Robotics I

Test 2 — December 17, 2009

Consider the robot in Figure 1, having four revolute joints. The Denavit-Hartenberg frames are already placed, with frame 0 located at the intersection of the first and second joint axis. The configuration shown corresponds (approximately) to $\boldsymbol{\theta} \simeq \begin{pmatrix} 0 & 6\pi/10 & \pi & 6\pi/10 \end{pmatrix}^T$ [rad] (or, equivalently, $\boldsymbol{\theta} \simeq \begin{pmatrix} 0 & 108 & 180 & 108 \end{pmatrix}^T$ [deg]).

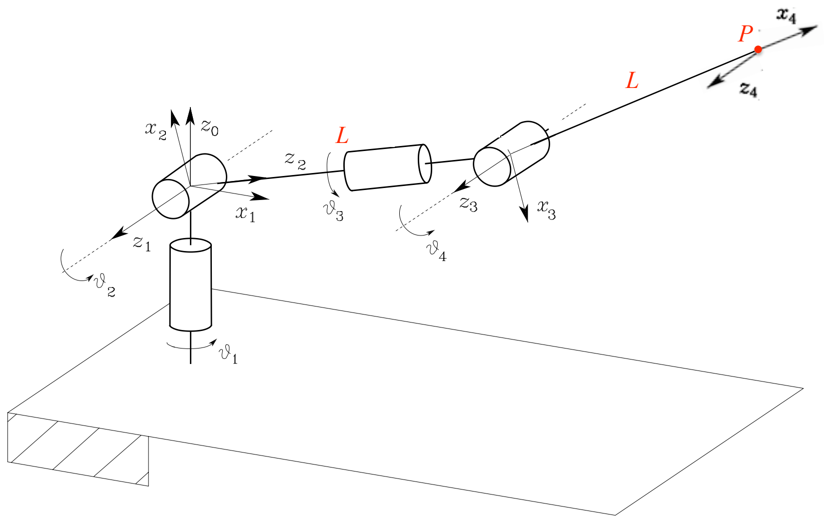

Figure 1: A 4R spatial manipulator

Let the robot be in the configuration $\boldsymbol{\theta}^* = \begin{pmatrix} 0 & 3\pi/4 & \pi & \pi \end{pmatrix}^T$ [rad], and set $L = 1$ [m] in the following if you plan to work in a numerical way.

1. Obtain the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{\theta}^*)$.
2. Show that the following Cartesian linear/angular velocity vector is feasible:

$$\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix} = \begin{pmatrix} 0 & 0 & -L & 0 & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

3. Determine the minimum norm joint velocity vector $\dot{\boldsymbol{\theta}}$ realizing the above Cartesian velocity.
4. Compute the joint torque vector $\boldsymbol{\tau}$ that keeps the robot in static equilibrium when the following Cartesian force/torque vector is applied from the environment to the end-effector:

$$\begin{pmatrix} \boldsymbol{F}^T & \boldsymbol{M}^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Consider only the velocity $\boldsymbol{v}$ of point P. Verify whether the associated 3×4 Jacobian $\boldsymbol{J}_L(\boldsymbol{\theta})$ is singular or not in the configuration $\boldsymbol{\theta}^*$.

[120 minutes; open books]

Solution

December 17, 2009

The 4R spatial manipulator is made by the subset of first four joints of the DLR manipulator considered in the textbook (p. 79, Fig. 2.29)[1]. However, the fourth (and last) reference frame is different, due to the missing axes 5, 6, and 7. The Denavit-Hartenberg parameters are given in Table 1 (the first three rows are those of Table 2.7 in the textbook, with $d_3 = L$).

i	α_i	a_i	d_i	θ_i
1	$\frac{\pi}{2}$	0	0	θ_1
2	$\frac{\pi}{2}$	0	0	θ_2
3	$\frac{\pi}{2}$	0	L	θ_3
4	0	L	0	θ_4

