Robotics 1

January 12, 2021

There are 10 questions. Provide answers with short texts, completed with drawings and derivations needed for the solutions. Students with confirmed midterm grade should do only the second set of 5 questions.

Question #1 [students without midterm]

The orientation of a rigid body $\mathcal{B}$ is defined by the rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}.$$

Determine the angles (α, β, γ) of a YXZ Euler sequence providing the same orientation. Check the correctness of the obtained result by a direct computation. Find also the singular cases for this minimal representation and provide an example of a rotation matrix $\boldsymbol{R}_s$ that falls in this class.

Question #2 [students without midterm]

Let matrix $\boldsymbol{R}$ of Question #1 be the current orientation of body $\mathcal{B}$. If $\boldsymbol{\Omega} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$ is the instantaneous angular velocity of $\mathcal{B}$ expressed in the *body* frame, compute the time derivative $\dot{\boldsymbol{R}}$.

Question #3 [students without midterm]

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & -a \\ -\sin\theta\cos\alpha & \cos\theta\cos\alpha & \sin\alpha & -d\sin\alpha \\ \sin\theta\sin\alpha & -\cos\theta\sin\alpha & \cos\alpha & -d\cos\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

What is this matrix? Is it correct? Provide a convincing explanation.

Question #4 [students without midterm]

Two views of a spatial 3R robot are shown in Fig. 1, together with the world reference frame RF_w. Assign the Denavit-Hartenberg frames and provide the associated table of parameters so that the configuration in Fig. 1(a) is $\boldsymbol{q}_a = (-\pi/2, \pi/2, 0)$ [rad] and the configuration in Fig. 1(b) is $\boldsymbol{q}_b = (0, 0, \pi/2)$ [rad]. The frame assignment must also include all four lengths L_0, L_1, L_2, and L_3 defined in the figure. Compute the symbolic expression ${}^w\boldsymbol{p}(\boldsymbol{q})$ of the end-effector position P.

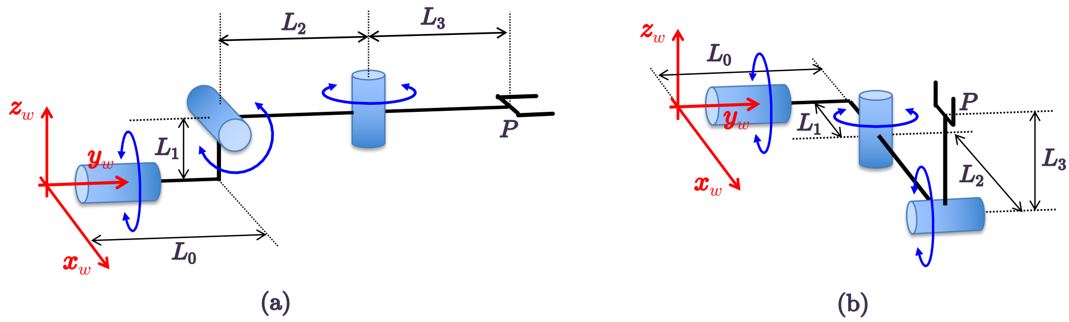

Figure 1: A spatial 3R robot in two different configurations $\boldsymbol{q}_a$ (a) and $\boldsymbol{q}_b$ (b).

Question #5 [students without midterm]

An electrical motor mounts on its axis a multi-turn absolute encoder with 11 bits. The first 3 bits are used for counting turns, while the following 8 bits measure a single turn. The motor drives a robot link through an harmonic drive having a flexspline with 120 external teeth. What is the angular resolution of this equipment at the link side? What is the maximum unidirectional angular displacement at the motor side that can be measured by the encoder? Which motor angle $\theta_m \in [0, 2\pi)$ corresponds to the Gray code [000|01100001]? Express all results in radians.

Question #6 [all students]

For the 4-dof planar RRPR robot in Fig. 2, with the joint variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ defined therein, derive the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ associated to the 3-dimensional task vector $\boldsymbol{r} = (p_x, p_y, \alpha)$, where $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ gives the position of the final flange center P and $\alpha \in \mathbb{R}$ is the orientation of the last robot link w.r.t. the axis $\boldsymbol{x}_0$. Find all singular configurations $\boldsymbol{q}_s$ of this task Jacobian matrix. For one such $\boldsymbol{q}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ and determine a basis for $\mathcal{R}\{\boldsymbol{J}_s\}$ and one for $\mathcal{N}\{\boldsymbol{J}_s\}$.

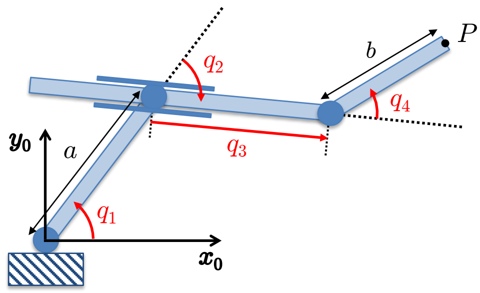

Figure 2: A 4-dof RRPR robot, with joint variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$.

Question #7 [all students]

Two planar 2R robots, named A and B and having both unitary link lengths, are in the static equilibrium shown in Fig. 3. The two D-H configurations w.r.t. their base frames are, respectively, $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$ [rad] and $\boldsymbol{q}_B = (\pi/2, -\pi/2)$ [rad]. Robot A pushes against robot B as in the figure, with a force $\boldsymbol{F} \in \mathbb{R}^2$ having norm $\|\boldsymbol{F}\| = 10$ [N]. Compute the joint torques $\boldsymbol{\tau}_A \in \mathbb{R}^2$ and $\boldsymbol{\tau}_B \in \mathbb{R}^2$ (both in [Nm]) that keep the two robots in equilibrium.

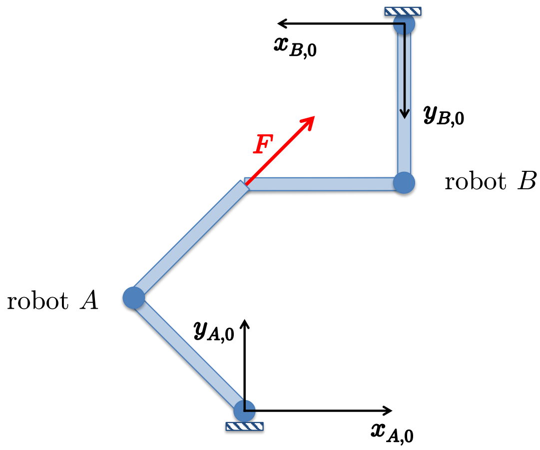

Figure 3: A static equilibrium condition for two planar 2R robots pushing against each other.

