Robotics 1

February 13, 2023

Exercise 1

Consider the planar RPPR robot in Fig. 1.

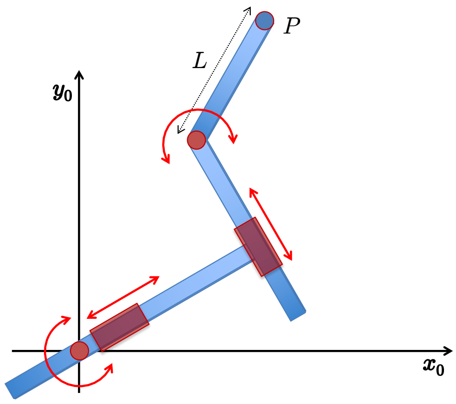

Figure 1: A planar RPPR robot

- Assign the frames according to the standard Denavit-Hartenberg (DH) convention and provide the corresponding table of parameters.
- Suppose that the two prismatic joints have a limited range: $|q_i| < D$, $i = 2, 3$. Determine the maximum possible distance Δ of the end-effector point P from the origin of the base frame and the robot configuration(s) $\boldsymbol{q}$ at which this value is attained.

Exercise 2

Given two different rotation matrices ${}^0\boldsymbol{R}_c$ and ${}^0\boldsymbol{R}_d$, suppose that a minimal representation with a set of ZYZ Euler angles $\boldsymbol{\alpha} \in I\!R^3$ has been extracted from each matrix, i.e., $\boldsymbol{\alpha}_c$ and $\boldsymbol{\alpha}_d$. Then, the relative error between the two orientations can be defined as $\boldsymbol{e}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_d - \boldsymbol{\alpha}_c$, i.e., the difference between the values of these two sets of Euler angles. As an alternative, one can define the relative rotation matrix ${}^c\boldsymbol{R}_d$ and extract from this matrix the same set of ZYZ Euler angles $\boldsymbol{\alpha}_{cd} \in I\!R^3$.

Is it true that $\boldsymbol{e}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_{cd}$ holds? If you believe so, provide a simple proof of this result. If you don't, provide then a numerical counterexample (without any representation singularity).

Exercise 3

A planar 2R robot has its direct kinematics defined as

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{1}$$

with link lengths $l_1 = 0.5$, $l_2 = 0.4$ [m]. Write a code that solves numerically the inverse kinematics problem for this robot using Newton iterative method. For a desired position $\boldsymbol{p}_d = (0.4, -0.3)$, determine two different initial configurations $\boldsymbol{q}^{[0]}$ so that the method converges in no more than $k_{\max} = 3$ iterations to the two inverse kinematics solutions, respectively $\boldsymbol{q}^a$ and, $\boldsymbol{q}^b$, with a final accuracy of at least $\varepsilon = 10^{-4}$ on the norm of the Cartesian error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})$. Provide the values of $\boldsymbol{q}^{[k]}$ for $k = 0, 1, 2, 3$ in the two situations, as well as the final values of the error norm $\|\boldsymbol{e}\|$.

Exercise 4

The kinematics of a 4-dof robot manipulator is characterized by the DH parameters in Tab. 1. Build the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that relates the joint velocities $\dot{\boldsymbol{q}} \in I\!R^4$ to the six-dimensional *twist* vector composed by a velocity $\boldsymbol{v} = \boldsymbol{v}_4 \in I\!R^3$ of the origin of the last (end-effector) DH frame and by an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_4 \in I\!R^3$ of the same frame:

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Correspondingly, the transpose of this matrix relates the six-dimensional end-effector *wrench* vector composed by a force $\boldsymbol{f} = \boldsymbol{f}_4 \in I\!R^3$ applied at the origin of the last (end-effector) DH frame and by a moment $\boldsymbol{\mu} = \boldsymbol{\mu}_4 \in I\!R^3$ applied on the same frame to the joint forces/torques $\boldsymbol{\tau} \in I\!R^4$:

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{\mu} \end{pmatrix}.$$

Find all the singular configurations of this Jacobian, i.e., all $\boldsymbol{q}_s$ such that rank $\boldsymbol{J}(\boldsymbol{q}_s) < 4$. At a singular configuration $\boldsymbol{q}_s$, determine:

i) a basis for the joint velocities $\dot{\boldsymbol{q}} \in I\!R^4$ that produce no end-effector twists;

ii) a basis for the end-effector twists $\boldsymbol{t} \in I\!R^6$ that are not realizable;

iii) all non-zero end-effector wrenches $\boldsymbol{w} \in I\!R^6$ that are statically balanced by $\boldsymbol{\tau} = \boldsymbol{0} \in I\!R^4$.

