Robotics I

February 5, 2019

Exercise 1

Consider the spatial 4R robot shown in Fig. 1. The second and third joint axes are always horizontal.

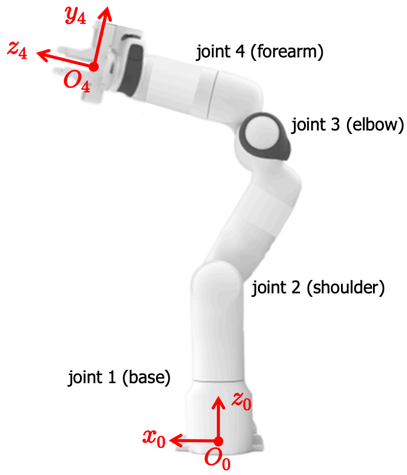

Figure 1: A 4-dof spatial robot with all revolute joints.

a. Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters, specifying the signs of all constant symbolic parameters. Keep base frame (frame 0) and last frame (the end-effector frame 4) as defined in Fig. 1 (these frames satisfy already the DH requirements!). Draw the frames and fill in the table on the extra sheet provided separately.

b. Write explicitly the resulting DH homogeneous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$ to ${}^3\boldsymbol{A}_4(q_4)$ and compute in an efficient way the direct kinematics $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q}) \in \mathbb{R}^3$ for the position of the origin O_4.

c. Discuss if and how the number of symbolic parameters in the direct kinematics of this robot could be reduced. What would be the consequences?

d. Sketch the robot in the stretched upward configuration and specify which is the associated configuration $\boldsymbol{q}_s$ in your DH convention. Compute then $\boldsymbol{p}_s = \boldsymbol{p}_4(\boldsymbol{q}_s)$.

e. In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, determine the expression in the base frame of the absolute position of a Tool Center Point (TCP) which is defined in the end-effector frame by ${}^4\boldsymbol{p}_{4,TCP} = \begin{pmatrix} 0 & 0.1 & 0.2 \end{pmatrix}^T$ [m].

Exercise 2

Make reference to the robot in Exercise 1.

a. Derive the expression of the 6×4 geometric Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of this robot, relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector frame.

b. Find all configurations at which the upper 3×4 block $\boldsymbol{J}_L(\boldsymbol{q})$ of the geometric Jacobian loses rank.

c. Find all configurations at which the lower 3×4 block $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian loses rank.

d. In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, check if the linear Cartesian velocity $\boldsymbol{v}_b = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ is feasible. Provide a joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that instantaneously realizes $\boldsymbol{v}_b$ or, at least, that minimizes the norm of the error w.r.t. the Cartesian velocity $\boldsymbol{v}_b$. If such a joint velocity exists, is it unique?

Exercise 3

Consider the planar 2R robot in Fig. 2, shown together with the geometric data of its desired task. The robot end-effector should follow a desired trajectory made by a circular path of radius R centered at $\boldsymbol{C}_0 = \begin{pmatrix} C_{0,x} & C_{0,y} \end{pmatrix}^T$, to be executed clockwise with a continuous, possibly time-varying desired scalar speed $v(t) > 0$, starting at time $t = 0$ from the path point $\boldsymbol{P}_0 = \begin{pmatrix} P_{0,x} & P_{0,y} \end{pmatrix}^T = \begin{pmatrix} C_{0,x}+R & C_{0,y} \end{pmatrix}^T$.

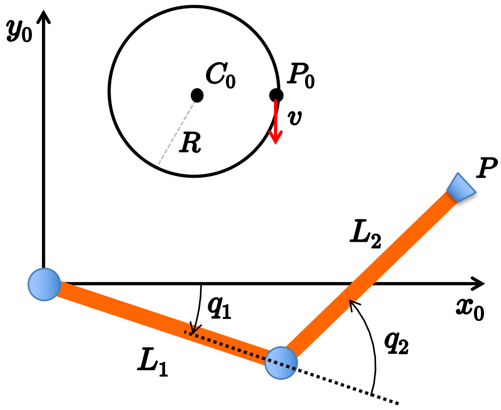

Figure 2: A planar 2R robot and its nominal end-effector trajectory.

