Robotics 1

Remote Exam – June 5, 2020

Exercise #1

Consider the 4-dof robot in Fig. 1, with all revolute joints. Some axes of a Denavit-Hartenberg (D-H) frame assignment are already given, together with an end-effector frame placed on the gripper. Assuming that all angles defined as usual in the interval $(-\pi, +\pi]$, complete the assignment of the frames so that $\alpha_i \geq 0$, for $i = 1, \dots, 4$. Provide the associated table of D-H parameters and specify the value of q_1 and the signs of q_2, q_3, and q_4 in the configuration shown. Finally, find the homogeneous transformation between the last D-H frame and the end-effector frame.

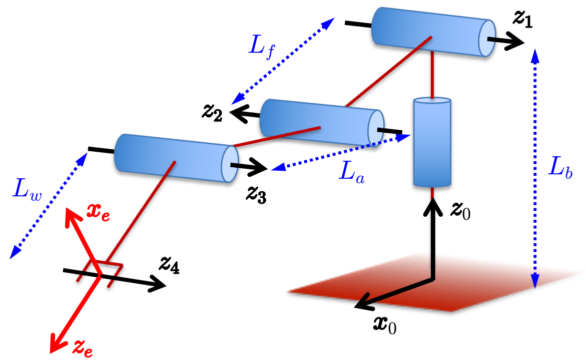

Figure 1: A 4-dof robot with given D-H axes $\boldsymbol{z}_i$ and an end-effector frame on the gripper.

Exercise #2

For the planar RRP robot in Fig. 2, define the direct kinematics $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ from the joint variables $\boldsymbol{q} = (q_1, q_2, q_3)$ to the task variables $\boldsymbol{r} = (p_x, p_y, \phi)$, derive the associated Jacobian $\boldsymbol{J}(\boldsymbol{q})$, and find all its kinematic singularities. With $l_1 = 0.5$ [m], compute in static conditions the joint torque/force vector $\boldsymbol{\tau}$ (with units [Nm,Nm,N]) that balances a force/moment vector $\boldsymbol{F} = \begin{pmatrix} 0 & 1.5 & -4.5 \end{pmatrix}^T$ [N,N,Nm] applied to the robot end-effector, first in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0 & 3 \end{pmatrix}^T$ [rad,rad,m] and then in a singular configuration $\boldsymbol{q}_s$ among those found.

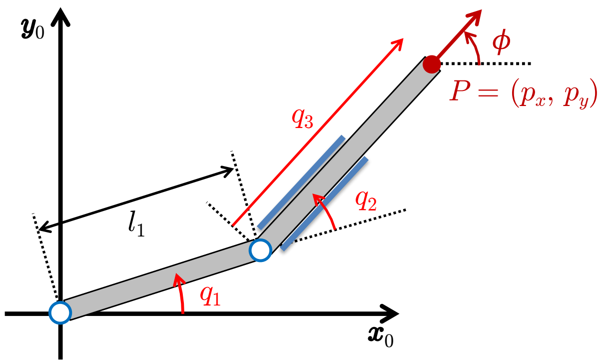

Figure 2: A RRP planar robot.

Exercise #3

The Jacobian of a 3R spatial robot relating $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the velocity $\boldsymbol{v} \in \mathbb{R}^3$ of its end-effector is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1(c_2 + c_{23}) & -c_1(s_2 + s_{23}) & -c_1 s_{23} \\ c_1(c_2 + c_{23}) & -s_1(s_2 + s_{23}) & -s_1 s_{23} \\ 0 & c_2 + c_{23} & c_{23} \end{pmatrix},$$

where the shorthand notation has been used (e.g., $c_{23} = \cos(q_2 + q_3)$). This matrix may have rank 1, 2, or 3, depending on the configuration $\boldsymbol{q}$. In each of these cases, define a basis for the null space $\mathcal{N}\{\boldsymbol{J}\}$ and for the range space $\mathcal{R}\{\boldsymbol{J}\}$ of the Jacobian. Find a configuration $\boldsymbol{q}_s$ with rank $\boldsymbol{J}(\boldsymbol{q}_s) = 2$ such that the end-effector velocity $\boldsymbol{v}_s = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$ is feasible. Determine then a joint velocity $\dot{\boldsymbol{q}}_s$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s$. Sketch graphically the situation.

