Robotics I

B: preferred for 5 credits

January 12, 2010

Exercise 1

Consider the Cartesian path defined by

$$\boldsymbol{p} = \boldsymbol{p}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} R\cos s \\ R\sin s \\ h\,s \end{pmatrix}, \qquad s \in [0, +\infty)$$

where $R > 0$ and $h > 0$. This path is a spiral around the $\boldsymbol{z}$-axis. Define a timing law $s = s(t)$ having a *trapezoidal speed* profile in $t \in [0, T]$, for a given and sufficiently large final time $T > 0$, such that the resulting planned trajectory $\boldsymbol{p}_d(t) = \boldsymbol{p}(s(t))$ satisfies the following conditions:

- $\dot{\boldsymbol{p}}_d(0) = \dot{\boldsymbol{p}}_d(T) = \boldsymbol{0}$;
- $\|\dot{\boldsymbol{p}}_d(t)\| \leq V$, for a given $V > 0$;
- $\|\ddot{\boldsymbol{p}}_d(t)\| \leq A$, for a given and sufficiently large $A > 0$.

Provide in particular the reached height $z_d(T)$ in closed form.

Moreover, define a *coordinated motion* for the *orientation* along the above path, by specifying a moving frame that has its $\boldsymbol{x}_o$ axis always pointing and orthogonal to the central axis of the spiral (the $\boldsymbol{z}$-axis) and its $\boldsymbol{z}_o$ always parallel to it. What is the maximum value reached by the norm of the angular velocity, $\|\boldsymbol{\omega}\|$, associated to the planned trajectory?

Finally, evaluate the solution found for the following numerical data:

$$R = 0.3 \text{ [m]}, \quad h = 0.1 \text{ [m]}, \quad V = 1 \text{ [m/s]}, \quad A = 5 \text{ [m/s}^2\text{]}, \quad T = 4 \text{ [s]}.$$

Exercise 2B

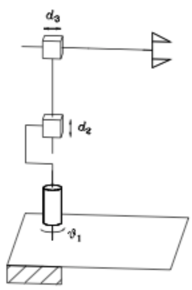

Figure 1: A cylindrical manipulator

Derive the 6×3 geometric Jacobian for the cylindrical manipulator in Fig. 1 and find the singularities of its linear velocity part. Consider a desired motion $\boldsymbol{p}_d(t)$ of the end-effector position that is twice-differentiable w.r.t. time. Taking the joint accelerations $\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{\theta}_1 & \ddot{d}_2 & \ddot{d}_3 \end{pmatrix}^T$ as control inputs and assuming that only $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are measured, define a Cartesian kinematic controller *at the acceleration level* that assigns (out of singularities) the closed-loop behavior to the system

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} = \boldsymbol{0},$$

where $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$, and $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are positive definite, diagonal matrices.

[150 minutes; open books]

Solutions

January 12, 2010

Exercise 1

The velocity vector along the path is given by

$$\dot{\boldsymbol{p}}_d = \frac{d\boldsymbol{p}_d(t)}{dt} = \frac{d\boldsymbol{p}(s)}{ds}\frac{ds(t)}{dt} = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix}\dot{s},$$

and thus

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{R^2+h^2}\,|\dot{s}(t)|.$$

The constraint $\|\dot{\boldsymbol{p}}_d(t)\| \leq V$ on the Cartesian velocity becomes

$$|\dot{s}(t)| \leq \frac{V}{\sqrt{R^2+h^2}} =: V_{\max}$$

for the speed profile $\dot{s}$.

The acceleration vector along the path is given by

$$\ddot{\boldsymbol{p}}_d = \frac{d^2\boldsymbol{p}_d(t)}{dt^2} = \frac{d\boldsymbol{p}(s)}{ds}\ddot{s}(t) + \frac{d^2\boldsymbol{p}(s)}{ds^2}\dot{s}^2(t) = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix}\ddot{s} + \begin{pmatrix} -R\cos s \\ -R\sin s \\ 0 \end{pmatrix}\dot{s}^2,$$

and thus

$$\|\ddot{\boldsymbol{p}}_d(t)\| = \sqrt{(R^2+h^2)\,\ddot{s}^2(t) + (R\,\dot{s}^2(t))^2}.$$

The constraint $\|\ddot{\boldsymbol{p}}_d(t)\| \leq A$ on the Cartesian acceleration can be rewritten as

$$(R^2+h^2)\,\ddot{s}^2(t) \leq A^2 - (R\,\dot{s}^2(t))^2$$

for the acceleration profile $\ddot{s}$. Since this constraint has to be satisfied for all $t \in [0, T]$, one should consider the worst case, i.e., $|\dot{s}| = V_{\max}$. We obtain

$$|\ddot{s}(t)| \leq \sqrt{\frac{A^2 - (\frac{RV^2}{R^2+h^2})^2}{R^2+h^2}} =: A_{\max}.$$