Assuming that the robot is commanded by the joint velocity $\dot{\boldsymbol{q}}$, define a single control law that guarantees the following properties:

- when the initial robot configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ at $t = 0$ is matched with the Cartesian point $\boldsymbol{P}_0$, there is a perfect reproduction of the desired trajectory for all $t \geq 0$;
- if there is no such initial matching, the Cartesian trajectory tracking error will converge to zero exponentially and in a decoupled way with respect to its components expressed in a reference frame $RF_r(t) = (\boldsymbol{x}_r(t), \boldsymbol{y}_r(t))$ that is moving with the desired position and has the axis $\boldsymbol{x}_r(t)$ always tangent to the path.

a. Using next the following numerical data

$$L_1 = L_2 = 0.5, \qquad \boldsymbol{C}_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \qquad R = 0.15 \ \text{[m]}, \qquad v = 3 \ \text{[m/s]},$$

determine the value $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ of an initial configuration and the value of the commanded velocity $\dot{\boldsymbol{q}}(0)$ at $t = 0$ that are needed for perfect reproduction of the desired trajectory.

b. In addition, with the robot in the initial configuration

$$\boldsymbol{q}_{\text{off}} = \begin{pmatrix} 0 \\ \pi/6 \end{pmatrix} \ \text{[rad]} \ \neq \boldsymbol{q}_0,$$

using the two time constants $\tau_{r,x} = 0.1$ and $\tau_{r,y} = 0.05$ [s] for the desired exponential transients of the trajectory tracking error components in the frame $RF_r(t)$, determine the initial value $\dot{\boldsymbol{q}}(0)$ of the control law that satisfies the above mentioned properties.

[210 minutes, open books]

Solution

February 5, 2019

Exercise 1

The robot is a modified (imaginary) version of the Franka Emika, with 4 degrees of freedom only. A possible DH frame assignment is shown in Fig. 3, with the associated parameters given in Tab. 1. The signs of the non-zero symbolic constants are also reported in the table.

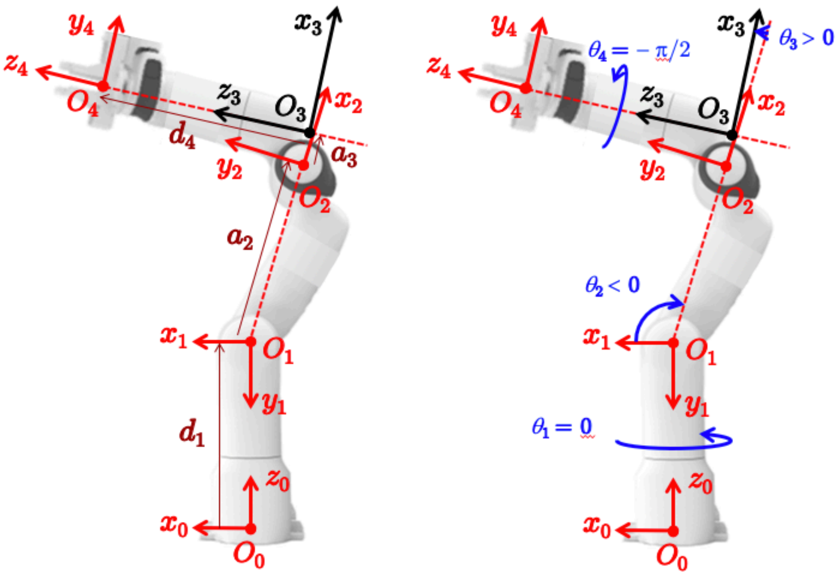

Figure 3: A possible DH frame assignment for the 4-dof robot of Fig. 1. Constant parameters are shown on the left and joint variables shown on the right.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 > 0$	q_1
2	0	$a_2 > 0$	0	q_2
3	$-\pi/2$	$a_3 > 0$	0	q_3
4	0	0	$d_4 > 0$	q_4

Table 1: Parameters associated to the DH frames in Fig. 3.