Hint: It is convenient to work by expressing the geometric Jacobian in the DH frame RF_1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	0	q_3	0
4	0	a_4	0	q_4

Table 1: Table of DH parameters of a 4-dof robot.

[240 minutes, open books]

Solution

February 13, 2023

Exercise 1

A possible assignment of standard DH frames for the considered RPPR robot arm is shown in Fig. 2. The corresponding DH parameters are reported in Tab. 2.

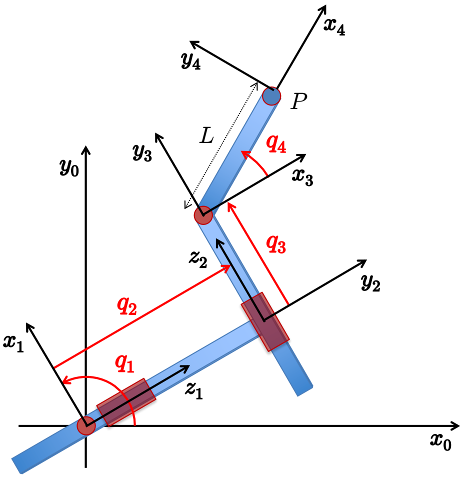

Figure 2: Assignment of DH frames for the RPPR robot in Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$\pi/2$	0	q_2	$\pi/2$
3	$\pi/2$	0	q_3	$\pi/2$
4	0	L	0	q_4

Table 2: Table of DH parameters for the RPPR robot with frames assigned as in Fig. 2.

The position of point P in the plane (x_0, y_0) is

$$\boldsymbol{p} = \begin{pmatrix} q_2 s_1 + q_3 c_1 + L s_{14} \\ -q_2 c_1 + q_3 s_1 - L c_{14} \end{pmatrix} = \begin{pmatrix} s_1 & c_1 \\ -c_1 & s_1 \end{pmatrix} \begin{pmatrix} q_2 + L c_4 \\ q_3 + L s_4 \end{pmatrix}.$$

Thus, its distance from the origin of the base frame is

$$\|\boldsymbol{p}\| = \sqrt{q_2^2 + q_3^2 + L^2 + 2L\,(q_2 c_4 + q_3 s_4)}.$$

For $|q_2| \leq D$ and $|q_3| \leq D$, the maximum distance is then easily evaluated as

$$\Delta = \max_{\boldsymbol{q}\in I\!R^4:\,|q_i|\leq D,\,i=1,2} \|\boldsymbol{p}\| = \sqrt{D^2 + D^2 + L^2 + 2L\left(D\,\frac{\sqrt{2}}{2} + D\,\frac{\sqrt{2}}{2}\right)} = \sqrt{2}D + L,$$

which is attained for

$$q_2 = \pm D, \qquad q_3 = \pm D, \qquad q_4 = \text{atan2}\left\{q_3, q_2\right\} \quad \left(= \left\{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\right\}\right),$$

with an arbitrary value of q_1. Four possible classes of solutions are obtained depending on the combination of signs: $\boldsymbol{q} = (q_1, D, D, \pi/4)$, $\boldsymbol{q} = (q_1, D, -D, -\pi/4)$, $\boldsymbol{q} = (q_1, -D, D, 3\pi/4)$, and $\boldsymbol{q} = (q_1, -D, -D, -3\pi/4)$.

Exercise 2

In general, the difference between the set of angles $\boldsymbol{\alpha}_c$ and $\boldsymbol{\alpha}_d$ of any minimal representation that one can extract from two rotation matrices, respectively $\boldsymbol{R}_c$ and $\boldsymbol{R}_d$, is different from the set of angles $\boldsymbol{\alpha}_{cd}$ of the same minimal representation that are extracted from the relative rotation matrix ${}^c\boldsymbol{R}_d = \boldsymbol{R}_c^T \boldsymbol{R}_d$. This is indeed true for any choice of angles $\boldsymbol{\alpha} \in I\!R^3$ used for the minimal representation of orientation. This result is due to the fact that the extraction of a minimal representation from a rotation matrix is a nonlinear operation.

