2 = \pi$

$$\boldsymbol{J}(q_1, \pi) = \begin{pmatrix} -(A - D) \sin q_1 & D \sin q_1 \\ (A - D) \cos q_1 & -D \cos q_1 \end{pmatrix} \quad \Rightarrow \quad \mathcal{N}\{\boldsymbol{J}\} = \nu \begin{pmatrix} D \\ A - D \end{pmatrix}, \ \mathcal{R}\{\boldsymbol{J}\} = \rho \begin{pmatrix} \sin q_1 \\ -\cos q_1 \end{pmatrix},$$

where ν and ρ are two scaling factors.

Using the numerical data in (1), we have $A = 0.6$ and $D = 0.5$ [m]. For $\boldsymbol{p}_e = (0.6 \ \ -0.5)^T$ [m], we obtain from (5–6) the two inverse kinematic solutions

$$\boldsymbol{q}_{sol,a} = \begin{pmatrix} 1.3895 \\ \pi/2 \end{pmatrix} \text{[rad]} = \begin{pmatrix} -79.61° \\ 90° \end{pmatrix}, \qquad \boldsymbol{q}_{sol,b} = \begin{pmatrix} 0 \\ -\pi/2 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 0° \\ -90° \end{pmatrix}.$$

In these two (nonsingular) configurations, we solve the inverse differential kinematics problem for $\boldsymbol{v}_e = (1 \;\; 0)^T$ [m/s] as

$$\dot{\boldsymbol{q}}_{sol,a} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{sol,a})\, \boldsymbol{v}_e = \begin{pmatrix} 0.5 & -0.0902 \\ 0.6 & 0.4918 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.6393 \\ -2 \end{pmatrix} \text{ [rad/s]},$$

$$\dot{\boldsymbol{q}}_{sol,b} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{sol,b})\, \boldsymbol{v}_e = \begin{pmatrix} 0.5 & 0.5 \\ 0.6 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ [rad/s]},$$

The norms of these two joint velocity vectors are different, namely

$$\left\|\dot{\boldsymbol{q}}_{sol,a}\right\| = 2.5860, \qquad \left\|\dot{\boldsymbol{q}}_{sol,b}\right\| = 2.$$

This should not come out unexpected because the given problem (including the direction of $\boldsymbol{v}_e$ with the respect to the robot postures) has no special symmetries.

Exercise 2

The time-optimal profile for the desired rest-to-rest motion from $\boldsymbol{q}_0$ to $\boldsymbol{q}_f$ under the bounds given in (2) is bang-coast-bang in acceleration for the first joint and bang-bang only for the second. In fact, the joint displacements are

$$\boldsymbol{\Delta} = \boldsymbol{q}_f - \boldsymbol{q}_0 = \begin{pmatrix} -1 \\ 0.7 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix},$$

and the condition for the existence of a time interval when a joints is cruising at its maximum velocity is satisfied for the first joint, but not for the second:

$$|\Delta_1| = 1 > 0.35 = \frac{0.5^2}{0.8} = \frac{V_1^2}{A_1}, \qquad |\Delta_2| = 0.7 < 1.28 = \frac{0.8^2}{0.5} = \frac{V_2^2}{A_2}.$$

The minimum time for the first joint is then

$$T_1 = \frac{|\Delta_1|\, A_1 + V_1^2}{V_1 A_1} = 2.6250 \text{ [s]}.$$

For the second joint, we have instead a triangular velocity profile that is symmetric w.r.t. the middle instant of motion $T_2/2$. At $t = T_2/2$, the joint reaches its maximum velocity $V_{m,2} = A_2 T_2/2 < V_2$. The area under this velocity profile should be equal to the displacement (in absolute value) for the second joint. Thus,

$$\text{(area =)} \;\; \frac{T_2 V_{m,2}}{2} = \frac{A_2 T_2^2}{4} = |\Delta_2| \qquad \Rightarrow \qquad T_2 = \sqrt{\frac{4\,|\Delta_2|}{A_2}} = 2.3664 \text{ [s]}.$$

The minimum motion time is therefore

$$T_{min} = \max\{T_1, T_2\} = \max\{2.6250, 2.3664\} = 2.6250 \text{ [s]}.$$

and is imposed by the first joint. In order to have a coordinated joint motion, the second joint should then slow down a bit, by using an acceleration/deceleration $\pm A_{m,2}$ (whose absolute value is less than A_2) so as to complete its motion exactly at $t = T_{min}$. Thus,

$$\text{(area =)} \;\; \frac{A_{m,2}\, T_{min}^2}{4} = |\Delta_2| \qquad \Rightarrow \qquad A_{m,2} = \frac{4\,|\Delta_2|}{T_{min}^2} = 0.4063 \text{ [rad/s}^2\text{]}.$$

Accordingly, $V_{m,2} = A_{m,2} T_{min}/2 = 0.5333$ [rad/s^2] is the velocity reached by the second joint at the middle instant of motion. The velocity profiles of the two joints for the obtained coordinated motion in minimum time are shown in Fig. 5.

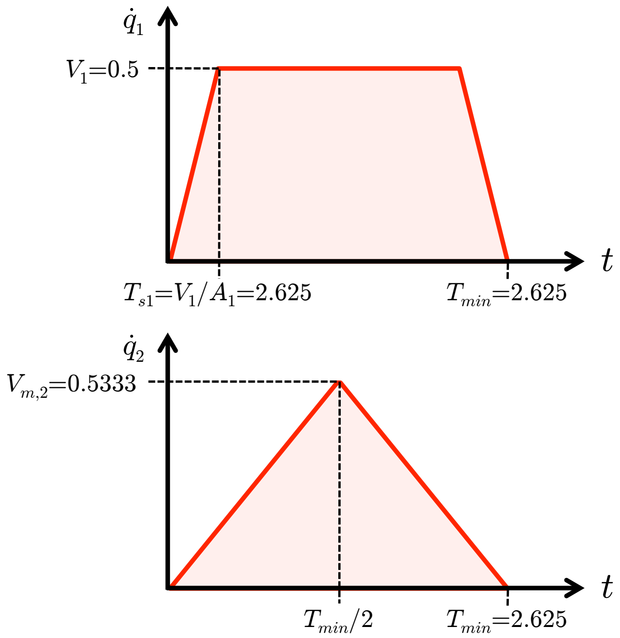

Figure 5: The profiles of the joint velocities achieving the desired coordinated motion in minimum time $T_{min} = 2.6250$ s.

* * * * *