Based on Tab. 1, the four DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & -\sin q_1 & 0 \\ \sin q_1 & 0 & \cos q_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2(q_2) \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & a_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & a_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^0\boldsymbol{R}_3(q_3) & {}^2\boldsymbol{p}_3(q_3) \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_3 & 0 & -\sin q_3 & a_3 \cos q_3 \\ \sin q_3 & 0 & \cos q_3 & a_3 \sin q_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4 \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_4 & -\sin q_4 & 0 & 0 \\ \sin q_4 & \cos q_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

An efficient symbolic computation for obtaining the end-effector position $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q})$ makes use of recursive matrix-vector products in homogeneous coordinates as

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{p}_4(\boldsymbol{q}) \\ 1 \end{pmatrix} &= {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] \\
&= \begin{pmatrix} \cos q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3) - d_4 \sin(q_2+q_3) \right) \\ \sin q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2+q_3) - d_4 \sin(q_2+q_3) \right) \\ d_1 - a_2 \sin q_2 - a_3 \sin(q_2+q_3) - d_4 \cos(q_2+q_3) \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}.
\end{aligned} \tag{1}
$$

From the DH rules of frame assignment, we could eliminate two parameters by setting them to zero, i.e., $d_1 = 0$ and $d_4 = 0$, with simplifications in (1). The consequences would be that:

- all position vectors computed through the direct kinematics would be expressed with respect to a frame $RF_{0'}$ oriented like the original frame 0, but placed at the robot shoulder;
- the position of the origin O_4 of the original frame 4 at the robot end-effector would be given in the new frame $4'$ by

$$
{}^{4'}\boldsymbol{p}_{O_4} = \begin{pmatrix} 0 & 0 & d_4 \end{pmatrix}^T.
$$

Figure 4 shows the robot in the stretched upward configuration $\boldsymbol{q}_s = \begin{pmatrix} * & -\pi/2 & -\pi/2 & * \end{pmatrix}^T$, where '$*$' could be any value. Taking $* = 0$, the end-effector position is evaluated from (1) as

$$
\boldsymbol{p}_s = \boldsymbol{p}_4(\boldsymbol{q}_s) = \begin{pmatrix} -a_3 \\ 0 \\ d_1 + a_2 + d_4 \end{pmatrix}.
$$

Figure 4: The robot in a stretched configuration.

Finally, the absolute position of the TCP in the configuration $\boldsymbol{q}_0 = \mathbf{0}$, given its numerical value ${}^4\boldsymbol{p}_{4,TCP}$ in the end-effector frame, is computed as

$$
\begin{pmatrix} \boldsymbol{p}_{4,TCP}(\boldsymbol{q}_0) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(0) \left[{}^1\boldsymbol{A}_2(0) \left[{}^2\boldsymbol{A}_3(0) \left[{}^3\boldsymbol{A}_4(0) \begin{pmatrix} 0 \\ 0.1 \\ 0.2 \\ 1 \end{pmatrix} \right] \right] \right] = \begin{pmatrix} a_2 + a_3 \\ -0.1 \\ d_1 - d_4 - 0.2 \\ 1 \end{pmatrix} \text{ [m]}.
$$

Exercise 2

In order to derive the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

of the spatial 4-dof robot of Fig. 1, the simplest way is to compute the 3×4 upper block $\boldsymbol{J}_L(\boldsymbol{q})$ by partial differentiation of the position vector $\boldsymbol{p}_4(\boldsymbol{q})$ in eq. (1), and the 3×4 lower block $\boldsymbol{J}_A(\boldsymbol{q})$ by using the standard formulas. From eq. (1), we obtain

$$\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1(a_2c_2 + a_3c_{23} - d_4s_{23}) & -c_1(a_2s_2 + a_3s_{23} + d_4c_{23}) & -c_1(a_3s_{23} + d_4c_{23}) & 0 \\ c_1(a_2c_2 + a_3c_{23} - d_4s_{23}) & -s_1(a_2s_2 + a_3s_{23} + d_4c_{23}) & -s_1(a_3s_{23} + d_4c_{23}) & 0 \\ 0 & d_4s_{23} - a_3c_{23} - a_2c_2 & d_4s_{23} - a_3c_{23} & 0 \end{pmatrix}. \tag{2}$$