In order to have a feasible $A_{\max} > 0$, the value of A should be sufficiently large, i.e.,

$$A > \frac{RV^2}{R^2+h^2}. \tag{1}$$

At this stage, given the total time T and the computed limits $V_{\max}$ and $A_{\max}$, the timing law with trapezoidal speed profile is fully specified. In particular, we have for the acceleration/deceleration interval time

$$T_s = \frac{V_{\max}}{A_{\max}} = \frac{V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}.$$

In order to have a complete trapezoidal profile (with at least one instant where $V_{\max}$ is reached), the total time T should be sufficiently large, i.e.,

$$T \geq 2T_s = \frac{2V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}. \tag{2}$$

The total displacement of the parameter s at time $t = T$ is then

$$s_{\max} := s(T) = (T - T_s)V_{\max} = TV_{\max} - \frac{V_{\max}^2}{A_{\max}} = \frac{TV}{\sqrt{R^2 + h^2}} - \frac{V^2}{\sqrt{(R^2 + h^2)A^2 - \frac{(RV^2)^2}{R^2+h^2}}}.$$

Therefore, the reached height at the final time $t = T$ is

$$z_d(T) = h\, s(T) = h\, s_{\max}.$$

For completeness, we compute also the curvature of the given parametric path:

$$\kappa(s) = \frac{\left\| \dfrac{d\boldsymbol{p}}{ds} \times \dfrac{d^2\boldsymbol{p}}{ds^2} \right\|}{\left\| \dfrac{d\boldsymbol{p}}{ds} \right\|^3} = \frac{R}{R^2 + h^2}.$$

Indeed, $\kappa(s)$ is constant for all s and collapses to $1/R$ for $h = 0$.

For planning the requested orientation trajectory, which has to be coordinated with the position trajectory, we define a moving frame as a function of the same parameter s. This is given by

$$\boldsymbol{R}(s) = \begin{pmatrix} \boldsymbol{x}_o(s) & \boldsymbol{y}_0(s) & \boldsymbol{z}_o(s) \end{pmatrix} = \begin{pmatrix} -\cos s & \sin s & 0 \\ -\sin s & -\cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this moving frame is *not* the Frenet frame associated to the parametrized path. Using the notations $\boldsymbol{p}'(s) = d\boldsymbol{p}(s)/ds$ and $\boldsymbol{p}''(s) = d^2\boldsymbol{p}(s)/ds^2$, the Frenet frame is specified as

$$\boldsymbol{R}_{\text{Frenet}}(s) = \begin{pmatrix} \boldsymbol{t}(s) & \boldsymbol{n}(s) & \boldsymbol{b}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{p}'(s)}{\|\boldsymbol{p}'(s)\|} & \dfrac{\boldsymbol{p}''(s)}{\|\boldsymbol{p}''(s)\|} & \boldsymbol{t}(s) \times \boldsymbol{n}(s) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{R}{\sqrt{R^2+h^2}} \sin s & -\cos s & \frac{h}{\sqrt{R^2+h^2}} \sin s \\ \frac{R}{\sqrt{R^2+h^2}} \cos s & -\sin s & -\frac{h}{\sqrt{R^2+h^2}} \cos s \\ \frac{h}{\sqrt{R^2+h^2}} & 0 & \frac{R}{\sqrt{R^2+h^2}} \end{pmatrix}.$$

In fact, the two frames coincide (modulo a rotation of $\pi/2$ around the $\boldsymbol{z}$-axis) only when $h = 0$.

Setting $\boldsymbol{R}_d(t) = \boldsymbol{R}(s(t))$, the angular velocity vector is computed from

$$\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_d \boldsymbol{R}_d^T = \dot{s}(t) \begin{pmatrix} \sin s(t) & \cos s(t) & 0 \\ -\cos s(t) & \sin s(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\cos s(t) & -\sin s(t) & 0 \\ \sin s(t) & -\cos s(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}(t) & 0 \\ \dot{s}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As expected (being the rotation of the moving frame only around the $\boldsymbol{z}$-axis and counterclockwise),

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{s}(t) \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{\omega}\| = |\dot{s}(t)|,$$

and the maximum value of the norm of the angular velocity vector is obviously $V_{\max}$.

With the given numerical data, which satisfy both inequalities (1) and (2), we obtain:

$$V_{\max} = \sqrt{10} = 3.1623, \quad A_{\max} = 4\sqrt{10} = 12.6491, \quad T_s = 0.25,$$

$$s_{\max} = 3.75\sqrt{10} = 11.8585, \quad z_d(T) = 0.375\sqrt{10} = 1.1859\,.$$

In the following, we show plots of the planned trajectory obtained in Matlab (code available).