Exercise #4

A 2R planar robot has to perform in a coordinated way a rest-to-rest motion from $\boldsymbol{q}_s = \begin{pmatrix} 0 & -\pi/2 \end{pmatrix}^T$ to $\boldsymbol{q}_g = \begin{pmatrix} -\pi/2 & \pi/2 \end{pmatrix}^T$, while guaranteeing continuity of acceleration at all times. Plan a joint trajectory in the presence of bounds $|\dot{q}_i| \leq V_i$ on joint velocities and $|\ddot{q}_i| \leq A_i$ on joint accelerations (for $i = 1, 2$), so as to complete the motion task in minimum time T^* within the chosen class of trajectories. Provide the value of T^* for $V_1 = 1$, $V_2 = 2$ [rad/s] and $A_1 = 1.5$, $A_2 = 2$ [rad/s^2].

Exercise #5

This is in the form of a Questionnaire. Please answer with formulas and/or clear and short texts.

A) Which of the following matrices represents a rotation and which not? Motivate your answers.

$$\boldsymbol{R}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{R}_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad \boldsymbol{R}_3 = \begin{pmatrix} -\sqrt{0.5} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{0.5} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

B) An Harmonic Drive with a circular spline having 150 inner teeth is used as reduction element in a robot joint. An absolute encoder is mounted on the motor side of the joint. How many bits should have this encoder in order to provide an angular resolution better than or equal to 0.0002 rad on the link side of the transmission?

C) A time series $\{q_k\} = \{q(kT_c)\}$ of joint position measurements is collected every time step $T_c = 0.03$ s (about 33 Hz) in the interval $t \in [0, 0.6]$ s from the profile $q(t) = -3\cos\omega t$, with $\omega = 2$. Compute the joint velocity estimate $\dot{q}_k$ for $k = 20$ using 1-step and 4-step backward difference formulas (BDF methods). What is the related percentage error using each method? What is the relation between the time step and the accuracy? Write a short code (e.g., in MATLAB) and comment the obtained numerical results.

[180 minutes (3 hours); open books]

Solution

June 5, 2020

Exercise #1

A completed frame assignment with non-negative values of the twist angles α_i, for $i = 1, \dots, 4$, is shown in Fig. 3. The associated Denavit-Hartenberg parameters are reported in Tab. 1, together with the signs of the variables $q_i = \theta_i$, for $i = 1, \dots, 4$, when the robot is in the configuration shown in the figure (where $q_1 = 0$). Note that this solution is not yet unique since the axes from $\boldsymbol{x}_2$ to $\boldsymbol{x}_4$ could have been chosen each in the opposite direction (with no change of the constant parameters α_i's in the table!). However, this one is more natural as it assigns positive lengths to the non-zero parameters a_i. The transformation between D-H frame 4 and end-effector frame is

$$
{}^4\boldsymbol{T}_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

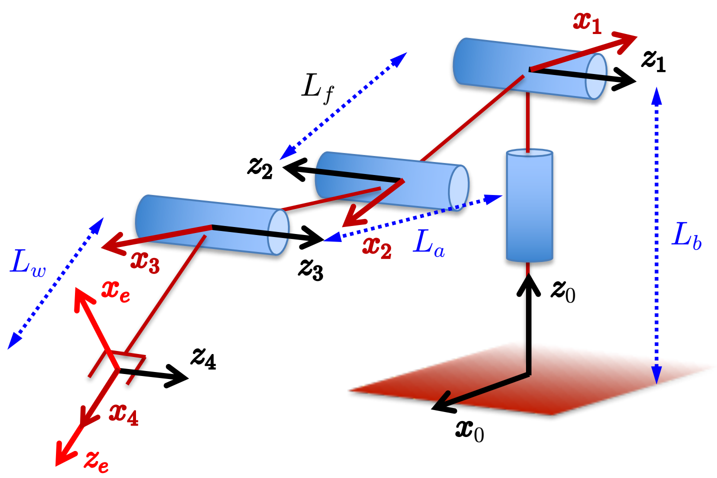

Figure 3: Complete assignment of the D-H frames for the 4-dof robot. All frames are right-handed, so the $\boldsymbol{y}_i$ axes follow automatically (and are not shown).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	L_b	$q_1 = \pi$
2	π	L_f	0	$q_2 < 0$
3	π	L_a	0	$q_3 > 0$
4	0	L_w	0	$q_4 > 0$

Table 1: The D-H table of parameters for the frame assignment in Fig. 3.