where the usual compact notation has been used for trigonometric functions (e.g., $c_{23} = \cos(q_2 + q_3)$). For later analysis, it is also convenient to express the Jacobian in the rotated frame 1, or

$${}^1\boldsymbol{J}_L(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} 0 & -(a_2s_2 + a_3s_{23} + d_4c_{23}) & -(a_3s_{23} + d_4c_{23}) & 0 \\ 0 & a_2c_2 + a_3c_{23} - d_4s_{23} & a_3c_{23} - d_4s_{23} & 0 \\ a_2c_2 + a_3c_{23} - d_4s_{23} & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Further, being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ for all i, we have

$$\begin{aligned} \boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_2(q_1,q_2)\,{}^2\boldsymbol{z}_2 & {}^0\boldsymbol{R}_3(q_1,q_2,q_3)\,{}^3\boldsymbol{z}_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -s_1 & -s_1 & -c_1s_{23} \\ 0 & c_1 & c_1 & -s_1s_{23} \\ 1 & 0 & 0 & -c_{23} \end{pmatrix}, \end{aligned} \tag{4}$$

where ${}^0\boldsymbol{R}_j(q_1,\ldots,q_j) = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\ldots{}^{j-1}\boldsymbol{R}_j(q_j)$, for $j \geq 1$.

We immediately see that the last column of the two 3×4 Jacobian matrices in (2) and (3) is identically zero. Thus, the rank of these matrices will drop from the maximum value 3 if and only if the determinant of the first 3×3 square blocks (denoted with an additional bar, i.e., $\bar{\boldsymbol{J}}$) is zero. Using (3), we have

$$\det \bar{\boldsymbol{J}}_L(\boldsymbol{q}) = \det {}^1\bar{\boldsymbol{J}}_L(\boldsymbol{q}) = a_2\,(a_3s_3 + d_4c_3)\,(a_2c_2 + a_3c_{23} - d_4s_{23})\,. \tag{5}$$

Singularities of $\boldsymbol{J}_L(\boldsymbol{q})$ occur when one (or both) of the factors in the right-hand side of (5) is zero. The first factor depends only on q_3 and vanishes when the forearm is 'almost' stretched or folded[1]. Actually, being in practice $a_3 \ll d_4$, the roots are relatively close to $q_3 = \pm\pi/2$, where $c_3 \simeq 0$). Indeed, according to Fig. 3, only the stretched configuration (corresponding to the negative solution for q_3) is of interest: the other would lead to a self-collision between link 2 and link 3. The second factor in the right-hand side of (5) vanishes when the origin O_4 lies on axis of joint 1. In fact, from eq. (1) we have that

$$|a_2c_2 + a_3c_{23} - d_4s_{23}| = \sqrt{p_x^2 + p_y^2},$$

namely, the distance from the axis $\boldsymbol{z}_0$ of the origin of frame 4 on the robot end-effector.

[1] In general, the solutions of the trigonometric equation $a_3 \sin q_3 + d_4 \cos q_3 = 0$ can be found by the algebraic substitution $q_3 = \tan(x/2)$, which converts the problem into that of finding the roots of a quadratic polynomial (with a number of special cases). For instance, if $a_3 = 0.1$ and $d_4 = 0.5$, we find the two solutions $q_3^+ = 1.768$ and $q_3^- = -1.374$ [rad]. For much smaller values of the ratio a_3/d_4, the two roots converge to $q_3^\pm = \pm 1.57 = \pm\pi/2$ [rad].