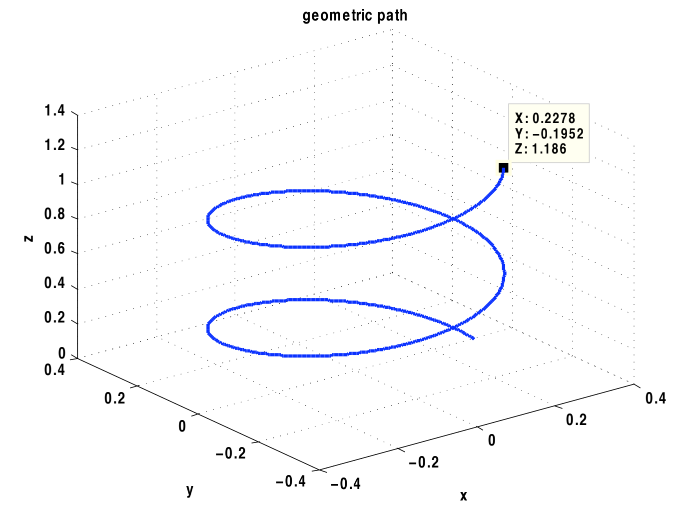

Figure 2: The spiral Cartesian trajectory (with coordinates of the final reached point at time $T = 4$ s)

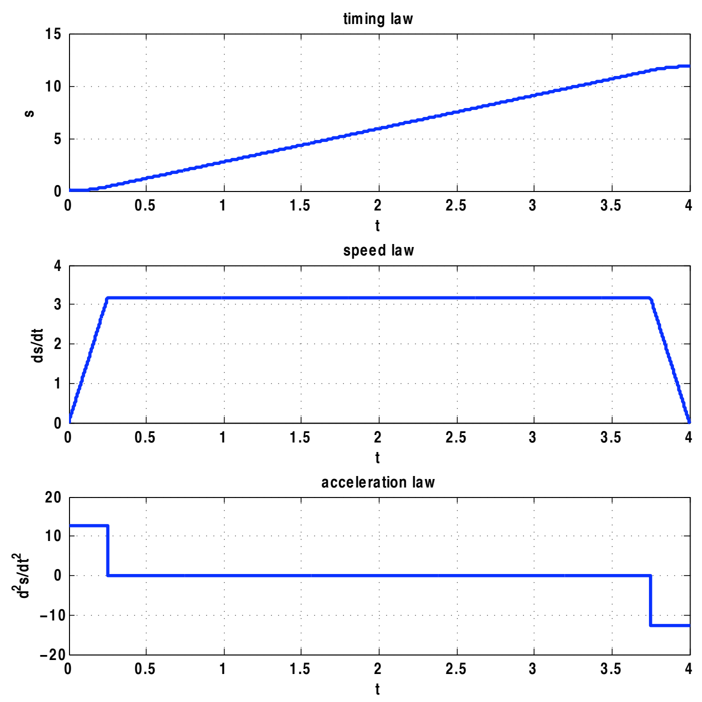

Figure 3: Timing law: Path parameter $s(t)$, speed $\dot{s}(t)$, and acceleration $\ddot{s}(t)$

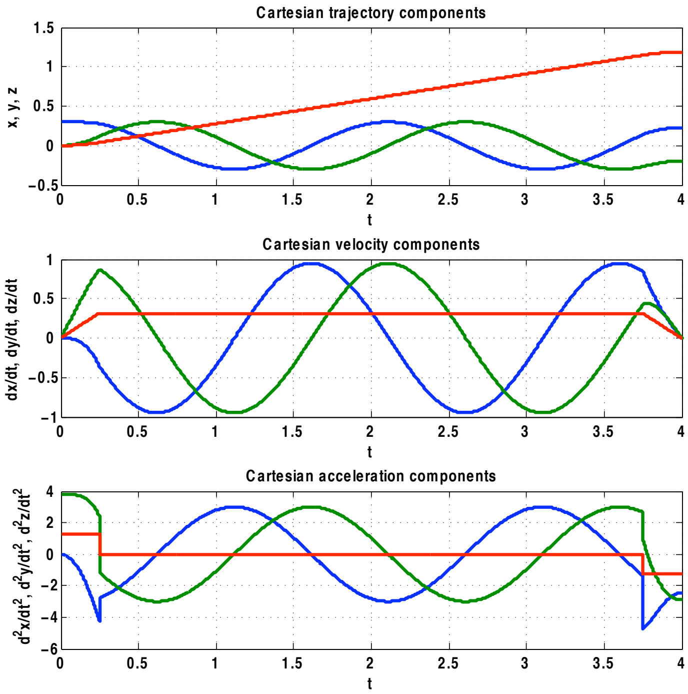

Figure 4: Components of Cartesian trajectory: Position, velocity, and acceleration (x in blue, y in green, z in red)