Exercise #2

The direct kinematics of this robot for the task at hand is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} l_1 \cos q_1 + q_3 \cos(q_1+q_2) \\ l_1 \sin q_1 + q_3 \sin(q_1+q_2) \\ q_1+q_2 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

and its 3×3 Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1 \sin q_1 - q_3 \sin(q_1+q_2) & -q_3 \sin(q_1+q_2) & \cos(q_1+q_2) \\ l_1 \cos q_1 + q_3 \cos(q_1+q_2) & q_3 \cos(q_1+q_2) & \sin(q_1+q_2) \\ 1 & 1 & 0 \end{pmatrix}.$$

It is easy to check that $\det \boldsymbol{J}(\boldsymbol{q}) = l_1 \cos q_2$ and so a singularity occurs iff $q_2 = \pm\pi/2$. In this singular configurations, the rank of the Jacobian is 2. The vector of torques (first two components) and force (third component, for the prismatic joint) in the joint space that statically balances the Cartesian vector $\boldsymbol{F}$ of forces/moment is given by $\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}$. Setting $l_1 = 0.5$ [m], we have in the (nonsingular) configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0 & 3 \end{pmatrix}^T$

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q}_0)\boldsymbol{F} = -\begin{pmatrix} -3.5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 4.5 \\ -1.5 \end{pmatrix} \text{ [Nm, Nm, N]}.$$

We can choose for comparison a singular configuration $\boldsymbol{q}_s$ which is similar to $\boldsymbol{q}_0$, namely with the same values for $q_1 = \pi/2$ and $q_3 = 3$, but indeed with $q_2 = \pm\pi/2$. For $\boldsymbol{q}_{s,1} = \begin{pmatrix} \pi/2 & -\pi/2 & 3 \end{pmatrix}^T$, we obtain

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q}_{s,1})\boldsymbol{F} = -\begin{pmatrix} -0.5 & 3 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ [Nm, Nm, N]}.$$

Therefore, no effort by the joint motors is needed[1] to balance the Cartesian vector $\boldsymbol{F}$, which lies in this case in the null space of $\boldsymbol{J}^T(\boldsymbol{q}_{s,1})$. On the other hand, with $\boldsymbol{q}_{s,2} = \begin{pmatrix} \pi/2 & \pi/2 & 3 \end{pmatrix}^T$ we obtain

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q}_{s,2})\boldsymbol{F} = -\begin{pmatrix} -0.5 & -3 & 1 \\ 0 & -3 & 1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} \text{ [Nm, Nm, N]},$$

and we need torques on the first two joints in order to balance the Cartesian vector $\boldsymbol{F}$. In fact, $\boldsymbol{F} \notin \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}_{s,2})\} = \alpha \begin{pmatrix} 0 & 1 & 3 \end{pmatrix}^T, \forall\alpha$.

Exercise #3

The given Jacobian matrix is associated to the direct kinematics of a 3R spatial robot with unitary lengths of links 2 and 3. In fact, the end-effector position for such robot (with $d_1 = 0$, without loss of generality) is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} c_1(c_2+c_{23}) \\ s_1(c_2+c_{23}) \\ s_2+s_{23} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{1}$$

[1] It would be useful to draw a picture of this case and to reason geometrically about the balance of moments at the first and the second joint.

and thus

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1(c_2+c_{23}) & -c_1(s_2+s_{23}) & -c_1 s_{23} \\ c_1(c_2+c_{23}) & -s_1(s_2+s_{23}) & -s_1 s_{23} \\ 0 & c_2+c_{23} & c_{23} \end{pmatrix}. \tag{2}$$

We compute first the determinant of $\boldsymbol{J}(\boldsymbol{q})$. To simplify the result, it is convenient to premultiply the Jacobian by the (nonsingular) rotation matrix

$${}^0\boldsymbol{R}_1^T(q_1) = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which implies expressing the Cartesian end-effector velocity $\boldsymbol{v}$ in the rotated frame 1 (attached to the first link of the robot). This yields the simpler form

$${}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 0 & -(s_2+s_{23}) & -s_{23} \\ c_2+c_{23} & 0 & 0 \\ 0 & c_2+c_{23} & c_{23} \end{pmatrix}.$$

We obtain then[2]

$$\det \boldsymbol{J}(\boldsymbol{q}) = \det {}^1\boldsymbol{J}(\boldsymbol{q}) = -(c_2+c_{23}) \det \begin{pmatrix} -(s_2+s_{23}) & -s_{23} \\ c_2+c_{23} & c_{23} \end{pmatrix} = -\sin q_3 \,(\cos q_2+\cos(q_2+q_3)). \tag{3}$$

Therefore, singularities occur for $\sin q_3 = 0$, i.e., when the forearm is stretched ($q_3 = 0$, type I) or fully folded ($q_3 = \pi$, type II); or for $c_2 + c_{23} = 0$ (type I), which corresponds to $p_x^2 + p_y^2 = 0$ from (1), i.e., when the end-effector lies on the joint axis 1; or at the intersection of the two previous situations (type II). Singularities of type I are associated to a single loss of rank (i.e., in these configurations, rank $\boldsymbol{J}(\boldsymbol{q}) = 2$), whereas singularities of type II are associated to a double loss of rank (i.e., rank $\boldsymbol{J}(\boldsymbol{q}) = 1$). We remark also that, because of the equal length of links 2 and 3, when the forearm is fully folded the robot end-effector will certainly be on the axis of joint 1. This explains why $q_3 = \pi$ is a singularity of type II.