Singularities of the lower part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of the robot are simpler to determine. Discarding the third column in (4), which is identical to the second one, a singularity will occur if and only if the determinant of the remaining matrix (denoted again with an additional bar) will vanish, i.e.,

$$\det \bar{\boldsymbol{J}}_A(\boldsymbol{q}) = s_{23} = 0. \tag{6}$$

When this happens, feasible angular velocities of the robot end-effector are characterized by

$$\boldsymbol{\omega} \in \mathcal{R}\{\boldsymbol{J}_A(\boldsymbol{q})\}_{s_{23}=0} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \right\}.$$

In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, the linear part of the geometric Jacobian is evaluated from (2) as

$$\boldsymbol{J}_{L,0} = \boldsymbol{J}_L(\boldsymbol{0}) = \begin{pmatrix} 0 & -d_4 & -d_4 & 0 \\ a_2 + a_3 & 0 & 0 & 0 \\ 0 & -(a_2 + a_3) & -a_3 & 0 \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0} & \boldsymbol{0} \end{pmatrix}, \tag{7}$$

where we have partitioned the first three columns from the fourth (zero) column. It is easy to see that $\boldsymbol{J}_{L,0}$ (namely $\bar{\boldsymbol{J}}_{L,0}$) has full rank equal to 3. Therefore, any Cartesian linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ can be instantaneously realized by the robot in the given configuration. Moreover, joint velocities that lie in the null space of $\boldsymbol{J}_{L,0}$ take the form

$$\dot{\boldsymbol{q}}_a = \rho \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{for some } \rho \lesseqqgtr 0. \tag{8}$$

As a result, the problem of realizing the given linear Cartesian velocity $\boldsymbol{v}_b = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ has an infinite number of solutions. Among these, the joint velocity solution of minimum norm is given by

$$\begin{aligned} \dot{\boldsymbol{q}}_b &= \boldsymbol{J}_{L,0}^{\#} \boldsymbol{v}_b = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0} & \boldsymbol{0} \end{pmatrix}^{\#} \boldsymbol{v}_b = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0}^{\#} \\ \boldsymbol{0}^T \end{pmatrix} \boldsymbol{v}_b = \begin{pmatrix} \bar{\boldsymbol{J}}_{L,0}^{-1} \\ \boldsymbol{0}^T \end{pmatrix} \boldsymbol{v}_b \\ &= \begin{pmatrix} 0 & \dfrac{1}{a_2 + a_3} & 0 \\ \dfrac{a_3}{a_2 d_4} & 0 & -\dfrac{1}{a_2} \\ -\dfrac{a_2 + a_3}{a_2 d_4} & 0 & \dfrac{1}{a_2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{a_3}{a_2 d_4} - \dfrac{1}{a_2} \\ \dfrac{1}{a_2} - \dfrac{a_2 + a_3}{a_2 d_4} \\ 0 \end{pmatrix}, \end{aligned} \tag{9}$$

where we have used the properties of pseudoinverse matrices. All joint velocities that provide zero Cartesian velocity error (i.e., such that $\boldsymbol{J}_{L,0}\,\dot{\boldsymbol{q}} = \boldsymbol{v}_b$) are obtained by adding to the particular solution (9) a null-space joint velocity (8) for some scalar ρ:

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_b + \dot{\boldsymbol{q}}_a = \begin{pmatrix} 0 \\ \dfrac{a_3}{a_2 d_4} - \dfrac{1}{a_2} \\ \dfrac{1}{a_2} - \dfrac{a_2 + a_3}{a_2 d_4} \\ \rho \end{pmatrix}.$$

Exercise 3

The reference trajectory $\boldsymbol{p}_d(t)$ in the Cartesian plane is a circular path $\boldsymbol{p}_d(s)$ traced clockwise with a linear speed $v(t) \geq 0$, and is specified thus as

$$\boldsymbol{p}_d(s) = \boldsymbol{C}_0 + R \begin{pmatrix} \cos \dfrac{s}{R} \\ -\sin \dfrac{s}{R} \end{pmatrix}, \quad s \geq 0, \qquad s(t) = \int_0^t v(\tau) d\tau, \quad t \geq 0, \tag{10}$$

satisfying $s(0) = 0$ and $\boldsymbol{p}_d(0) = \boldsymbol{P}_0$. Note that the path is parametrized in this case by the arc length s, and that the minus sign on the second component of the vector in (10) accounts for the clockwise tracing of the circle. From (10), it follows

$$\dot{\boldsymbol{p}}_d(t) = \frac{d\boldsymbol{p}(s)}{ds}\, \dot{s}(t) = -v(t) \begin{pmatrix} \sin \dfrac{s(t)}{R} \\ \cos \dfrac{s(t)}{R} \end{pmatrix}, \quad t \geq 0, \tag{11}$$

providing $\|\dot{\boldsymbol{p}}_d(t)\| = v(t)$.