Table 1: Denavit-Hartenberg parameters

The associated homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(\theta_1) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(\theta_1) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ \sin\theta_2 & 0 & -\cos\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(\theta_2) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3(\theta_3) & {}^2\boldsymbol{p}_{23} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(\theta_4) = \begin{pmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & L\cos\theta_4 \\ \sin\theta_4 & \cos\theta_4 & 0 & L\sin\theta_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_4(\theta_4) & {}^3\boldsymbol{p}_{34}(\theta_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
$$

The 6×4 geometric Jacobian

$$
\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{\theta}) \\ \boldsymbol{J}_A(\boldsymbol{\theta}) \end{pmatrix}
$$

can be computed symbolically or numerically for a given configuration. We present first the general symbolic derivation, and then a more direct numerical approach.

[1] Note that in Fig. 2.29 the $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ axes are drawn in a wrong way. The associated Table 2.7 of DH parameters is instead correct for the full 7R arm.

The 3×4 upper part $\boldsymbol{J}_L$ of the geometric Jacobian relates $\dot{\boldsymbol{\theta}}$ to the velocity $\boldsymbol{v}$ of point P. It can be obtained either by (analytic) differentiation of $\boldsymbol{p}_{04}$, i.e., by computing this vector as

$$\begin{pmatrix} \boldsymbol{p}_{04}(\boldsymbol{\theta}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(\theta_1)\,{}^1\boldsymbol{A}_2(\theta_2)\,{}^2\boldsymbol{A}_3(\theta_3)\,{}^3\boldsymbol{A}_4(\theta_4) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$$

and obtaining then

$$\boldsymbol{J}_L(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{p}_{04}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}},$$

or by the geometric formula

$$\boldsymbol{J}_L(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{04} & \boldsymbol{z}_1 \times \boldsymbol{p}_{04} & \boldsymbol{z}_2 \times \boldsymbol{p}_{04} & \boldsymbol{z}_3 \times (\boldsymbol{p}_{04} - \boldsymbol{p}_{03}) \end{pmatrix},$$

where we used the fact that $\boldsymbol{p}_{00} = \boldsymbol{p}_{01} = \boldsymbol{p}_{02} = \boldsymbol{0}$ (the origins of frames 0, 1, and 2 coincide).

Thus, for deriving its explicit symbolic form we need

$$\boldsymbol{p}_{04} = L \begin{pmatrix} \cos\theta_1 \sin\theta_2 + \cos\theta_1 \sin\theta_2 \sin\theta_4 + (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ \sin\theta_1 \sin\theta_2 + \sin\theta_1 \sin\theta_2 \sin\theta_4 - (\cos\theta_1 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ -\cos\theta_2 - \cos\theta_2 \sin\theta_4 + \sin\theta_2 \cos\theta_3 \cos\theta_4 \end{pmatrix},$$

and, when following the geometric construction, also

$$\boldsymbol{p}_{04} - \boldsymbol{p}_{03} = L \begin{pmatrix} \cos\theta_1 \sin\theta_2 \sin\theta_4 + (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ \sin\theta_1 \sin\theta_2 \sin\theta_4 - (\cos\theta_1 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ -\cos\theta_2 \sin\theta_4 + \sin\theta_2 \cos\theta_3 \cos\theta_4 \end{pmatrix}$$

as well as

$$\begin{aligned}
\boldsymbol{z}_0 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\boldsymbol{z}_1 &= {}^0\boldsymbol{R}_1(\theta_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \\ 0 \end{pmatrix} \\
\boldsymbol{z}_2 &= {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \\ -\cos\theta_2 \end{pmatrix} \\
\boldsymbol{z}_3 &= {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\,{}^2\boldsymbol{R}_3(\theta_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta_1 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3 \\ \cos\theta_1 \cos\theta_3 + \sin\theta_1 \cos\theta_2 \sin\theta_3 \\ \sin\theta_2 \sin\theta_3 \end{pmatrix}.
\end{aligned}$$