Indeed, at configurations $\boldsymbol{q}$ where the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ has full rank, we will have $\mathcal{N}\{\boldsymbol{J}\} = \boldsymbol{0}$ and $\mathcal{R}\{\boldsymbol{J}\} = \mathbb{R}^3$. Let us now perform the analysis of subspaces for the various singular cases.

a) $q_3 = 0$ (and $c_2 \neq 0$, otherwise this would become case **d)** below —a type-II singularity). The Jacobian (2) takes the form

$$\boldsymbol{J}_{\mathrm{I},0}(q_1,q_2) = \begin{pmatrix} -2s_1c_2 & -2c_1s_2 & -c_1s_2 \\ 2c_1c_2 & -2s_1s_2 & -s_1s_2 \\ 0 & 2c_2 & c_2 \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 & -2s_2 & -s_2 \\ 2c_2 & 0 & 0 \\ 0 & 2c_2 & c_2 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\mathrm{I},0} = 2$ and

$$\mathcal{N}\{\boldsymbol{J}_{\mathrm{I},0}\} = \left\{ \alpha \begin{pmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\}, \quad \mathcal{R}\{\boldsymbol{J}_{\mathrm{I},0}\} = \left\{ \beta_1 \begin{pmatrix} -s_1c_2 \\ c_1c_2 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} -c_1s_2 \\ -s_1s_2 \\ c_2 \end{pmatrix} \right\}, \quad \forall \alpha, \beta_1, \beta_2.$$

[2]When using the Symbolic Toolbox of MATLAB, the straight command `detJ=simplify(det(J))` will produce as output: $-(\cos q_3 + 1)(\sin(q_2 + q_3) - \sin q_2)$. This expression is indeed equivalent to (3), but more difficult to analyze. The procedure followed in the text is suggested by the observation of the internal structure of matrix $\boldsymbol{J}(\boldsymbol{q})$ in (2). It also lends itself to a more intuitive interpretation of the singular configurations of this 3R spatial robot.

b) $q_3 = \pi$. The Jacobian (2) becomes

$$\boldsymbol{J}_{\mathrm{II}.\pi}(q_1, q_2) = \begin{pmatrix} 0 & 0 & c_1 s_2 \\ 0 & 0 & s_1 s_2 \\ 0 & 0 & -c_2 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\mathrm{II},\pi} = 1$ and

$$\mathcal{N}\{\boldsymbol{J}_{\mathrm{II},\pi}\} = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \mathcal{R}\{\boldsymbol{J}_{\mathrm{II},\pi}\} = \left\{ \beta \begin{pmatrix} c_1 s_2 \\ s_1 s_2 \\ -c_2 \end{pmatrix} \right\}, \quad \forall \alpha_1, \alpha_2, \beta.$$

c) $c_2 + c_{23} = 0$ (and $c_2 \neq 0$, otherwise this would also imply either $q_3 = 0$ (case **d)** below) or $q_3 = \pi$ (case **b)** above) —both singularities of type II). The Jacobian (2) becomes

$$\boldsymbol{J}_{\mathrm{I.axis1}}(\boldsymbol{q}) = \begin{pmatrix} 0 & -c_1(s_2 + s_{23}) & -c_1 s_{23} \\ 0 & -s_1(s_2 + s_{23}) & -s_1 s_{23} \\ 0 & 0 & -c_2 \end{pmatrix} = {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 & -(s_2 + s_{23}) & -s_{23} \\ 0 & 0 & 0 \\ 0 & 0 & -c_2 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\mathrm{I,axis1}} = 2$ and

$$\mathcal{N}\{\boldsymbol{J}_{\mathrm{I,axis1}}\} = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \ \mathcal{R}\{\boldsymbol{J}_{\mathrm{II,axis1}}\} = \left\{ \beta_1 \begin{pmatrix} c_1(s_2 + s_{23}) \\ s_1(s_2 + s_{23}) \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} c_1 s_{23} \\ s_1 s_{23} \\ c_2 \end{pmatrix} \right\}, \ \forall \alpha, \beta_1, \beta_2.$$

d) $q_3 = 0$ AND $c_2 + c_{23} = 0 \ \Rightarrow \ q_2 = \pm\pi/2$. The Jacobian (2) becomes

$$\boldsymbol{J}_{\mathrm{II,double}}(q_1) = \begin{pmatrix} 0 & \mp 2c_1 & \mp c_1 \\ 0 & \mp 2s_1 & \mp s_1 \\ 0 & 0 & 0 \end{pmatrix}$$

with rank $\boldsymbol{J}_{\mathrm{II,double}} = 1$ and

$$\mathcal{N}\{\boldsymbol{J}_{\mathrm{II,double}}\} = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\}, \ \mathcal{R}\{\boldsymbol{J}_{\mathrm{II,double}}\} = \left\{ \beta \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} \right\}, \quad \forall \alpha_1, \alpha_2, \beta.$$