The robot direct and differential kinematics are given by

$$\boldsymbol{p} = \begin{pmatrix} L_1 \cos q_1 + L_2 \cos(q_1 + q_2) \\ L_1 \sin q_1 + L_2 \sin(q_1 + q_2) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}) \tag{12}$$

and

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\, \dot{\boldsymbol{q}} = \begin{pmatrix} -\left(L_1 \sin q_1 + L_2 \sin(q_1 + q_2)\right) & -L_2 \sin(q_1 + q_2) \\ L_1 \cos q_1 + L_2 \cos(q_1 + q_2) & L_2 \cos(q_1 + q_2) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{13}$$

In nominal conditions, when the initial robot configuration $\boldsymbol{q}_0$ is matched with the Cartesian point $\boldsymbol{P}_0$, i.e., $\boldsymbol{f}(\boldsymbol{q}_0) = \boldsymbol{p}_d(0) = \boldsymbol{P}_0$, the joint velocity command that guarantees perfect reproduction of the desired trajectory $\boldsymbol{p}_d(t)$ for all $t \geq 0$ is given just by the feedforward command

$$\dot{\boldsymbol{q}}(t) = \dot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t))\, \dot{\boldsymbol{p}}_d(t), \qquad \text{with} \quad \boldsymbol{q}_d(t) = \boldsymbol{q}_0 + \int_0^t \dot{\boldsymbol{q}}_d(\tau) d\tau. \tag{14}$$

On the other hand, if there is an initial Cartesian error $\boldsymbol{e}_0 = \boldsymbol{P}_0 - \boldsymbol{f}(\boldsymbol{q}_0) \neq \boldsymbol{0}$, a feedback control action is needed. Let the Cartesian trajectory tracking error be

$$\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t), \qquad t \geq 0, \tag{15}$$

with $\boldsymbol{p}(t) = \boldsymbol{f}(\boldsymbol{q}(t))$ from eq. (12). The error vector $\boldsymbol{e} \in \mathbb{R}^2$ can be expressed in a rotated (planar and right-handed) frame $RF_r(t)$, having the origin attached to the desired Cartesian position $\boldsymbol{p}_d(t)$ of the robot end-effector and the $\boldsymbol{x}_r(t)$-axis pointing along the (positive) tangent direction to the path, see Fig. 5. Define the rotated tracking error as

$$^{R}\boldsymbol{e}(t) = \boldsymbol{R}(t)\boldsymbol{e}(t), \qquad \text{with} \quad \boldsymbol{R}(t) = \begin{pmatrix} -\sin \dfrac{s(t)}{R} & \cos \dfrac{s(t)}{R} \\ -\cos \dfrac{s(t)}{R} & -\sin \dfrac{s(t)}{R} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_r(t) & \boldsymbol{y}_r(t) \end{pmatrix}. \tag{16}$$

Note that the time derivative of the (planar) rotation matrix $\boldsymbol{R}(t)$ in (16) is

$$\begin{aligned} \dot{\boldsymbol{R}}(t) &= \begin{pmatrix} -\cos \dfrac{s(t)}{R} & -\sin \dfrac{s(t)}{R} \\ \sin \dfrac{s(t)}{R} & -\cos \dfrac{s(t)}{R} \end{pmatrix} \frac{\dot{s}(t)}{R} = \begin{pmatrix} -\sin \dfrac{s(t)}{R} & \cos \dfrac{s(t)}{R} \\ -\cos \dfrac{s(t)}{R} & -\sin \dfrac{s(t)}{R} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{\dot{s}(t)}{R} \\ -\dfrac{\dot{s}(t)}{R} & 0 \end{pmatrix} \\ &= \boldsymbol{R}(t)\, \boldsymbol{S}(\omega(t)), \qquad \text{with} \quad \omega(t) = \frac{\dot{s}(t)}{R}, \end{aligned} \tag{17}$$

where $\boldsymbol{S}(.)$ is a (planar) skew-symmetric matrix[2].