Performing symbolic computations[2], and factoring out the length L, we obtain

$$\boldsymbol{J}_L(\boldsymbol{\theta}) = L \cdot \begin{pmatrix} \boldsymbol{J}_{L,1} & \boldsymbol{J}_{L,2} & \boldsymbol{J}_{L,3} & \boldsymbol{J}_{L,4} \end{pmatrix},$$

[2] When using the Matlab Symbolic Toolbox, take advantage of the `simplify` instruction to reduce the length/complexity of terms.

where:

$$\boldsymbol{J}_{L,1} = \begin{pmatrix} -\sin\theta_1 \sin\theta_2 - \sin\theta_1 \sin\theta_2 \sin\theta_4 + (\cos\theta_1 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ \cos\theta_1 \sin\theta_2 + \cos\theta_1 \sin\theta_2 \sin\theta_4 + (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3)\cos\theta_4 \\ 0 \end{pmatrix}$$

$$\boldsymbol{J}_{L,2} = \begin{pmatrix} \cos\theta_1(\cos\theta_2 + \cos\theta_2 \sin\theta_4 - \sin\theta_2 \cos\theta_3 \cos\theta_4) \\ \sin\theta_1(\cos\theta_2 + \cos\theta_2 \sin\theta_4 - \sin\theta_2 \cos\theta_3 \cos\theta_4) \\ \sin\theta_2 + \sin\theta_2 \sin\theta_4 + \cos\theta_2 \cos\theta_3 \cos\theta_4 \end{pmatrix}$$

$$\boldsymbol{J}_{L,3} = \begin{pmatrix} (\sin\theta_1 \cos\theta_3 - \cos\theta_1 \cos\theta_2 \sin\theta_3)\cos\theta_4 \\ -(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \cos\theta_2 \sin\theta_3)\cos\theta_4 \\ -\sin\theta_2 \sin\theta_3 \cos\theta_4 \end{pmatrix}$$

$$\boldsymbol{J}_{L,4} = \begin{pmatrix} \cos\theta_1 \sin\theta_2 \cos\theta_4 - (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3)\sin\theta_4 \\ \sin\theta_1 \sin\theta_2 \cos\theta_4 + (\cos\theta_1 \sin\theta_3 - \sin\theta_1 \cos\theta_2 \cos\theta_3)\sin\theta_4 \\ -\cos\theta_2 \cos\theta_4 - \sin\theta_2 \cos\theta_3 \sin\theta_4 \end{pmatrix}.$$

The 3×4 lower part $\boldsymbol{J}_A$ of the geometric Jacobian, relating $\dot{\boldsymbol{\theta}}$ to the angular velocity $\boldsymbol{\omega}$ of frame 4, is given instead by

$$\boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix},$$

where the previous symbolic expressions for $\boldsymbol{z}_i$, $i = 0, 1, 2, 3$, are used.

At this stage, the elements of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ should be evaluated at the given configuration

$$\boldsymbol{\theta}^* = \begin{pmatrix} 0 & 3\pi/4 & \pi & \pi \end{pmatrix}^T.$$

In this configuration, the end-effector (the origin of frame 4) is positioned along the axis of joint 1.