In order to find a joint velocity that realizes the desired end-effector velocity $\boldsymbol{v}_s = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$ when the robot is in a type-I singularity, we should consider only the above two cases **a)** and **c)**. In case **a)**, however, $\boldsymbol{v}_s \notin \mathcal{R}\{\boldsymbol{J}_{\mathrm{I},0}\}$ for any possible pair (q_1, q_2) (with $c_2 \neq 0$). This is checked by organizing a 3×3 matrix with vector $\boldsymbol{v}_s$ next to basis vectors that span $\mathcal{R}\{\boldsymbol{J}_{\mathrm{I},0}\}$, and computing

$$\det \begin{pmatrix} -s_1 c_2 & -c_1 s_2 & -1 \\ c_1 c_2 & -s_1 s_2 & 1 \\ 0 & c_2 & 0 \end{pmatrix} = c_2^2 \,(c_1 + s_1) \neq 0, \quad \forall (q_1, q_2), \text{ with } q_2 \neq \pm\frac{\pi}{2}.$$

Therefore $\boldsymbol{v}_s$ is independent from (and thus cannot be generated by) any combination of columns of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ in such configurations. Conversely, in case **c)** the similar check yields

$$\det \begin{pmatrix} c_1(s_2 + s_{23}) & c_1 s_{23} & -1 \\ s_1(s_2 + s_{23}) & s_1 s_{23} & 1 \\ 0 & c_2 & 0 \end{pmatrix} = c_2(s_1 - c_1)(s_2 + s_{23}).$$

This determinant can be zeroed by choosing $s_1 = c_1$, i.e., for $q_1 = -\pi/4$ or for $q_1 = 3\pi/4$, both admissible values in this case. Thus, for these two values of the first joint angle, it follows that $\boldsymbol{v}_s \in \mathcal{R}\{\boldsymbol{J}_{\text{II,axis1}}\}$ in case **c)**, i.e., when the robot end-effector is placed on the axis of joint 1 and its forearm is not stretched ($q_3 \neq 0$) nor folded ($q_3 \neq \pi$). A solution can then be found by pseudoinversion of the Jacobian. Taking for instance $q_1 = -\pi/4$, we have

$$\boldsymbol{J}_s(q_2, q_3) = \boldsymbol{J}(\boldsymbol{q})|_{\{q_1=-\pi/4, c_2+c_{23}=0, c_2\neq 0\}} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}(s_2+s_{23}) & -\frac{1}{\sqrt{2}}s_{23} \\ 0 & \frac{1}{\sqrt{2}}(s_2+s_{23}) & \frac{1}{\sqrt{2}}s_{23} \\ 0 & 0 & -c_2 \end{pmatrix}.$$

Its pseudoinverse can be computed also symbolically in MATLAB (still using the `pinv` function):

$$\boldsymbol{J}_s^{\#}(q_2, q_3) = \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{\sqrt{2}}{2(s_2+s_{23})} & \dfrac{\sqrt{2}}{2(s_2+s_{23})} & \dfrac{s_{23}}{c_2(s_2+s_{23})} \\ 0 & 0 & -\dfrac{1}{c_2} \end{pmatrix}.$$

The joint velocity that solves the problem is then

$$\dot{\boldsymbol{q}}_s = \boldsymbol{J}_s^{\#}(q_2, q_3)\boldsymbol{v}_s = \begin{pmatrix} 0 \\ \dfrac{\sqrt{2}}{s_2+s_{23}} \\ 0 \end{pmatrix}. \tag{4}$$

As a result, only joint 2 will move in order to realize the desired $\boldsymbol{v}_s$. Indeed, it is immediate to check that $\boldsymbol{J}_s(q_2, q_3)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s$. Wishing to obtain one of the many possible numerical solutions, we can set, e.g., $\boldsymbol{q} = \begin{pmatrix} -\pi/4 & \pi/4 & \pi/2 \end{pmatrix}^T$ and obtain from (4) the joint velocity $\dot{\boldsymbol{q}}_s = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$. This situation is sketched in Fig. 4.