Question #8 [all students]

With reference to Fig. 4, a planar 2R robot with link lengths $l_1 = 0.5$ and $l_2 = 0.4$ [m] should intercept and follow a target that moves at constant speed $v = 0.3$ [m/sec] along a line passing through the point $P_0 = (-0.8, 1.1)$ [m] and making an angle $\beta = -20°$ with the axis $\boldsymbol{x}_0$. The robot starts at rest from the configuration $\boldsymbol{q}_s = (\pi, 0)$ [rad] (in DH terms) as soon as the target enters the workspace. The rendez-vous occurs after $T = 2$ s, with the robot end effector and the target having the same final velocity. Plan a coordinated joint space trajectory for this task.

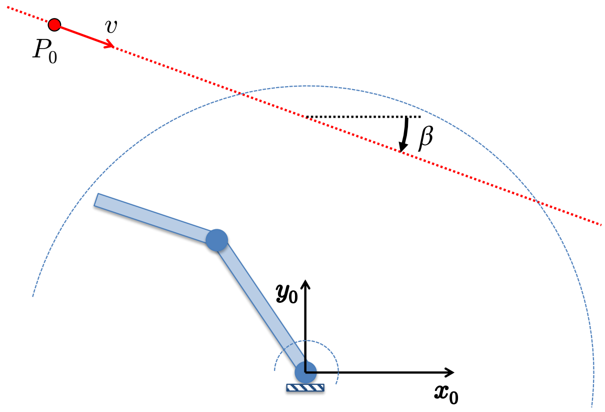

Figure 4: A rendez-vous task for a planar 2R robot and a target in uniform linear motion.

Question #9 [all students]

Consider the following trajectories for the two revolute joints of a robot:

$$q_1(t) = \frac{\pi}{4} + \frac{\pi}{4}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad q_2(t) = -\frac{\pi}{2}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right), \qquad t \in [0, T].$$

Compute the boundary values for the position, velocity, and acceleration at $t = 0$ and $t = T$, and the instants and values of maximum absolute velocity and maximum absolute acceleration for both joints. Assume that the robot motion is bounded by $|\dot{q}_i| \le V_i$ and $|\ddot{q}_i| \le A_i$, for $i = 1, 2$, with

$$V_1 = 4 \text{ [rad/s]}, \qquad V_2 = 8 \text{ [rad/s]}, \qquad A_1 = 20 \text{ [rad/s}^2\text{]}, \qquad A_2 = 40 \text{ [rad/s}^2\text{]}.$$

Determine the minimum feasible motion time T. Sketch the associated time profiles of the position, velocity and acceleration for the two joints.

Question #10 [all students]

Consider again the task in Question #8. The robot is commanded by the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$. Once the rendez-vous has been accomplished, design a feedback control law that will let the robot follow the moving target and react to position errors e_t and e_n that may occur along the tangent and normal directions to the linear path, respectively with the prescribed decoupled dynamics $\dot{e}_t = -3\, e_t$ and $\dot{e}_n = -10\, e_n$. Provide the explicit expression of all terms in the control law.

[240 minutes (4 hours) for the full exam; open books]

[150 minutes (2.5 hours) for students with midterm; open books]

Solution

January 12, 2021

Question #1 [students without midterm]

The orientation of a rigid body $\mathcal{B}$ is defined by the rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}.$$

Determine the angles (α, β, γ) of a YXZ Euler sequence providing the same orientation. Check the correctness of the obtained result by a direct computation. Find also the singular cases for this minimal representation and provide an example of a rotation matrix $\boldsymbol{R}_s$ that falls in this class.

Reply #1

The rotation matrix obtained with the angles (α, β, γ) of a YXZ Euler sequence is computed from

$$\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \; \boldsymbol{R}_X(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}, \; \boldsymbol{R}_Z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Z(\gamma) \\ &= \begin{pmatrix} \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \sin\alpha\sin\beta\sin\gamma - \cos\alpha\sin\gamma & \sin\alpha\cos\beta \\ \cos\beta\sin\gamma & \cos\beta\cos\gamma & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma & \cos\alpha\cos\beta \end{pmatrix}. \end{aligned} \tag{1}$$

We solve the inverse problem for this minimal representation,

$$\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) = \boldsymbol{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix},$$

by using first the elements of the last column in this matrix equality. We obtain

$$\beta = \text{ATAN2}\left\{\sin\beta, \cos\beta\right\} = \text{ATAN2}\left\{-R_{23}, \pm\sqrt{R_{13}^2 + R_{33}^2}\right\}. \tag{2}$$

Provided that $|\cos\beta| = \sqrt{R_{13}^2 + R_{33}^2} \neq 0$ (regular case), we solve for the other two angles as

$$\alpha = \text{ATAN2}\left\{\frac{R_{13}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{22}}{\cos\beta}\right\},$$

obtaining a pair of solutions, one for each sign chosen for $\cos\beta$ in (2). For the given matrix $\boldsymbol{R}$, we have

$$\sqrt{R_{13}^2 + R_{33}^2} = 0.5 = |\cos\beta| \neq 0,$$

and thus a regular case. The two solutions are

$$(\alpha_1, \beta_1, \gamma_1) = (\pi,\, -1.0472,\, \pi/2), \qquad (\alpha_2, \beta_2, \gamma_2) = (0,\, -2.0944,\, -\pi/2) \qquad \text{[rad]}.$$

To verify the result, use (1) to obtain indeed $\boldsymbol{R}_{YXZ}(\alpha_1,\beta_1,\gamma_1) = \boldsymbol{R}_{YXZ}(\alpha_2,\beta_2,\gamma_2) = \boldsymbol{R}$. Finally, a singular case is encountered, e.g., for the rotation matrix

$$\boldsymbol{R}_s = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = \{R_{s,ij}\} \qquad \Rightarrow \qquad \cos\beta_s = 0.$$

In this case $\beta_s = \text{ATAN2}\{-R_{s,23}, 0\} = \pi/2$ and only the difference $\alpha - \gamma$ of the two other angles is defined as

$$\alpha_s - \gamma_s = \text{ATAN2}\{R_{s.12}, R_{s,11}\} = -\pi/2,$$

leading to an infinity of solutions $(\alpha,\beta,\gamma) = (\alpha_s, \pi/2, \alpha_s + \pi/2)$, $\forall \alpha_s$. ■

Question #2 [students without midterm]

Let matrix $\boldsymbol{R}$ *of Question #1 be the current orientation of body* $\mathcal{B}$*. If* $\boldsymbol{\Omega} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$ *is the instantaneous angular velocity of* $\mathcal{B}$ *expressed in the* body *frame, compute the time derivative* $\dot{\boldsymbol{R}}$*.*