The choice of a counterexample in which $\boldsymbol{e}_{\boldsymbol{\alpha}} \neq \boldsymbol{\alpha}_{cd}$ with the ZYZ Euler angles is arbitrary, but should keep in mind that the representation must not run into a singularity for any of the involved rotation matrices. This means that the two elements (1,3) and (2,3) in last column of the matrices $\boldsymbol{R}_c$, $\boldsymbol{R}_d$ and ${}^c\boldsymbol{R}_d$ should not be simultaneously zero.

Consider for example the two elementary rotation matrices by $\pi/4$ around the x and z axes,

$$\boldsymbol{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \qquad \boldsymbol{R}_z = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let

$$\boldsymbol{R}_c = \boldsymbol{R}_x \boldsymbol{R}_z = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -0.5 & 0.5 & \frac{\sqrt{2}}{2} \\ 0.5 & -0.5 & \frac{\sqrt{2}}{2} \end{pmatrix}, \qquad \boldsymbol{R}_d = \boldsymbol{R}_z \boldsymbol{R}_x = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0.5 & 0.5 \\ -\frac{\sqrt{2}}{2} & 0.5 & 0.5 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

From these, we obtain the relative orientation

$${}^c\boldsymbol{R}_d = \boldsymbol{R}_c^T \boldsymbol{R}_d = \begin{pmatrix} 0.8536 & -0.25 & 0.4571 \\ 0.1464 & 0.9571 & 0.25 \\ -0.5 & -0.1464 & 0.8536 \end{pmatrix}.$$

All three matrices $\boldsymbol{R}_c$, $\boldsymbol{R}_d$ and ${}^c\boldsymbol{R}_d$ satisfy the condition for not having a singularity in their ZYZ Euler representation, i.e.,

$$\sin\theta = \pm\sqrt{R_{13}^2 + R_{23}^2} \neq 0,$$

where R_{ij} denotes an element of the various rotation matrices. Thus, we can extract the set of angles $\boldsymbol{\alpha} = (\phi, \theta, \psi)$ using the inverse relationships in the regular case:

$$\theta = \text{atan2}\left\{\sin\theta, R_{33}\right\}, \qquad \phi = \text{atan2}\left\{\frac{R_{31}}{\sin\theta}, \frac{R_{32}}{\sin\theta}\right\}, \qquad \psi = \text{atan2}\left\{\frac{R_{13}}{\sin\theta}, \frac{-R_{23}}{\sin\theta}\right\}.$$

As a result, for each rotation matrix we obtain two regular solutions, namely

$$\boldsymbol{\alpha}_c^{I} = \begin{pmatrix} 2.3562 \\ 0.7854 \\ 3.1416 \end{pmatrix}, \quad \boldsymbol{\alpha}_c^{II} = \begin{pmatrix} -0.7854 \\ -0.7854 \\ 0 \end{pmatrix}; \qquad \boldsymbol{\alpha}_d^{I} = \begin{pmatrix} 3.1416 \\ 0.7137 \\ 2.3562 \end{pmatrix}, \quad \boldsymbol{\alpha}_d^{II} = \begin{pmatrix} 0 \\ -0.7137 \\ -0.7854 \end{pmatrix};$$

and

$$\boldsymbol{\alpha}_{cd}^{I} = \begin{pmatrix} 1.8557 \\ 0.5121 \\ 3.1416 \end{pmatrix}, \qquad \boldsymbol{\alpha}_{cd}^{II} = \begin{pmatrix} -1.2859 \\ -0.5121 \\ 0 \end{pmatrix}.$$

The four possible errors between the Euler angles are[1]

$$\boldsymbol{e}_{\boldsymbol{\alpha}}^{I,I} = \boldsymbol{\alpha}_d^{I} - \boldsymbol{\alpha}_c^{I} = \begin{pmatrix} 0.7854 \\ -0.0717 \\ -0.7854 \end{pmatrix}, \qquad \boldsymbol{e}_{\boldsymbol{\alpha}}^{I,II} = \boldsymbol{\alpha}_d^{II} - \boldsymbol{\alpha}_c^{I} = \begin{pmatrix} -2.3562 \\ -1.4991 \\ -3.9270 \end{pmatrix},$$
$$\boldsymbol{e}_{\boldsymbol{\alpha}}^{II,I} = \boldsymbol{\alpha}_d^{I} - \boldsymbol{\alpha}_c^{II} = \begin{pmatrix} 3.9270 \\ 1.4991 \\ 2.3562 \end{pmatrix}, \qquad \boldsymbol{e}_{\boldsymbol{\alpha}}^{II,II} = \boldsymbol{\alpha}_d^{II} - \boldsymbol{\alpha}_c^{II} = \begin{pmatrix} 0.7854 \\ 0.0717 \\ -0.7854 \end{pmatrix}.$$