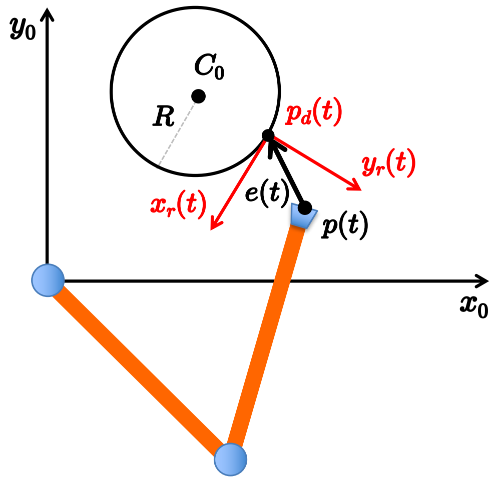

Figure 5: The reference frame $RF_r(t)$ used in the definition of the rotated tracking error.

The target dynamics of the rotated tracking error ${}^R\boldsymbol{e}$ is specified by

$$ {}^R\dot{\boldsymbol{e}} = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} {}^R\boldsymbol{e} = -\boldsymbol{K}\, {}^R\boldsymbol{e}, \qquad \text{with} \quad k_x > 0,\, k_y > 0, \tag{18} $$

namely, as a linear and decoupled behavior along the axes of the rotated frame, with errors exponentially converging to zero:

$$ {}^R e_x(t) = {}^R e_x(0) \exp(-k_x t), \qquad {}^R e_y(t) = {}^R e_y(0) \exp(-k_y t). \tag{19} $$

In these exponential evolutions, the time constants are the inverse of the gains: $\tau_{r,x} = 1/k_x$, $\tau_{r,y} = 1/k_y$. Since

$$ {}^R\dot{\boldsymbol{e}} = \boldsymbol{R}\,\dot{\boldsymbol{e}} + \dot{\boldsymbol{R}}\,\boldsymbol{e} = \boldsymbol{R}\left(\dot{\boldsymbol{e}} + \boldsymbol{S}\boldsymbol{e}\right) = \boldsymbol{R}\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{S}\boldsymbol{e}\right), $$

in order to obtain (18), the control law should be speficied as

$$ \begin{aligned} \dot{\boldsymbol{q}}(t) &= \boldsymbol{J}^{-1}(\boldsymbol{q}(t))\left(\dot{\boldsymbol{p}}_d(t) + \boldsymbol{S}(\omega(t))\boldsymbol{e}(t) + \boldsymbol{R}^T(t)\boldsymbol{K}\, {}^R\boldsymbol{e}(t)\right) \\ &= \boldsymbol{J}^{-1}(\boldsymbol{q}(t))\left(\dot{\boldsymbol{p}}_d(t) + \left(\boldsymbol{S}(\omega(t)) + \boldsymbol{R}^T(t)\boldsymbol{K}\boldsymbol{R}(t)\right)(\boldsymbol{p}_d(t) - \boldsymbol{f}(\boldsymbol{q}(t)))\right). \end{aligned} \tag{20} $$

We note that the time-varying gain matrix $\boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R}$ is required in order to obtain a constant and decoupled error dynamics in the rotated frame. Also, when the initial tracking error is zero, one has $\boldsymbol{q}(t) = \boldsymbol{q}_d(t)$ for all $t \geq 0$, and the control law (20) collapses into a feedforward command only, as given by (14).

We move next to the application of the above formulas with the numerical data given for the problem. An initially matched robot configuration $\boldsymbol{q}_0$ is determined by the desired initial Cartesian point $\boldsymbol{P}_0$ and the robot inverse kinematics. From the center $\boldsymbol{C}_0$ of the circular path and its radius R, we have

$$ \boldsymbol{C}_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \quad R = 0.15 \qquad \Rightarrow \qquad \boldsymbol{P}_0 = \begin{pmatrix} 0.35 \\ 0.3 \end{pmatrix} \text{ [m]}. $$