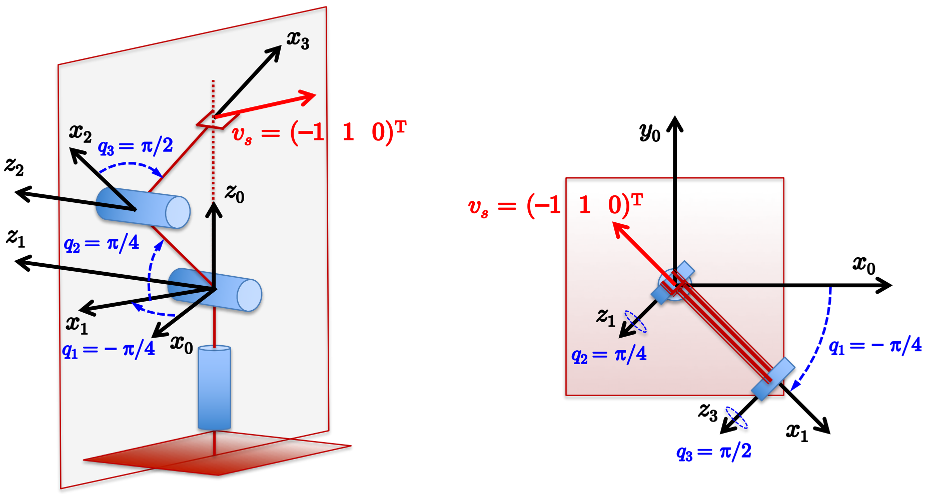

Figure 4: 3D view [left] and top view [right] of a singular configuration of type I for the 3R spatial robot, where the desired end-effector velocity $\boldsymbol{v}_s$ can actually be realized.

Exercise #4

The class of interpolating trajectories that we will consider is given by quintic polynomials, allowing to impose rest-to-rest motion (zero initial and final velocity), but also zero initial and final acceleration and providing thus the required continuity over the entire motion interval, for $t \in [0, T]$. For a generic joint, we will use the doubly normalized quintic polynomial

$$q_i(\tau) = q_i(0) + (q_i(1) - q_i(0)) \left(10\tau^3 - 15\tau^4 + 6\tau^5\right), \qquad i = 1, 2,$$

for $\tau = t/T \in [0, 1]$ and with $\boldsymbol{q}(0) = \boldsymbol{q}_s$, $\boldsymbol{q}(1) = \boldsymbol{q}_g$. For $i = 1, 2$, the associated first and second time derivatives are

$$\dot{q}_i(\tau) = \frac{q_{g,i} - q_{s,i}}{T} \left(30\tau^2 - 60\tau^3 + 30\tau^4\right)$$

and

$$\ddot{q}_i(\tau) = \frac{q_{g,i} - q_{s,i}}{T^2} \left(60\tau - 180\tau^2 + 120\tau^3\right),$$

which are both automatically zero at $\tau = 0$ and $\tau = 1$. For this class of trajectories, the instants of maximum velocity and acceleration (in absolute value) can be found analytically. The maximum acceleration occurs when the jerk (third derivative) is zero, i.e., symmetrically to the trajectory midpoint:

$$\dddot{q}_i(\tau) = \frac{60(q_{g,i} - q_{s,i})}{T^3} \left(1 - 6\tau + 6\tau^2\right) = 0 \quad \Rightarrow \quad \tau_a = 0.5 \pm \frac{\sqrt{3}}{6} \quad (\text{with } \tau_a \in [0, 1]).$$

Therefore[3], we impose the acceleration bounds as

$$\max_{\tau \in [0,1]} |\ddot{q}_i(\tau)| = |\ddot{q}_i(\tau_a)| = \frac{|q_{g,i} - q_{s,i}|}{T^2} \left|60\,\tau_a - 180\,\tau_a^2 + 120\,\tau_a^3\right| \leq A_i, \qquad i = 1, 2. \tag{5}$$

Similarly, the maximum velocity occurs when the acceleration is zero. In turn, the acceleration is a cubic function of time for which we already know that two roots are in $\tau = 0$ and $\tau = 1$ (where velocity has its minimum, i.e., zero). It is easy to see that the third root is at $\tau_v = 0.5$ (the trajectory midpoint), where in fact the velocity reaches its maximum (in absolute value). Thus, we have for the velocity bounds

$$\max_{\tau \in [0,1]} |\dot{q}_i(\tau)| = |\dot{q}_i(\tau_v)| = \frac{|q_{g,i} - q_{s.i}|}{T} \left(30\,\tau_v^2 - 60\,\tau_v^3 + 30\,\tau_v^4\right) \leq V_i, \qquad i = 1, 2. \tag{6}$$