Reply #2

The result can be obtained in two equivalent ways, using a skew symmetric matrix $\boldsymbol{S}$ built with the angular velocity of the body $\mathcal{B}$. Either we express the angular velocity in the base frame as $\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Omega}$ and then use $\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}$ (as in the lecture slides). Or we use directly the alternative form $\dot{\boldsymbol{R}} = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega})$ (as in Exercise #1 in the June 11, 2012 exam). Being

$$\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Omega} = \begin{pmatrix} -1 \\ 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix},$$

we obtain in both cases

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R} = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & 0.5 \\ \frac{\sqrt{3}}{2} & 0 & 1 \\ -0.5 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}$$

$$= \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega}) = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -0.5 \\ -0.5 & -0.5 & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$ ■

Question #3 [students without midterm]

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & -a \\ -\sin\theta\cos\alpha & \cos\theta\cos\alpha & \sin\alpha & -d\sin\alpha \\ \sin\theta\sin\alpha & -\cos\theta\sin\alpha & \cos\alpha & -d\cos\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

What is this matrix? Is it correct? Provide a convincing explanation.

Reply #3

Matrix $\boldsymbol{M}$ simply represents the inverse of the generic Denavit-Hartenberg homogeneous transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} \cos\theta & -\sin\theta\cos\alpha & \sin\theta\sin\alpha & a\cos\theta \\ \sin\theta & \cos\theta\cos\alpha & -\cos\theta\sin\alpha & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}(\alpha,\theta) & \boldsymbol{p}(a,d,\theta) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.$$

In fact,

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \boldsymbol{R}^T(\alpha,\theta) & -\boldsymbol{R}^T(\alpha,\theta)\boldsymbol{p}(a,d,\theta) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & -a \\ -\sin\theta\cos\alpha & \cos\theta\cos\alpha & \sin\alpha & -d\sin\alpha \\ \sin\theta\sin\alpha & -\cos\theta\sin\alpha & \cos\alpha & -d\cos\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \boldsymbol{M}. \qquad \blacksquare$$

Question #4 [students without midterm]

Two views of a spatial 3R robot are shown in Fig. 1, together with the world reference frame RF_w. Assign the Denavit-Hartenberg frames and provide the associated table of parameters so that the configuration in Fig. 1(a) is $\boldsymbol{q}_a = (-\pi/2, \pi/2, 0)$ [rad] and the configuration in Fig. 1(b) is $\boldsymbol{q}_b = (0, 0, \pi/2)$ [rad]. The frame assignment must also include all four lengths L_0, L_1, L_2, and L_3 defined in the figure. Compute the symbolic expression ${}^w\boldsymbol{p}(\boldsymbol{q})$ of the end-effector position P.

Reply #4

An assignment of the Denavit-Hartenberg (DH) frames is illustrated in the two configurations shown in Fig. 5. This assignment is consistent with the values $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ that joint variables should take in the configurations of Fig. 1(a) and (b). The associated set of DH parameters is given in Table 1. The origins of the DH frames 0 and 3 have been chosen coincident with the origin O_w of the world frame and with the end-effector position P, respectively. In this way, all four kinematic lengths L_i, $i = 0, 1, 2, 3$, appear in the DH table. The fourth and fifth columns in the table return the values of the joint variables for the two configurations $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$.

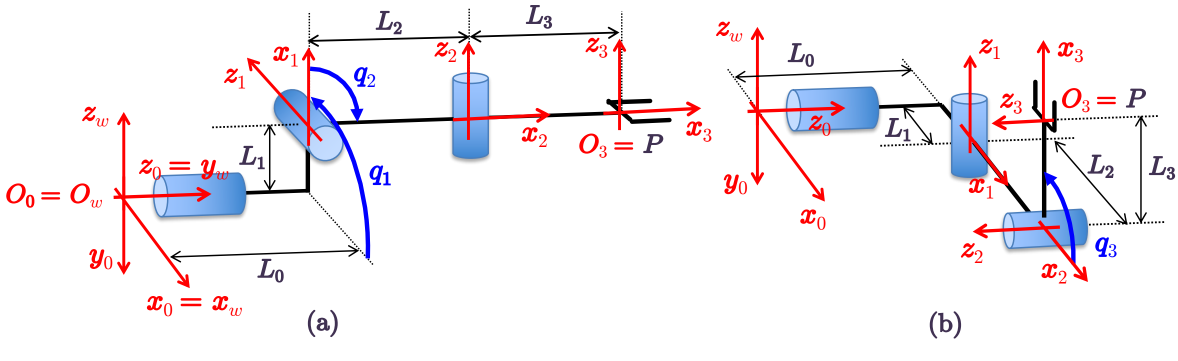

Figure 5: Assignment of the DH frames for the spatial 3R robot, shown here for the two configurations $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ of Fig. 1.

i	α_i	a_i	d_i	θ_i (a)	θ_i (b)
1	$\pi/2$	L_1	L_0	$q_{a1} = -\pi/2$	$q_{b1} = 0$
2	$\pi/2$	L_2	0	$q_{a2} = \pi/2$	$q_{b2} = 0$
3	0	L_3	0	$q_{a3} = 0$	$q_{b3} = \pi/2$

Table 1: Table of DH parameters for the spatial 3R robot.

By building the DH homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, 2, 3$, from Table 1, it is straightforward to compute the position of the origin of the end-effector frame $O_3 = P$ as

$$
{}^0\boldsymbol{p}_H(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_1(q_2)\,{}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \cos q_1\,(L_1 + (L_2 + L_3\cos q_3)\cos q_2) + L_3 \sin q_1 \sin q_3 \\ \sin q_1\,(L_1 + (L_2 + L_3\cos q_3)\cos q_2) - L_3 \cos q_1 \sin q_3 \\ L_0 + (L_2 + L_3 \cos q_3)\sin q_2 \\ 1 \end{pmatrix}.
$$

In order to change the expression of this position vector from the base frame of the robot (the 0th DH frame) to the world frame, we need the additional rotation matrix

$$
{}^w\boldsymbol{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
$$

Since the world frame and 0th DH frame have the same origin, we have

$$
{}^w\boldsymbol{p} = {}^w\boldsymbol{R}_0\,{}^0\boldsymbol{p} = \begin{pmatrix} \cos q_1\,(L_1 + (L_2 + L_3\cos q_3)\cos q_2) + L_3 \sin q_1 \sin q_3 \\ L_0 + (L_2 + L_3 \cos q_3)\sin q_2 \\ -\sin q_1\,(L_1 + (L_2 + L_3\cos q_3)\cos q_2) + L_3 \cos q_1 \sin q_3 \end{pmatrix}. \qquad \blacksquare
$$

Question #5 [students without midterm]