As anticipated, none of these angular errors coincide with the two possible values of ZYZ Euler angles $\boldsymbol{\alpha}_{cd}^{I}$ and $\boldsymbol{\alpha}_{cd}^{II}$ extracted from the relative rotation matrix ${}^{c}\boldsymbol{R}_d$.

Exercise 3

From (1), the analytic Jacobian of the planar 2R robot is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(l_1 s_1 + l_2 s_{12}\right) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix}. \tag{2}$$

The basic step of Newton method at the k-th iteration is

$$\boldsymbol{q}^{[k+1]} = \boldsymbol{q}^{[k]} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{[k]}) \left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[k]})\right), \tag{3}$$

with inversion of the Jacobian and multiplication by the current position error $\boldsymbol{e}^{[k]} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[k]})$. In order to guarantee convergence, the method needs to be initialized with a configuration $\boldsymbol{q}^{[0]}$ that is close enough to a solution. A MATLAB code for the solution of the given inverse kinematics (IK) problem using Newton method is reported further below (without output instructions).

For the initialization, based on the desired $\boldsymbol{p}_d$ and on the link lengths of this robot, one can use intuition to guess a configuration that is close enough to the 'elbow up' IK solution. For instance, with the initial guess

$$\boldsymbol{q}^{[0]} = \begin{pmatrix} 40^\circ \\ -90^\circ \end{pmatrix} = \begin{pmatrix} 0.6981 \\ -1.5708 \end{pmatrix} \text{ [rad]},$$

the method fails to converge with the desired error accuracy $\varepsilon = 10^{-4}$ within the requested $k_{\max} = 3$ iterations (i.e., after three evaluations of the basic step (3)). The final configuration at $k = 3$ is

$$\boldsymbol{q}^{[3]} = \begin{pmatrix} 0.1837 \\ -1.9858 \end{pmatrix} \text{ [rad]} \quad \Rightarrow \quad \boldsymbol{f}(\boldsymbol{q}^{[3]}) = \begin{pmatrix} 0.3999 \\ -0.2980 \end{pmatrix} \neq \boldsymbol{p}_d \quad \Rightarrow \quad \|\boldsymbol{e}^{[3]}\| = 2 \cdot 10^{-3} \text{ [m]}.$$

However, the final configuration that was reached gives a clue for a good new guess. With

$$\boldsymbol{q}^{[0]} = \begin{pmatrix} 20^\circ \\ -120^\circ \end{pmatrix} = \begin{pmatrix} 0.3491 \\ -2.0944 \end{pmatrix} \text{ [rad]},$$

[1]While most of these angles take values in $(-\pi, \pi]$, there are two angular errors exceeding this range, namely $3.9270 = \pi + 0.7854$ and $-3.9270 = -\pi - 0.7854$. Indeed, when defining an angular quantities over a 2π range, one should organize error computations so as to lead to the smallest difference (in this case, ± 0.7854). Even in this way, the results for $\boldsymbol{e}_{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}_{cd}$ are different.

the method converges in fact in $k = 2$ iterations, generating the solution

$$\Rightarrow \quad \boldsymbol{q}^{[1]} = \begin{pmatrix} 0.1736 \\ -1.9961 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{[a]} = \boldsymbol{q}^{[2]} = \begin{pmatrix} 0.1797 \\ -1.9824 \end{pmatrix} \text{ [rad]},$$

with a final norm of the Cartesian error $\|\boldsymbol{e}\| = \left\|\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[a]})\right\| = 7 \cdot 10^{-5}$ m.