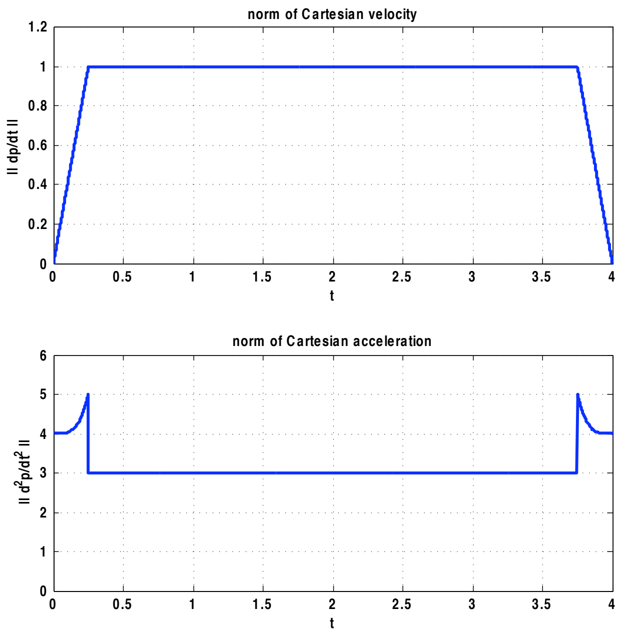

Figure 5: Norms of the Cartesian velocity and acceleration: The given bounds $\|\dot{\boldsymbol{p}}_d(t)\| \leq 1$ and $\|\ddot{\boldsymbol{p}}_d(t)\| \leq 5$ are always satisfied during motion

Exercise 2B

The Jacobian for the cylindrical (RPP) manipulator with $\boldsymbol{q} = (\theta_1, d_2, d_3)$ is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p} & \boldsymbol{z}_1 & \boldsymbol{z}_2 \\ \boldsymbol{z}_0 & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},$$

with the axes of the three joints being

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_2 = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix},$$

and the end-effector position vector given by

$$\boldsymbol{p} = \boldsymbol{k}(\boldsymbol{q}) = \begin{pmatrix} d_3\cos\theta_1 \\ d_3\sin\theta_1 \\ d_2 \end{pmatrix}. \tag{3}$$

Then, the expression of the geometric Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -d_3\sin\theta_1 & 0 & \cos\theta_1 \\ d_3\cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which reveals that it is inherently impossible to rotate about the axes $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$.

The Jacobian relative to the end-effector linear velocity can be extracted by considering only the first three rows, i.e.,

$$\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -d_3\sin\theta_1 & 0 & \cos\theta_1 \\ d_3\cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \end{pmatrix},$$

which coincides indeed with the differentiation w.r.t. $\boldsymbol{q}$ of the direct kinematics function $\boldsymbol{k}(\boldsymbol{q})$ in (3). Its determinant is

$$\det \boldsymbol{J}_L(\boldsymbol{q}) = d_3,$$

vanishing at the singularity $d_3 = 0$. This occurs when the end-effector is located along the axis of joint 1, a situation conceptually similar to the shoulder singularity of an anthropomorphic 3R arm.

Since $\dot{\boldsymbol{p}} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}$, the differential kinematics at the acceleration level is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}_L(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}},$$

where

$$\dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -\dot{d}_3\sin\theta_1 - d_3\,\dot{\theta}_1\cos\theta_1 & 0 & -\dot{\theta}_1\sin\theta_1 \\ \dot{d}_3\cos\theta_1 - d_3\,\dot{\theta}_1\sin\theta_1 & 0 & \dot{\theta}_1\cos\theta_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{pmatrix} = \begin{pmatrix} -2\,\dot{d}_3\,\dot{\theta}_1\sin\theta_1 - d_3\,\dot{\theta}_1^2\cos\theta_1 \\ 2\,\dot{d}_3\,\dot{\theta}_1\cos\theta_1 - d_3\,\dot{\theta}_1^2\sin\theta_1 \\ 0 \end{pmatrix}.$$

Therefore, designing the joint acceleration vector as

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}_L^{-1}(\boldsymbol{q})(\ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{p}}_d - \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{k}(\boldsymbol{q})) - \dot{\boldsymbol{J}}_L(\boldsymbol{q})\dot{\boldsymbol{q}}) \tag{4}$$

yields

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}) + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}),$$

namely the desired closed-loop behavior. Note that (4) is implemented using only the measurements of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, beside the knowledge of the desired trajectory (up to its second time derivative) and of the arm direct and differential kinematics.

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