An electrical motor mounts on its axis a multi-turn absolute encoder with 11 bits. The first 3 bits are used for counting turns, while the following 8 bits measure a single turn. The motor drives a robot link through an harmonic drive having a flexspline with 120 external teeth. What is the angular resolution of this equipment at the link side? What is the maximum unidirectional angular displacement at the motor side that can be measured by the encoder? Which motor angle $\theta_m \in [0, 2\pi)$ *corresponds to the Gray code* [000|01101001]*? Express all results in radians.*

Reply #5

Being $N_b = 8$ bits devoted to a single turn of the absolute encoder, its angular resolution on the motor side is $\Delta_m = 2\pi/2^{N_b} = 2\pi/256 = 0.0245$ [rad]. The reduction ratio of an harmonic drive with $N_f = 120$ teeth on the flexspline is $N_r = N_f/2 = 60$. Thus, the angular resolution on the link side is $\Delta_l = \Delta_m/N_r = 4.0906 \cdot 10^{-4}$ [rad] (about 2 hundreds of a degree). Being $N_t = 3$ bits devoted to the counting of full motor turns, when the motor rotates in a single direction (say, starting from $\theta_{m,init} = 0$ and counterclockwise), the maximum angular displacement that will be measured is $\Delta_{max} = 2\pi \cdot 2^{N_t} - \Delta_m = 50.2409$ [rad] (the Δ_m can also be neglected, leading to $\Delta_{max} = 50.2655$ [rad]). Finally, the given Gray code refers to the first motor turn (the $N_t = 3$ most significant bits are zero). The conversion to binary of the least significant $N_b = 8$ bits (a byte), $x_{gray} = [01101001]$, can be done using logical exclusive-or operations (as shown in the lecture slides). We obtain $x_{bin} = [01001110]$ and then $x_{dec} = 2^7 + 2^4 + 2^3 + 2^2 = 78$. The following simple Matlab code does the conversions:

```
xgray=[0 1 1 0 1 0 0 1] \% from MSB to LSB
\% Gray to binary
xbin(1)=xgray(1);
for i=1:Nbits-1
  xbin(i+1)=xor(xbin(i),xgray(i+1));
end
\% binary to decimal
xdec=xbin(Nbits);
for i=1:Nbits-1
  xdec=xdec+xbin(Nbits-i)*2^i;
end
```

The measured motor angle is thus $\theta_m = x_{dec} \cdot \Delta_m = 1.9144$ [rad] (about 109°). ■

Question #6 [all students]

For the 4-dof planar RRPR robot in Fig. 2, with the joint variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ defined therein, derive the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ associated to the 3-dimensional task vector $\boldsymbol{r} = (p_x, p_y, \alpha)$, where $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ gives the position of the final flange center P and $\alpha \in \mathbb{R}$ is the orientation of the last robot link w.r.t. the axis $\boldsymbol{x}_0$. Find all singular configurations $\boldsymbol{q}_s$ of this task Jacobian matrix. For one such $\boldsymbol{q}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ and determine a basis for $\mathcal{R}\{\boldsymbol{J}_s\}$ and one for $\mathcal{N}\{\boldsymbol{J}_s\}$.

Reply #6

The task kinematics of this robot is given by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} a\cos q_1 + q_3\cos(q_1+q_2) + b\cos(q_1+q_2+q_4) \\ a\sin q_1 + q_3\sin(q_1+q_2) + b\sin(q_1+q_2+q_4) \\ q_1+q_2+q_4 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

The associated 3×4 task Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -a\,s_1 - q_3 s_{12} - b\,s_{124} & -q_3 s_{12} - b\,s_{124} & c_{12} & -b\,s_{124} \\ a\,c_1 + q_3 c_{12} + b\,c_{124} & q_3 c_{12} + b\,c_{124} & s_{12} & b\,c_{124} \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

where we have used the trigonometric shorthand notation (e.g., $s_{124} = \sin(q_1+q_2+q_4)$) for compactness. As usual, in order to perform a singularity analysis of the Jacobian, it is convenient to get rid of the angle q_1 from its expression[1]. This is obtained by premultiplying $\boldsymbol{J}$ by the transpose of the rotation matrix $\boldsymbol{R}(q_1)$:

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \boldsymbol{R}^T(q_1)\,\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_3 s_2 - b\,s_{24} & -q_3 s_2 - b\,s_{24} & c_2 & -b\,s_{24} \\ a + q_3 c_2 + b\,c_{24} & q_3 c_2 + b\,c_{24} & s_2 & b\,c_{24} \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

[1] An intrinsic property of the manipulator, like the loss of mobility at the task level in certain configurations, will never depend on the arbitrary choice of the base (zero-th) frame of the robot, and thus on the value of q_1.

To study the rank of matrix $\boldsymbol{J}$ (or, equivalently, of ${}^1\boldsymbol{J}$), we have two possible ways. The first, and more cumbersome, is to compute the determinant of the square 3×3 matrix $\boldsymbol{J}\boldsymbol{J}^T$. Performing computations in Matlab yields

$$\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right) = \det\left({}^1\boldsymbol{J}(\boldsymbol{q})\,{}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = 2a^2c_2^2 + 2q_3^2 + a^2q_3^2 + 2aq_3c_2 - a^2q_3^2c_2^2.$$

Because of the undefined sign of the fourth addend and of the minus sign in the last term, it is not immediate to conclude on necessary and sufficient conditions for zeroing this determinant. The second way is to analyze the four minors obtained by deleting each time one column from the Jacobian. This can be done either on $\boldsymbol{J}$ or on ${}^1\boldsymbol{J}$, leading to identical results in both cases (also when using the symbolic code in Matlab). For compactness, we illustrate the method on the ${}^1\boldsymbol{J}$ matrix only. We have

$${}^1\boldsymbol{J}_{-1}(\boldsymbol{q}) = \begin{pmatrix} -q_3s_2 - b\,s_{24} & c_2 & -b\,s_{24} \\ q_3c_2 + b\,c_{24} & s_2 & b\,c_{24} \\ 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \det {}^1\boldsymbol{J}_{-1}(\boldsymbol{q}) = -q_3$$

$${}^1\boldsymbol{J}_{-2}(\boldsymbol{q}) = \begin{pmatrix} -q_3s_2 - b\,s_{24} & c_2 & -b\,s_{24} \\ a + q_3c_2 + b\,c_{24} & s_2 & b\,c_{24} \\ 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \det {}^1\boldsymbol{J}_{-2}(\boldsymbol{q}) = -q_3 - a\,c_2$$

$${}^1\boldsymbol{J}_{-3}(\boldsymbol{q}) = \begin{pmatrix} -q_3s_2 - b\,s_{24} & -q_3s_2 - b\,s_{24} & -b\,s_{24} \\ a + q_3c_2 + b\,c_{24} & q_3c_2 + b\,c_{24} & b\,c_{24} \\ 1 & 1 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \det {}^1\boldsymbol{J}_{-3}(\boldsymbol{q}) = a\,q_3s_2$$

$${}^1\boldsymbol{J}_{-4}(\boldsymbol{q}) = \begin{pmatrix} -q_3s_2 - b\,s_{24} & -q_3s_2 - b\,s_{24} & c_2 \\ a + q_3c_2 + b\,c_{24} & q_3c_2 + b\,c_{24} & s_2 \\ 1 & 1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \det {}^1\boldsymbol{J}_{-4}(\boldsymbol{q}) = a\,c_2.$$