From the inverse kinematics equations of a planar 2R robot,

$$ c_2 = \frac{P_{0,x}^2 + P_{0,y}^2 - L_1^2 - L_2^2}{2L_1L_2}, \qquad s_2 = \pm\sqrt{1 - c_2^2} \qquad \Rightarrow \qquad q_2 = \text{ATAN2}\{s_2, c_2\}, $$

[2]If the path were linear, the orientation of the frame attached to the path would be constant and so $\boldsymbol{R}$. Being $\dot{\boldsymbol{R}} = \boldsymbol{0}$, also $\boldsymbol{S}$ would vanish in the following formulas.

and

$$c_1 = \frac{P_{0,x}(L_1 + L_2c_2) + P_{0,y}L_2s_2}{L_1^+ L_2^2 + 2L_1L_2c_2}, \quad s_1 = \frac{P_{0,y}(L_1 + L_2c_2) - P_{0,x}L_2s_2}{L_1^+ L_2^2 + 2L_1L_2c_2} \qquad \Rightarrow \qquad q_1 = \text{ATAN2}\{s_1, c_1\},$$

using also the robot link lengths $L_1 = L_2 = 0.5$ [m], we obtain the two solutions

$$\boldsymbol{q}_0^{up} = \begin{pmatrix} q_1^{up} \\ q_2^{up} \end{pmatrix} = \begin{pmatrix} 1.8003 \\ -2.1834 \end{pmatrix} \text{[rad]}, \qquad \boldsymbol{q}_0^{down} = \begin{pmatrix} q_1^{down} \\ q_2^{down} \end{pmatrix} = \begin{pmatrix} -0.3831 \\ 2.1834 \end{pmatrix} \text{[rad]}. \tag{21}$$

With these configurations and the desired initial speed $v = 3$ [m/s], we obtain from (13) and (14) the two alternative initial commanded velocities

$$\dot{\boldsymbol{q}}^{up}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_0^{up})\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 7.6055 \\ -12.2087 \end{pmatrix} \text{[rad/s]}, \quad \dot{\boldsymbol{q}}^{down}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_0^{down})\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} -10.7179 \\ -6.6039 \end{pmatrix} \text{[rad/s]}. \tag{22}$$

These joint velocities will achieve perfect reproduction of the desired trajectory at $t = 0$. The values in (22) are relatively large because of the large speed v that is being requested at the Cartesian level.

Instead, when the robot is initially in

$$\boldsymbol{q}_{\text{off}} = \begin{pmatrix} 0 \\ \pi/6 \end{pmatrix} \text{[rad]} \qquad \Rightarrow \qquad \boldsymbol{e}_0 = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}_{\text{off}}) = \begin{pmatrix} -0.583 \\ 0.05 \end{pmatrix} \text{[m]},$$

we need to use the control law (20) at $t = 0$. Therein, considering also the desired time constants $\tau_{r,x} = 0.1$ and $\tau_{r,y} = 0.05$ [s], the diagonal gain matrix $\boldsymbol{K}$, the rotation matrix $\boldsymbol{R}(0)$, the effective gain matrix $\boldsymbol{R}_{\text{eff}}(0) = \boldsymbol{R}^T(0)\boldsymbol{K}\boldsymbol{R}(0)$, and the skew-symmetric matrix $\boldsymbol{S}(\omega(0))$ are evaluated as

$$\boldsymbol{K} = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}, \qquad \boldsymbol{R}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\boldsymbol{R}_{\text{eff}}(0) = \begin{pmatrix} 20 & 20 \\ -20 & 10 \end{pmatrix}, \qquad \boldsymbol{S}(\omega(0)) = \begin{pmatrix} 0 & 20 \\ -20 & 0 \end{pmatrix}.$$

As a result, the joint velocity control that will be applied at the initial instant will be

$$\dot{\boldsymbol{q}}(0) = \begin{pmatrix} 20.2224 \\ -22.4186 \end{pmatrix} \text{[rad/s]}. \tag{23}$$

These are indeed extremely large values. However, they are needed in order to obtain the specified fast rate of exponential decay for the trajectory tracking error (the time constants are too small, and could be possibly increased to obtain smaller values in (23)).

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