As for the 'elbow down' IK solution, the initial guess

$$\boldsymbol{q}^{[0]} = \begin{pmatrix} -70° \\ 100° \end{pmatrix} = \begin{pmatrix} -1.2217 \\ 1.7453 \end{pmatrix} \text{ [rad]},$$

leads to convergence in exactly $k = k_{\max} = 3$ iterations, generating the solution

$$\Rightarrow \quad \boldsymbol{q}^{[1]} = \begin{pmatrix} -1.4589 \\ 2.0125 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{[2]} = \begin{pmatrix} -1.4672 \\ 1.9826 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{q}^{[b]} = \boldsymbol{q}^{[3]} = \begin{pmatrix} -1.4665 \\ 1.9823 \end{pmatrix} \text{ [rad]},$$

with a final norm of the Cartesian error $\|\boldsymbol{e}\| = \left\|\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{[b]})\right\| = 9 \cdot 10^{-8}$ m. Fig. 3 illustrates the fast convergence rate (in fact, quadratic) of the method for the case of the 'elbow down' IK solution: the norm of the error is plotted in logarithmic scale over iterations.

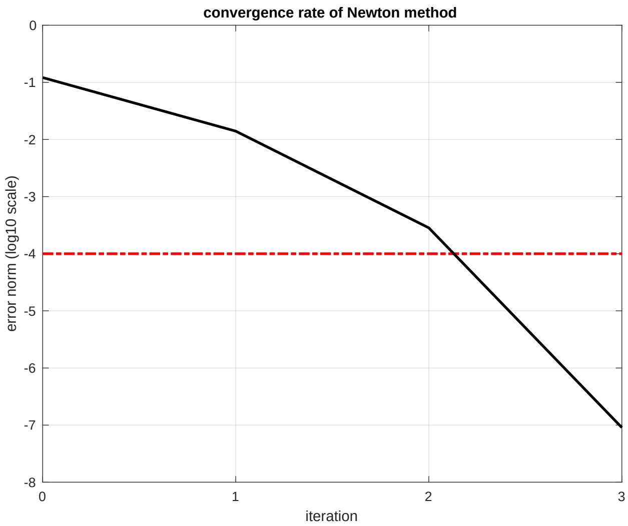

Figure 3: Error convergence of Newton method for the 'elbow down' IK solution.

MATLAB code for Newton method

```
% robot data
l1=0.5;l2=0.4;    %[m]
% desired task (end-effector position)
pd=[0.4;-0.3];    %[m]
% parameters in Newton method
% (final error tolerance and max number of iterations)
eps=0.0001; k_max=3;
% two (alternative) initial guesses
q0=[20;-120]*pi/180;     % for elbow up IK solution
%q0=[-70;100]*pi/180;    % for elbow down IK solution
```

```
% initialization
flag_sol=0;
k=1;
q(:,k)=q0;
% main loop
while k<=k_max,
  f=[l1*cos(q(1,k))+l2*cos(q(1,k)+q(2,k));
     l1*sin(q(1,k))+l2*sin(q(1,k)+q(2,k))];
  e(:,k)=p_d-f;
  norm_e(k)=norm(e(:,k));
  if norm_e(k)<=eps,
    k_sol=k;
    q_sol=q(:,k);
    e_sol=e(:,k);
    norm_e_sol=norm(e(:,k));
    flag_sol=1;
    break
  else
    J=[-(l1*sin(q(1,k))+l2*sin(q(1,k)+q(2,k)))  -l2*sin(q(1,k)+q(2,k));
        l1*cos(q(1,k))+l2*cos(q(1,k)+q(2,k))    l2*cos(q(1,k)+q(2,k))];
    % core Newton step
    q(:,k+1)=q(:,k)+inv(J)*e(:,k);
  end
  k=k+1;
end
```

Exercise 4

From Tab. 1, we compute the DH homogeneous transformations of this RRPR robot:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \qquad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} c_4 & -s_4 & 0 & a_4 c_4 \\ s_4 & c_4 & 0 & a_4 s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The position of the end-effector (the origin of frame 4) follows as

$$
\boldsymbol{p}_{4,hom} = \begin{pmatrix} \boldsymbol{p}_4 \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left({}^1\boldsymbol{A}_2(q_2) \left({}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} a_4 c_4 \\ a_4 s_4 \\ 0 \\ 1 \end{pmatrix} \right) \right) \quad \Rightarrow \quad \boldsymbol{p}_4 = \begin{pmatrix} c_1 \left(q_3 s_2 + a_4 c_{24} \right) \\ s_1 \left(q_3 s_2 + a_4 c_{24} \right) \\ -q_3 c_2 + a_4 s_{24} \end{pmatrix}. \tag{4}
$$