In order for the Jacobian to be singular, all four determinants above should simultaneously be zero. This occurs if and only if

$$q_3 = \cos q_2 = 0 \qquad \Longleftrightarrow \qquad q_2 = \pm\frac{\pi}{2},\ q_3 = 0.$$

Thus, the prismatic joint should be fully retracted (so that joint axes 2 and 4 coincide) and the second link should be orthogonal to the first one. Choosing for instance the configuration $\boldsymbol{q}_s = (q_1, \pi/2, 0, q_4)$, with arbitrary q_1 and q_4, leads to

$${}^1\boldsymbol{J}_s = {}^1\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} -b\,c_4 & -b\,c_4 & 0 & -b\,c_4 \\ a - b\,s_4 & -b\,s_4 & 1 & -b\,s_4 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank} {}^1\boldsymbol{J}_s = 2.$$

A basis for the null space of the Jacobian $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ can be computed using directly ${}^1\boldsymbol{J}_s$:

$$\mathcal{N}\left\{\boldsymbol{J}_s\right\} = \mathcal{N}\left\{{}^1\boldsymbol{J}_s\right\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ a \\ 0 \end{pmatrix} \right\}.$$

Note in particular that the first vector prescribes an equal and opposite velocity to joints 2 and 4. The second basis vector involves instead also the third, prismatic joint. To provide a basis for the range space of $\boldsymbol{J}_s$, we first pick two independent columns of ${}^1\boldsymbol{J}_s$

$$\mathcal{R}\left\{{}^1\boldsymbol{J}_s\right\} = \left\{ \begin{pmatrix} -b\,c_4 \\ -b\,s_4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and then obtain, from $\boldsymbol{J}_s = \boldsymbol{R}(q_1)\,{}^1\boldsymbol{J}_s$,

$$\mathcal{R}\left\{\boldsymbol{J}_s\right\} = \left\{ \boldsymbol{R}(q_1) \begin{pmatrix} -b\,c_4 \\ -b\,s_4 \\ 1 \end{pmatrix}, \boldsymbol{R}(q_1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -b\,c_{14} \\ -b\,s_{14} \\ 1 \end{pmatrix}, \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \right\}. \qquad \blacksquare$$

Question #7 [all students]

Two planar 2R robots, named A and B and having both unitary link lengths, are in the static equilibrium shown in Fig. 3. The two D-H configurations w.r.t. their base frames are, respectively, $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$ *[rad] and* $\boldsymbol{q}_B = (\pi/2, -\pi/2)$ *[rad]. Robot A pushes against robot B as in the figure, with a force* $\boldsymbol{F} \in \mathbb{R}^2$ *having norm* $\|\boldsymbol{F}\| = 10$ *[N]. Compute the joint torques* $\boldsymbol{\tau}_A \in \mathbb{R}^2$ *and* $\boldsymbol{\tau}_B \in \mathbb{R}^2$ *(both in [Nm]) that keep the two robots in equilibrium.*

Reply #7

Evaluate the 2×2 Jacobians of the two 2R robots, respectively at $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$ [rad] and $\boldsymbol{q}_B = (\pi/2, -\pi/2)$, each expressed in its own DH base frame:

$$\boldsymbol{J}_A(\boldsymbol{q}_A) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_A} = \begin{pmatrix} -\sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix},$$

$$\boldsymbol{J}_B(\boldsymbol{q}_B) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The force vector $\boldsymbol{F}_A$ applied by robot A to robot B is oriented as the second link of robot A. When expressed in the base frame of robot A, it is

$${}^A\boldsymbol{F}_A = \|\boldsymbol{F}\| \cdot \begin{pmatrix} \cos(q_1+q_2) \\ \sin(q_1+q_2) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_A} = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}.$$

To obtain this Cartesian force at the end effector, robot A should produce a joint torque given by

$$\boldsymbol{\tau}_A = \boldsymbol{J}_A^T(\boldsymbol{q}_A)\,{}^A\boldsymbol{F}_A = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

On the other hand, robot B should balance the force applied by robot A at its end effector by reacting with an equal and opposite force $\boldsymbol{F}_B = -\boldsymbol{F}$, namely ${}^A\boldsymbol{F}_B = -\,{}^A\boldsymbol{F}_A$ when these forces are both expressed in the same base frame of robot A. However, when expressing the exchanged force in the base frame of robot B, we can easily see that[2]

$${}^B\boldsymbol{F}_B = {}^B\boldsymbol{R}_A\,{}^A\boldsymbol{F}_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \left(-\,{}^A\boldsymbol{F}_A\right) = {}^A\boldsymbol{F}_A = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}.$$

[2]Here, we use planar 2×2 rotation matrices, i.e., $\boldsymbol{R} \in SO(2)$.

Therefore, to obtain this Cartesian force at the end-effector, robot B should produce a joint torque given by

$$\boldsymbol{\tau}_B = \boldsymbol{J}_B^T(\boldsymbol{q}_B)\,{}^B\boldsymbol{F}_B = \begin{pmatrix} 0 \\ 5\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 7.0711 \end{pmatrix} \text{ [Nm]}.$$

It is also worth reasoning on the two zeros that appear in $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$, in relation with the geometry of this static interaction task. Such analysis is left to the reader. ■

Question #8 [all students]

With reference to Fig. 4, a planar 2R robot with link lengths $l_1 = 0.5$ and $l_2 = 0.4$ [m] should intercept and follow a target that moves at constant speed $v = 0.3$ [m/sec] along a line passing through the point $P_0 = (-0.8, 1.1)$ [m] and making an angle $\beta = -20°$ with the axis $\boldsymbol{x}_0$. The robot starts at rest from the configuration $\boldsymbol{q}_s = (\pi, 0)$ [rad] (in DH terms) as soon as the target enters the workspace. The rendez-vous occurs after $T = 2$ s, with the robot end effector and the target having the same final velocity. Plan a coordinated joint space trajectory for this task.