From (5) and (6), we solve for the minimum feasible motion time. One obtains

$$T^* = \max\left\{T_{\min,V_1}, T_{\min,V_2}, T_{\min,A_1}, T_{\min,A_2}\right\},$$

with

$$T_{\min,V_i} = \frac{|q_{g,i} - q_{s.i}|}{V_i} \left(30\,\tau_v^2 - 60\,\tau_v^3 + 30\,\tau_v^4\right), \qquad i = 1, 2,$$

and

$$T_{\min,A_i} = \sqrt{\frac{|q_{g,i} - q_{s,i}|}{A_i} \left|60\,\tau_a - 180\,\tau_a^2 + 120\,\tau_a^3\right|}, \qquad i = 1, 2.$$

[3] Any of the two signs can be used in the expression of τ_a, provided we recognize that the accelerations in the two instants will be equal in magnitude and opposite in sign. In the first instant (before the midpoint), the acceleration will have the same sign of $(q_{g,i} - q_{s,i})$. However, in order to avoid making distinctions, we will take the absolute value of each term in (5).

Plugging in the numerical data of the problem, we obtain

$$T_{\min,V_1} = 2.9452, \qquad T_{\min,V_2} = 2.9452, \qquad T_{\min,A_1} = 2.4589, \qquad T_{\min,A_2} = 3.0115,$$

so that $T^* = T_{\min,A_2} = 3.0115$ [s]. Accordingly, the saturating quantity is the acceleration of joint 2, which will reach $|\ddot{q}_2(\tau_a)| = A_2 = 2$ [rad/s^2]. The other maximum values attained are indeed all feasible:

$$|\ddot{q}_1(\tau_v)| = 0.9870 < 1 = V_1, \qquad |\ddot{q}_2(\tau_v)| = 1.9560 < 2 = V_2, \qquad |\ddot{q}_1(\tau_a)| = 1 < 1.5 = A_1.$$

The plots of trajectory position, velocity, and acceleration of both joints are shown in Figs. 5–7.

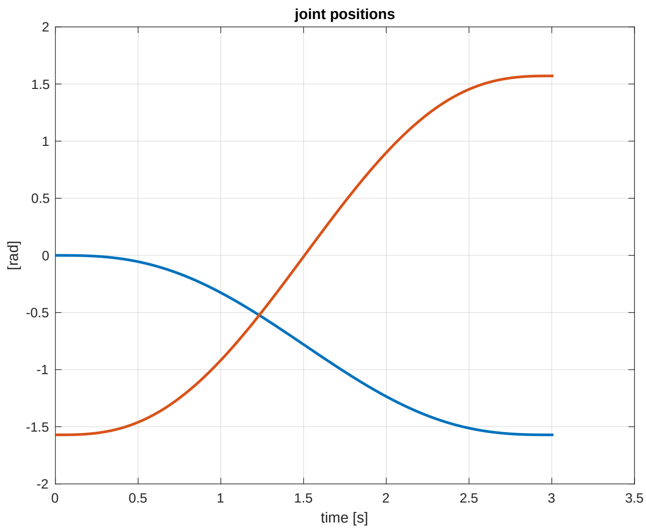

Figure 5: Position of joint 1 (in blue) and 2 (in red).

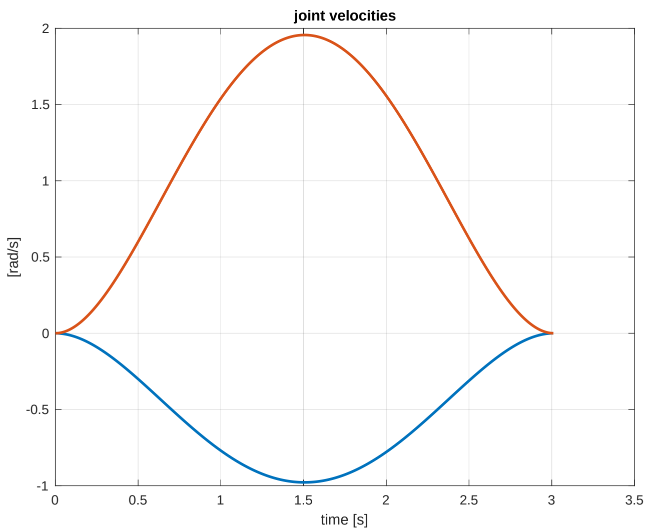

Figure 6: Velocity of joint 1 (in blue) and 2 (in red).