Alternatively (and in a much faster way for the problem at hand!), we may first evaluate numerically the homogeneous transformations at the configuration $\boldsymbol{\theta}^*$, using in this case also $L = 1$, and then perform all the required operations, including products of matrices and (vector) cross products, so as to obtain the numerical value of the geometric Jacobian. The Matlab code is:

```
% configuration data

th1=0;
th2=3*pi/4;
th3=pi;
th4=pi;
L=1;

% homogeneous transformations

A1 = [cos(th1) 0 sin(th1) 0;
      sin(th1) 0 -cos(th1) 0;
      0 1 0 0;
      0 0 0 1];
A2 = [cos(th2) 0 sin(th2) 0;
      sin(th2) 0 -cos(th2) 0;
```

```
      0 1 0 0;
      0 0 0 1];
A3 = [cos(th3) 0  sin(th3) 0;
      sin(th3) 0 -cos(th3) 0;
      0 1 0 L;
      0 0 0 1];
A4 = [cos(th4) -sin(th4) 0 L*cos(th4);
      sin(th4)  cos(th4) 0 L*sin(th4);
      0 0 1 0;
      0 0 0 1];

A12=A1*A2;
A13=A12*A3;
A14=A13*A4;

% geometric Jacobian

z0=[0 0 1]';
z1=A1(1:3,3);
z2=A12(1:3,3);
z3=A13(1:3,3);
p0=[0 0 0]';
p1=A1(1:3,4);
p2=A12(1:3,4);
p3=A13(1:3,4);
p4=A14(1:3,4);

J(1:3,1)=cross(z0,p4-p0);
J(1:3,2)=cross(z1,p4-p1);
J(1:3,3)=cross(z2,p4-p2);
J(1:3,4)=cross(z3,p4-p3);

J(4:6,1)=z0;
J(4:6,2)=z1;
J(4:6,3)=z2;
J(4:6,4)=z3;

% end
```

Whatever approach is followed, one ends up with the following matrix (where $L = 1$, if we have worked numerically):

$$\boldsymbol{J}(\boldsymbol{\theta}^*) = \begin{pmatrix} 0 & -L\sqrt{2} & 0 & -L\dfrac{\sqrt{2}}{2} \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & -L\dfrac{\sqrt{2}}{2} \\ 0 & 0 & \dfrac{\sqrt{2}}{2} & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & \dfrac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

It can be seen that the rank of $\boldsymbol{J}_L(\boldsymbol{\theta}^*)$ is 3, and thus the given configuration $\boldsymbol{\theta}^*$ is not singular for this sub-Jacobian. By inspection of this matrix, the desired linear/angular velocity vector $\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix}^T$ is realized by choosing

$$\dot{\boldsymbol{\theta}}_d = \begin{pmatrix} 0 & -\dfrac{\sqrt{2}}{2} & 0 & \sqrt{2} \end{pmatrix}^T,$$

obtaining in fact

$$\boldsymbol{J}(\boldsymbol{\theta}^*)\dot{\boldsymbol{\theta}}_d = \begin{pmatrix} 0 \\ 0 \\ -L \\ 0 \\ -\dfrac{\sqrt{2}}{2} \\ 0 \end{pmatrix}.$$

Moreover, one can see that the joint velocity vector $\dot{\boldsymbol{\theta}}_d$ is the only one providing the desired linear/angular velocity. Therefore, $\dot{\boldsymbol{\theta}}_d$ is the minimum norm solution (with $\|\dot{\boldsymbol{\theta}}_d\| = 1.5811$). As a check, it can be verified that

$$\boldsymbol{J}^{\#}(\boldsymbol{\theta}^*) \begin{pmatrix} \boldsymbol{v}_d \\ \boldsymbol{\omega}_d \end{pmatrix} = \dot{\boldsymbol{\theta}}_d,$$

where the pseudoinverse $\boldsymbol{J}^{\#}(\boldsymbol{\theta}^*)$ can be computed either by using the Matlab function `pinv` or by its explicit expression in case of a full (column) rank matrix $\boldsymbol{J}$ with more rows than columns,

$$\boldsymbol{J}^{\#} = (\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T,$$

which applies to the present case since the rank of $\boldsymbol{J}(\boldsymbol{\theta}^*)$ is 4. Finally, the joint torque vector $\boldsymbol{\tau}$ that balances the specified Cartesian force/torque vector $\begin{pmatrix} \boldsymbol{F}^T & \boldsymbol{M}^T \end{pmatrix}^T$ is computed as

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{\theta}^*) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ L\sqrt{2} \\ 0 \\ L\dfrac{\sqrt{2}}{2} \end{pmatrix},$$

i.e., it is given by the transpose of the first row of $\boldsymbol{J}(\boldsymbol{\theta}^*)$, changed of sign (the usual convention holds also for joint torques: positive torques are counterclockwise).

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