Reply #8

In this trajectory planning problem, we need first to define the boundary conditions for the rendez-vous between the moving target and the robot end-effector. The robot workspace has an external (circular) boundary of radius $R = l_1 + l_2 = 0.9$ [m] (while the internal boundary has radius $R_{min} = |l_1 - l_2| = 0.1$ [m]). The target moves on a line that intercepts the external boundary in two points P_1 and P_2, the first of which is of interest. These points are found by solving the system of equations

$$\begin{cases} (x - x_0)\sin\beta - (y - y_0)\cos\beta = 0 & \text{[line through } P_0 = (x_0, y_0) \text{ with angular coefficient } \beta] \\ x^2 + y^2 = R^2 & \text{[circle with center in the origin and radius } R] \end{cases} \tag{3}$$

The solution is obtained using the two (real) roots x_1 and x_2 of the following second-order polynomial equation[3] derived from (3):

$$x^2 - 2(x_0 \sin\beta - y_0 \cos\beta)\sin\beta\, x + (x_0 \sin\beta - y_0 \cos\beta)^2 - R^2 \cos^2\beta = 0.$$

With each of these roots, we compute also the y-coordinate of the intercepting points:

$$y_i = y_0 + (x_i - x_0)\tan\beta, \qquad i = 1, 2.$$

From the given data, we get

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -0.1930 \\ 0.8791 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.7129 \\ 0.5494 \end{pmatrix} \quad \text{[m]}.$$

Note that $\|P_1\| = \|P_2\| = R = 0.9$. Clearly, the target enters the workspace in P_1 (and will exit in P_2). We shall set the initial time $t = 0$ of the robot motion at the instant of target entrance. Moreover, at $T = 2$ s, the target will be in the planned rendez-vous point

$$P_{rv} = P_1 + vT \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = \begin{pmatrix} 0.3708 \\ 0.6739 \end{pmatrix} \text{ [m]}.$$

Since $\|P_{rv}\| = 0.7692 < R$, the rendez-vouz will occur well inside the robot workspace. This Cartesian point specifies, via kinematic inversion, the goal configuration that the robot should reach at time $t = T$. From the usual inverse kinematics of the 2R robot, coded in Matlab as follows

[3] Indeed, this second-order equation may have either two real roots or two complex conjugate roots. In the latter case, no intercepting points exist. When P_0 is inside the circle ($\|P_0\| < R$), there are always two real roots.

```
p_x=P_rv(1);p_y=P_rv(2);
c2=(p_x^2+p_y^2-(l_1^2+l_2^2))/(2*l_1*l_2)
s2=-sqrt(1-c2^2) \% choose the minus sign to have the elbow-up solution as goal,
                 \% being this ‘closer’ to the start configuration qs
q2_g=atan2(s2,c2)
q1_g=atan2(p_y*(l_1+l_2*c2)-p_x*l_2*s2,p_x*(l_1+l_2*c2)+p_y*l_2*s2)
```

we obtain

$$\boldsymbol{q}_g = \boldsymbol{q}(T) = \begin{pmatrix} 1.5495 \\ -1.0996 \end{pmatrix} \text{[rad]} \quad \left(= \begin{pmatrix} 88.78^\circ \\ -63.00^\circ \end{pmatrix} \right).$$

In addition, the Cartesian velocity of the target at the goal instant $t = T$ of rendez-vous (actually, also at any other instant) is

$$\boldsymbol{v}_g = \dot{\boldsymbol{p}}(T) = v \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = 0.3 \begin{pmatrix} 0.9397 \\ -0.3420 \end{pmatrix} = \begin{pmatrix} 0.2819 \\ -0.1026 \end{pmatrix} \text{[m/s]}.$$

Therefore, by inverting the robot Jacobian at the rendez-vous configuration, the joint velocity at the goal is computed as

$$\dot{\boldsymbol{q}}_g = \dot{\boldsymbol{q}}(T) = \boldsymbol{J}^{-1}(\boldsymbol{q}_g)\,\boldsymbol{v}_g = \left.\begin{pmatrix} -l_1\sin q_1 - l_2\sin(q_1+q_2) & -l_2\sin(q_1+q_2) \\ l_1\cos q_1 + l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2) \end{pmatrix}^{-1}\right|_{\boldsymbol{q}=\boldsymbol{q}_g} \boldsymbol{v}_g$$

$$= \begin{pmatrix} -0.6739 & -0.1740 \\ 0.3708 & 0.3602 \end{pmatrix}^{-1} \boldsymbol{v}_g = \begin{pmatrix} -2.0212 & -0.9762 \\ 2.0809 & 3.7814 \end{pmatrix} \begin{pmatrix} 0.2819 \\ -0.1026 \end{pmatrix} = \begin{pmatrix} -0.4696 \\ 0.1986 \end{pmatrix} \text{[rad/s]}.$$

At this stage, there are 4 boundary conditions to be interpolated between $t = 0$ and $t = T = 2$ s for each joint. For the first joint, we have

$$q_{s,1} = q_1(0) = \pi, \quad \dot{q}_{s,1} = \dot{q}_1(0) = 0, \quad q_{g,1} = q_1(T) = 1.5495, \quad \dot{q}_{g,1} = \dot{q}_1(T) = -0.4696,$$

while for the second joint

$$q_{s,2} = q_2(0) = 0, \quad \dot{q}_{s,2} = \dot{q}_2(0) = 0, \quad q_{g,2} = q_2(T) = -1.0996, \quad \dot{q}_{g,2} = \dot{q}_2(T) = 0.1986.$$

Since there are no further conditions specified, we choose a cubic polynomial for each joint as interpolating function —the simplest solution with enough parameters to satisfy all boundary conditions. The motion of the robot joints should be coordinated. Thus, we will use a common parametrized time τ to define the joint trajectories $\boldsymbol{q}_d(\tau)$. Let the joint displacement vector be

$$\boldsymbol{\Delta} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \boldsymbol{q}_g - \boldsymbol{q}_s = \begin{pmatrix} -1.5921 \\ -1.0996 \end{pmatrix}.$$

For $\tau = t/T \in [0,1]$, we compute the desired trajectories as

$$q_{d,i}(\tau) = q_{s,i} + \Delta_i \left(\left(\frac{\dot{q}_{g,i}T}{\Delta_i} - 2 \right) \tau^3 + \left(3 - \frac{\dot{q}_{g,i}T}{\Delta_i} \right) \tau^2 \right), \qquad i = 1, 2, \tag{4}$$

with velocity profiles

$$\dot{q}_{d,i}(\tau) = \frac{\Delta_i}{T} \left(3 \left(\frac{\dot{q}_{g,i}T}{\Delta_i} - 2 \right) \tau^2 + 2 \left(3 - \frac{\dot{q}_{g,i}T}{\Delta_i} \right) \tau \right), \qquad i = 1, 2.$$