Since $\boldsymbol{v} = \boldsymbol{v}_4 = \dot{\boldsymbol{p}}_4$, the linear part of the geometric Jacobian can be obtained by differentiation as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_4}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1\left(q_3 s_2 + a_4 c_{24}\right) & c_1\left(q_3 c_2 - a_4 s_{24}\right) & c_1 s_2 & -c_1 a_4 s_{24} \\ c_1\left(q_3 s_2 + a_4 c_{24}\right) & s_1\left(q_3 c_2 - a_4 s_{24}\right) & s_1 s_2 & -s_1 a_4 s_{24} \\ 0 & q_3 s_2 + a_4 c_{24} & -c_2 & a_4 c_{24} \end{pmatrix}.$$

Setting $\boldsymbol{z}_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, the angular part of the geometric Jacobian is computed as

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 & \boldsymbol{0} & {}^0\boldsymbol{R}_3(q_1,q_2,q_3)\boldsymbol{z}_0 \end{pmatrix} = \begin{pmatrix} 0 & s_1 & 0 & s_1 \\ 0 & -c_1 & 0 & -c_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the 6×4 geometric Jacobian in the base frame is

$${}^0\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}.$$

To simplify the following analysis, it is convenient to express the Jacobian in the rotated frame RF_1. We have

$$\begin{aligned} {}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\bar{\boldsymbol{R}}_1^T(q_1)\,{}^0\boldsymbol{J}(\boldsymbol{q}) &= \begin{pmatrix} {}^0\boldsymbol{R}_1^T(q_1) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_1^T(q_1) \end{pmatrix} {}^0\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}_L(\boldsymbol{q}) \\ {}^0\boldsymbol{R}_1^v(q_1)\,\boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & q_3 c_2 - a_4 s_{24} & s_2 & -a_4 s_{24} \\ 0 & q_3 s_2 + a_4 c_{24} & -c_2 & a_4 c_{24} \\ -\left(q_3 s_2 + a_4 c_{24}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{5}$$

Even when the 6×4 geometric Jacobian is full (column) rank, there exist always directions along which no end-effector twists can be realized. When the range space of the Jacobian has dimension

$$\dim \mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q})\right) = \dim \mathcal{R}\left({}^0\boldsymbol{J}(\boldsymbol{q})\right) = 4,$$

being

$$\mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q})\right) \oplus \mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = \mathbb{R}^6,$$

it follows that the dimension of the complementary space is

$$\dim \mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = 6 - \dim \mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q})\right) = 2.$$

In particular, a basis for such unfeasible twists is given by

$$\mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\right) = \operatorname{span}\left\{{}^1\boldsymbol{t}_1, {}^1\boldsymbol{t}_2\right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ q_3 s_2 + a_4 c_{24} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \tag{6}$$

In order to evaluate the singular configurations of the robot manipulator, we compute

$$\det\left({}^1\boldsymbol{J}^T(\boldsymbol{q})\,{}^1\boldsymbol{J}(\boldsymbol{q})\right) = q_3^2\left(\left(q_3 s_2 + a_4 c_{24}\right)^2 + 1\right).$$

Therefore, the singularities of the geometric Jacobian occur only when $q_3 = 0$. Setting this value in the rotated Jacobian, one has

$${}^1\boldsymbol{J}(\boldsymbol{q}_s) = {}^1\boldsymbol{J}(\boldsymbol{q})\big|_{q_3=0} = \begin{pmatrix} 0 & -a_4 s_{24} & s_2 & -a_4 s_{24} \\ 0 & a_4 c_{24} & -c_2 & a_4 c_{24} \\ -a_4 c_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\operatorname{rank}{}^1\boldsymbol{J}(\boldsymbol{q}_s) = 3$. Thus, the null space of ${}^1\boldsymbol{J}(\boldsymbol{q}_s)$ is one-dimensional and coincides with the null space of ${}^0\boldsymbol{J}(\boldsymbol{q}_s)$ (because we are operating on the columns of the matrix, i.e., in the joint space, while products by $\boldsymbol{R}_1^T(q_1)$ affect the rows). This null space is spanned by

$$\mathcal{N}\left({}^0\boldsymbol{J}(\boldsymbol{q}_s)\right) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Indeed, for any joint velocity $\dot{\boldsymbol{q}}_0 \in \mathcal{N}\left({}^0\boldsymbol{J}(\boldsymbol{q}_s)\right)$, we have

$${}^0\boldsymbol{J}(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_0 = {}^1\boldsymbol{J}(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_0 = {}^1\boldsymbol{J}(\boldsymbol{q}_s)\begin{pmatrix} 0 \\ -\alpha \\ 0 \\ \alpha \end{pmatrix} = \boldsymbol{0}, \qquad \forall \alpha \in I\!R.$$