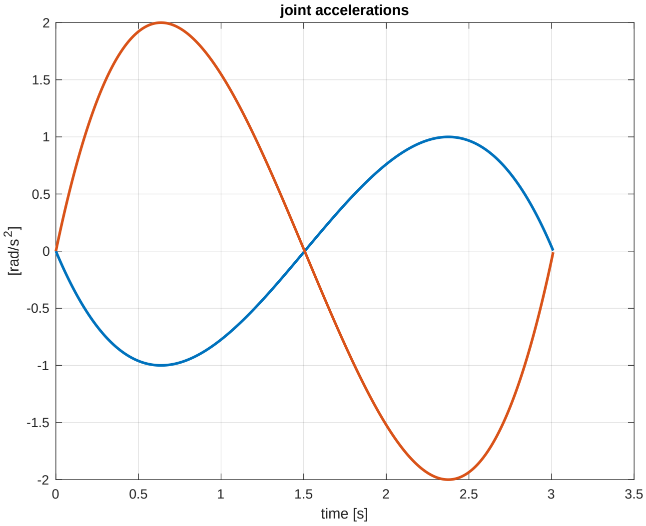

Figure 7: Acceleration of joint 1 (in blue) and 2 (in red). The acceleration of the second joint is the limiting factor in the minimum time solution.

Exercise #5

Three questions were posed.

A) Matrix $\boldsymbol{R}_1$ is not a rotation matrix: its columns are orthonormal, but the determinant is -1. On the other hand, matrices $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ are both elements of $SO(3)$, thus representing rotations.

B) In Harmonic Drives, the Flexspline element has always a number of outer teeth n_{FS} which is two less than the number of inner teeth n_{CS} of the Circular Spline. The reduction ratio is thus

$$n_r = \frac{n_{FS}}{n_{CS} - n_{FS}} = \frac{n_{CS} - 2}{2} = \frac{148}{2} = 74.$$

The angular resolution ρ_m of an absolute encoder mounted on the motor side and having N_t traces ($= n_b$ bits) is related to the angular resolution ρ_l at the link side of the reduction element by

$$\rho_m = \frac{2\pi}{2^{n_b}} = n_r \cdot \rho_l = 74 \cdot 0.0002 = 0.00148 \text{ [rad]}.$$

Therefore

$$n_b = \left\lceil \log_2 \frac{2\pi}{0.00148} \right\rceil = \lceil 8.7298 \rceil = 9 \text{ bits}.$$

C) Given the time evolution of the position profile $q(t) = -3\cos\omega t$, with $\omega = 2$ [rad/s], we have to compare the (known) true value of its analytical time derivative $\dot{q}(t) = 3\,\omega \sin\omega t$, evaluated at $t = t_k = kT_c = 20 \cdot 0.03 = 0.6$ s, with two approximations given by Backward Difference Formulas (BDF) in discrete time[4], namely the 1-step (Euler)

$$\dot{q}_{k,1} = \frac{1}{T_c}\left(q_k - q_{k-1}\right)$$

[4]Note that the coefficients of the combination of samples in BDFs of any order are always alternating in sign and sum up to 0.

and the 4-step one

$$\dot{q}_{k,4} = \frac{1}{T_c}\left(\frac{25}{12}\,q_k - 4\,q_{k-1} + 3\,q_{k-2} - \frac{4}{3}\,q_{k-3} + \frac{1}{4}\,q_{k-4}\right),$$

both evaluated for $k = 20$, i.e., at the time sample $t_k = 0.6$ s. The percentage errors of the two approximations are given by

$$e_f = \left|\frac{\dot{q}_k - \dot{q}_{k,f}}{\dot{q}_k}\right| \cdot 100\ (\%), \qquad f = 1 \text{ or } 4.$$

This simple MATLAB code provides the result:

```
tc=0.03; om=2; k=20; % input data
t=k*tc; % time instant of evaluation = 0.6
t1=t-tc; t2=t1-tc; t3=t2-tc; t4=t3-tc; % backward time instants (up to 4)
qt=-3*cos(om*t); % position at the time instant of evaluation
qt1=-3*cos(om*t1); qt2=-3*cos(om*t2); % positions at previous time instants
qt3=-3*cos(om*t3); qt4=-3*cos(om*t4);
dq=3*om*sin(om*t) % exact value of velocity
dq1=(qt-qt1)/tc % approximation using 1-step BDF (Euler)
dq4=((25/12)*qt-4*qt1+3*qt2-(4/3)*qt3+(1/4)*qt4)/tc % approximation using 4-step BDF
e1=abs((dq-dq1)/dq)*100 % percentage error using 1-step BDF
e4=abs((dq-dq4)/dq)*100 % percentage error using 4-step BDF
```

The output is

$$\texttt{dq} = 5.5922, \qquad \texttt{dq1} = 5.5237, \qquad \texttt{dq4} = 5.5922, \qquad \texttt{e1} = 1.2260\,\% \qquad \texttt{e4} = 2.4770 \cdot 10^{-4}\,\%.$$

The 4-step approximation is much more accurate than the Euler method (in the short format output of MATLAB, it has the same first four decimal digits as the true value). Reducing T_c will reduce the estimation error (absolute or in percentage) for both methods.

* * * * *