Plugging the data in (4), we obtain

$$q_{d,1}(\tau) = 2.2449\,\tau^3 - 3.8370\,\tau^2 + \pi, \qquad \tau = t/T \in [0,1],$$

and

$$q_{d,2}(\tau) = 2.5964\,\tau^3 - 3.6960\,\tau^2, \qquad \tau = t/T \in [0,1].$$

The resulting position and velocity profiles are shown in Fig. 8. In Fig. 7, we illustrate with a stroboscopic view the approaching phase of the robot to the moving target during a time interval of $T = 2$ s. ■

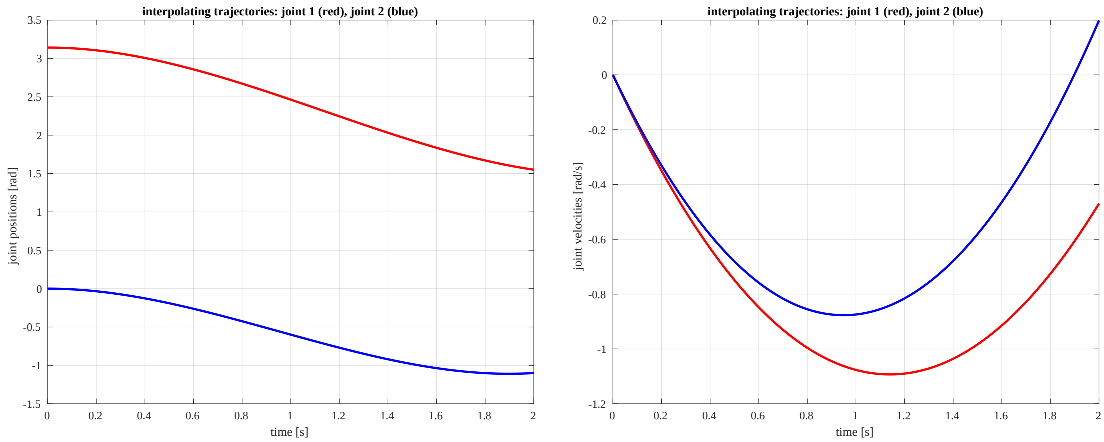

Figure 6: Joint position and velocity profiles of the planned interpolating trajectories for the rendez-vous between the robot end effector and the moving target.

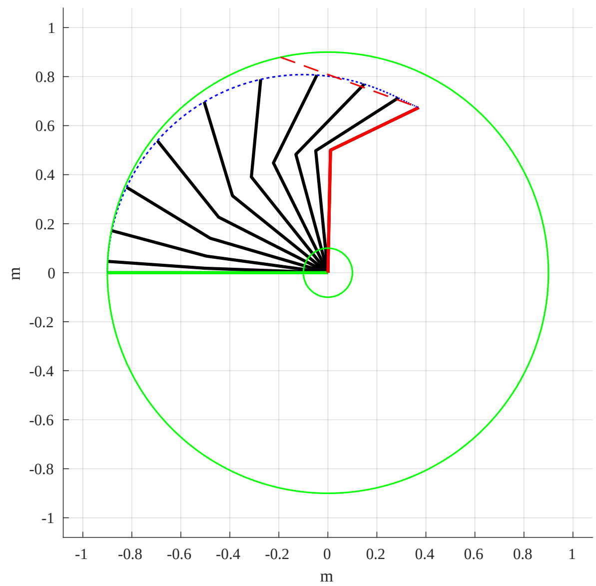

Figure 7: Stroboscopic view of robot and target motion in the rendez-vous task.

Question #9 [all students]

Consider the following trajectories for the two revolute joints of a robot:

$$q_1(t) = \frac{\pi}{4} + \frac{\pi}{4}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad q_2(t) = -\frac{\pi}{2}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right), \qquad t \in [0, T].$$

Compute the boundary values for the position, velocity, and acceleration at $t = 0$ and $t = T$, and the instants and values of maximum absolute velocity and maximum absolute acceleration for both joints. Assume that the robot motion is bounded by $|\dot{q}_i| \le V_i$ and $|\ddot{q}_i| \le A_i$, for $i = 1, 2$, with

$$V_1 = 4 \text{ [rad/s]}, \qquad V_2 = 8 \text{ [rad/s]}, \qquad A_1 = 20 \text{ [rad/s}^2\text{]}, \qquad A_2 = 40 \text{ [rad/s}^2\text{]}.$$

Determine the minimum feasible motion time T. Sketch the associated time profiles of the position, velocity and acceleration for the two joints.

Reply #9

Differentiating w.r.t. time the trajectories of the two joints yields

$$\dot{q}_1(t) = \frac{3\pi}{2T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right), \qquad \dot{q}_2(t) = -\frac{\pi^2}{2T}\sin\left(\frac{\pi t}{T}\right), \qquad t \in [0, T]$$

and further

$$\ddot{q}_1(t) = \frac{3\pi}{2T^2}\left(1 - 2\left(\frac{t}{T}\right)\right), \qquad \ddot{q}_2(t) = -\frac{\pi^3}{2T^2}\cos\left(\frac{\pi t}{T}\right), \qquad t \in [0, T].$$

By direct evaluation, we obtain the boundary values for the first joint

$$\begin{aligned} q_1(0) &= \frac{\pi}{4} & q_1(T) &= \frac{\pi}{2} \\ \dot{q}_1(0) &= 0 & \dot{q}_1(T) &= 0 \\ \ddot{q}_1(0) &= \frac{3\pi}{2T^2} & \ddot{q}_1(T) &= -\frac{3\pi}{2T^2}, \end{aligned}$$

and for the second joint

$$\begin{aligned} q_2(0) &= 0 & q_2(T) &= -\pi \\ \dot{q}_2(0) &= 0 & \dot{q}_2(T) &= 0 \\ \ddot{q}_2(0) &= -\frac{\pi^3}{2T^2} & \ddot{q}_2(T) &= \frac{\pi^3}{2T^2}. \end{aligned}$$

Moreover, it is easy to see that the following maximum absolute values are attained for the velocities

$$\max_{t\in[0,T]} |\dot{q}_1(t)| = |\dot{q}_1(T/2)| = \frac{3\pi}{8T}, \qquad \max_{t\in[0,T]} |\dot{q}_2(t)| = |\dot{q}_2(T/2)| = \frac{\pi^2}{2T},$$

and for the accelerations

$$\max_{t\in[0,T]} |\ddot{q}_1(t)| = |\ddot{q}_1(0)| = |\ddot{q}_1(T)| = \frac{3\pi}{2T^2}, \qquad \max_{t\in[0,T]} |\ddot{q}_2(t)| = |\ddot{q}_2(0)| = |\ddot{q}_2(T)| = \frac{\pi^3}{2T^2}.$$