In order to determine all end-effector twists $\boldsymbol{t} \in I\!R^6$ that are not realizable at a singular configuration $\boldsymbol{q}_s$, we should find $6 - \dim\mathcal{R}\left({}^1\boldsymbol{J}(\boldsymbol{q}_s)\right) = 3$ independent columns to be appended to the geometric Jacobian so that its rank will increase to its maximum possible value, namely 6. Working in the rotated frame, and following the same previous consideration about complementarity spaces, it is easy to see that we can use the two twist directions in (6) —which are never realizable, neither in a regular configuration nor in a singular configuration (where $q_3 = 0$) — and add a third independent column as follows

$$\mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q}_s)\right) = \operatorname{span}\left\{{}^1\boldsymbol{t}_1, {}^1\boldsymbol{t}_2, {}^1\boldsymbol{t}_3\right\} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ a_4 c_{24} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ s_2 \\ 0 \\ 0 \\ 0 \\ a_4 s_4 \end{pmatrix}\right\}. \tag{7}$$

Being outside the range space of ${}^1\boldsymbol{J}(\boldsymbol{q}_s)$, the three directions in (7) represent a basis for all generalized end-effector twists that are not realizable at $\boldsymbol{q}_s$. When expressed in the base frame, these become

$$
{}^0\boldsymbol{t}_1 = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{t}_1 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \\ 0 \\ 0 \\ a_4 c_{24} \end{pmatrix}, \quad {}^0\boldsymbol{t}_2 = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{t}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_1 \\ s_1 \\ 0 \end{pmatrix}, \quad {}^0\boldsymbol{t}_3 = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{t}_3 = \begin{pmatrix} c_1 c_2 \\ s_1 c_2 \\ s_2 \\ a_4 s_1 s_4 \\ -a_4 c_1 s_4 \\ 0 \end{pmatrix}.
$$

To determine all end-effector wrenches $\boldsymbol{w} \in I\!R^6$ for which the manipulator is statically balanced at $\boldsymbol{q}_s$ without the need of forces/torques $\boldsymbol{\tau} \in I\!R^4$ at the joints, we need to determine a basis for the null space of the transpose of the geometric Jacobian. However, such a basis has already been computed. Therefore, when working in the rotated frame, we have

$$
\mathcal{N}\left({}^1\boldsymbol{J}^T(\boldsymbol{q}_s)\right) = \text{span}\left\{{}^1\boldsymbol{w}_1, {}^1\boldsymbol{w}_2, {}^1\boldsymbol{w}_3\right\} = \text{span}\left\{{}^1\boldsymbol{t}_1, {}^1\boldsymbol{t}_2, {}^1\boldsymbol{t}_3\right\}.
$$

Similarly, when expressed in the base frame as ${}^0\boldsymbol{w}_i = {}^0\bar{\boldsymbol{R}}_1(q_1)\,{}^1\boldsymbol{w}_i$, for $i = 1, 2, 3$, these end-effector wrenches are

$$
{}^0\boldsymbol{w}_1 = {}^0\boldsymbol{t}_1, \qquad {}^0\boldsymbol{w}_2 = {}^0\boldsymbol{t}_2, \qquad {}^0\boldsymbol{w}_3 = {}^0\boldsymbol{t}_3.
$$

Indeed, for any end-effector wrench $\boldsymbol{w}_0 \in \mathcal{N}(\boldsymbol{J}^T(\boldsymbol{q}_s))$, the balancing forces/torques at the joints is

$$
\boldsymbol{\tau} = {}^0\boldsymbol{J}^T(\boldsymbol{q}_s)\,{}^0\boldsymbol{w}_0 = {}^1\boldsymbol{J}^T(\boldsymbol{q}_s)\,{}^1\boldsymbol{w}_0 = \boldsymbol{0}.
$$

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