In order to satisfy all the given bounds, the minimum value of the motion time T is determined as

$$T = \max\left\{\frac{3\pi}{8V_1}, \sqrt{\frac{3\pi}{2A_1}}, \frac{\pi^2}{2V_2}, \sqrt{\frac{\pi^3}{2A_2}}\right\} = \max\left\{0.2945,\ 0.4854,\ 0.6169,\ 0.6226\right\} = 0.6226 \text{ [s]},$$

which is enforced by the acceleration limit $A_2 = 40$ [rad/s^2] on the second joint. With $T = 0.6226$ s, the maximum values reached by the absolute velocities and accelerations are

$$\max_{t\in[0,T]} |\dot{q}_1(t)| = 1.8923, \quad \max_{t\in[0,T]} |\dot{q}_2(t)| = 7.9267, \quad \max_{t\in[0,T]} |\ddot{q}_1(t)| = 12.1585, \quad \max_{t\in[0,T]} |\ddot{q}_2(t)| = 40 = A_2.$$

The plots of the minimum time trajectories are reported in Fig. 8. ■

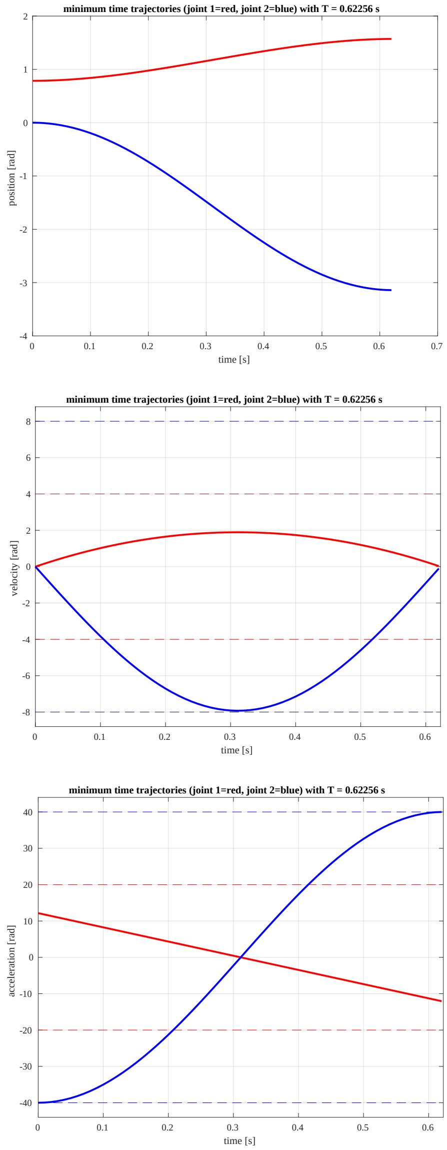

Figure 8: Minimum time solution for the considered class of trajectories under velocity/acceleration bounds: position, velocity, and acceleration profiles (continuous) and their limits (dashed).

Question #10 [all students]

Consider again the task in Question #8. The robot is commanded by the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$. Once the rendez-vous has been accomplished, design a feedback control law that will let the robot follow the moving target and react to position errors e_t and e_n that may occur along the tangent and normal directions to the linear path, respectively with the prescribed decoupled dynamics $\dot{e}_t = -3\, e_t$ and $\dot{e}_n = -10\, e_n$. Provide the explicit expression of all terms in the control law.

Reply #10

The control law contains a velocity feedforward term, in order to follow the moving target in nominal conditions, and a feedback action on the Cartesian error, which is rotated in the (Frenet) frame associated to the path. The target moves at constant speed $v = 0.3$ [m/sec] along a linear path making an angle $\beta = -20°$ with the axis $\boldsymbol{x}_0$. Thus, we have

$$\dot{\boldsymbol{p}}_d = v \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = 0.3 \begin{pmatrix} 0.9397 \\ -0.3420 \end{pmatrix} = \begin{pmatrix} 0.2819 \\ -0.1026 \end{pmatrix} \text{ [m/s]}.$$

The Cartesian error $\boldsymbol{e}_p \in \mathbb{R}^2$ is rotated into the tangential and normal components to the path (i.e., in the task frame) as

$$\boldsymbol{e}_p = \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{e}_{task} = \begin{pmatrix} e_t \\ e_n \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \boldsymbol{R}^T(\beta)\, \boldsymbol{e}_p, \quad (5)$$

where

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{pmatrix}$$

is the direct kinematics of the 2R robot. The complete control law is then

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\dot{\boldsymbol{p}}_d + \boldsymbol{R}(\beta) \boldsymbol{K}_{task}\, \boldsymbol{e}_{task} \right), = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\dot{\boldsymbol{p}}_d + \boldsymbol{R}(\beta) \boldsymbol{K}_{task}\, \boldsymbol{R}^T(\beta)\, \boldsymbol{e}_p \right), \quad (6)$$

where the robot Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and the (diagonal) task gain matrix $\boldsymbol{K}_{task} > 0$ are given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{pmatrix}, \qquad \boldsymbol{K}_{task} = \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}.$$

By replacing eq. (6) in the differential kinematics, we obtain

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\dot{\boldsymbol{p}}_d + \boldsymbol{R}(\beta) \boldsymbol{K}_{task}\, \boldsymbol{R}^T(\beta)\, \boldsymbol{e}_p \right)$$

$$\Rightarrow \quad \dot{\boldsymbol{e}}_p = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}} = -\boldsymbol{R}(\beta) \boldsymbol{K}_{task}\, \boldsymbol{R}^T(\beta)\, \boldsymbol{e}_p = -\boldsymbol{K}_p\, \boldsymbol{e}_p,$$

having defined the (full and symmetric) Cartesian gain matrix $\boldsymbol{K}_p = \boldsymbol{R}(\beta) \boldsymbol{K}_{task}\, \boldsymbol{R}^T(\beta) > 0$. Being $\boldsymbol{R}(\beta)$ a constant matrix, we immediately see that

$$\dot{\boldsymbol{e}}_{task} = \boldsymbol{R}^T(\beta)\, \dot{\boldsymbol{e}}_p = -\boldsymbol{R}^T(\beta) \boldsymbol{K}_p\, \boldsymbol{e}_p = -\boldsymbol{R}^T(\beta) \boldsymbol{K}_p\, \boldsymbol{R}(\beta)\, \boldsymbol{e}_{task} = -\boldsymbol{K}_{task}\, \boldsymbol{e}_{task},$$

or in scalar terms

$$\begin{pmatrix} \dot{e}_t \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} -3\, e_t \\ -10\, e_n \end{pmatrix},$$

which is exactly the desired decoupled and linear error dynamics. ∎

* * * * *