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Time-optimal control of a visco-elastic joint

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***Abstract*—We address the time-optimal, rest-to-rest motion problem under bounded control for a two-mass system interconnected by a visco-elastic joint. A complete geometric solution has recently been found in [1] for the purely elastic case, exploiting symmetries that are lost with the introduction of viscous damping. A semi-analytic solution is presented here by decoupling the fourth-order problem into its rigid body and visco-elastic second-order subproblems and imposing then motion coordination of the two subsystems. The presented numerical results confirm the optimality of the obtained bang-bang solutions, all having three control switchings.**

***Index Terms*—Time-optimal control, visco-elastic joint, motion coordination, robot control**

I. Introduction

The diffusion of lightweight collaborative robots with compliance at the joints [2] has lead to several studies aimed at optimizing perfomance in terms of speed and/or torque, while preserving the original operational safety [3]. Different criteria have been considered in an optimal control framework in case of variable stiffness actuation, e.g., maximization of the stored potential energy [4] or of the peak link velocity [5].

Less attention has been paid to the problem of transferring in minimum time a robot with elastic joints between two rest configurations under a maximum torque bound. To get a better insight, most of the literature has considered first the single-input, linear case. In [6], the safe brachistochrone problem was presented in particular for a two-mass system connected by a transmission of constant elasticity. The time-optimal problem is solved numerically under a bound on the motor input force and a safety-rated link velocity limit. An analytic solution to a similar problem was found in [5], assuming however as system input the motor velocity rather than the force. In [1], the complete time-optimal solution under an input force bound was determined in closed form, following a geometric approach in a suitable phase plane and exploiting the relevant symmetric properties of the problem. The optimal solution is of the bang-bang type with a single switching at the midpoint, when the task matches the natural motion of the mechanical system, or with three switchings otherwise (two of which are placed symmetrically w.r.t. the motion midpoint).

In this paper, the result in [1] is extended to visco-elastic joints, namely when the two masses are connected by a spring of constant stiffness that dissipates energy proportionally to the speed of spring deformation, i.e., in a viscous fashion. This damping effect destroys the symmetry of the problem.

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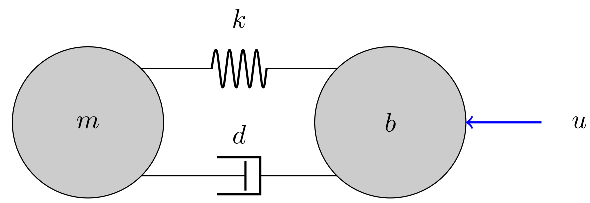

Fig. 1. Schematic representation of a visco-elastic joint.

A semi-analytic solution approach is presented that *i)* uses a transformation to decouple the system into its rigid and visco-elastic parts; *ii)* determines a parametrized class of bang-bang solutions in closed form for the rigid part; and *iii)* finds numerically the time-optimal input command by searching within the previous class of bang-bang solutions, imposing a coordinated motion to the visco-elastic subsystem.

II. Problem Statement

Consider two masses b and m, representing respectively the motor and the link, connected by a spring with stiffness k and viscous damping d. A bounded input command u is applied to the mass b. The dynamic equations of this linear mechanical system are

$$\begin{aligned} m\,\ddot{q} + d\,(\dot{q}-\dot{\theta}) + k\,(q-\theta) &= 0 \\ b\,\ddot{\theta} + d\,(\dot{\theta}-\dot{q}) + k\,(\theta-q) &= u. \end{aligned} \tag{1}$$

The model is composed by two coupled differential equations. However, introducing a change of coordinates from (θ, q) to (r, ϕ) defined by

$$r = \frac{m\,q + b\,\theta}{m+b} \qquad \phi = \theta - q, \tag{2}$$

the dynamics decouples in the following two second-order subsystems (both driven by u):

$$M\,\ddot{r} = u \tag{3}$$

$$\mu\,\ddot{\phi} + d\,\dot{\phi} + k\,\phi = \nu\,u, \tag{4}$$

where $M = m+b$ is the total mass, $\mu = mb/M$ is the reduced mass (since $\mu < \min\{m, b\}$), and $\nu = m/M$. We refer to eq. (3) as the Center of Mass (CoM) or *rigid body* subsystem. Furthermore, equation (4) involves the joint deformation ϕ and can be normalized as

$$\ddot{\phi} + 2\zeta\dot{\phi} + \omega^2\phi = \frac{u}{b}, \tag{5}$$

where the natural frequency $\omega = \sqrt{k/\mu}$ and the damping coefficient $\zeta = d/2\mu$ (with $d > 0$) of an asymptotically stable second-order dynamics appear. We refer to eq. (5) as the Reduced Mass (RM) or *visco-elastic* subsystem.

Defining as state $\boldsymbol{x} = (r, \phi, \dot{r}, \dot{\phi})$, the state-space representation of the system (3),(5) is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u, \tag{6}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & -2\zeta \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 1/M \\ 1/b \end{pmatrix}.$$

For the linear system (6), the rest-to-rest minimum-time control problem is formulated as [7]

$$\min_{|u|\leq 1} \int_0^{t_f} dt = t_f, \tag{7}$$

with boundary conditions at the initial time $t = 0$ and at the free final time $t = t_f$ given by

$$\boldsymbol{x}(0) = \boldsymbol{0} \qquad \boldsymbol{x}(t_f) = \begin{pmatrix} r_f & 0 & 0 & 0 \end{pmatrix}^T, \tag{8}$$

where r_f it the desired displacement to be realized. Note that the input bound on u is normalized in (7). In the following, we assume without loss of generality that $r_f > 0$.

Being the pair $(\boldsymbol{A}, \boldsymbol{B})$ in (6) controllable and having the matrix $\boldsymbol{A}$ all eigenvalues with non-positive real part,

$$\sigma(\boldsymbol{A}) = \left\{0, 0, -\zeta \pm \sqrt{\zeta^2 - \omega^2}\right\}, \tag{9}$$

the optimal control problem (7),(8) has a solution for any finite final state (see [8], Theo. 9.12, p. 288). The optimal solution can be obtained using Pontryagin maximum principle [7], with the optimal control law given by

$$u^*(t) = -\text{sign}\{\boldsymbol{\lambda}^{*T}(t)\boldsymbol{B}\}, \tag{10}$$

where $\boldsymbol{\lambda}^*(t)$ is the costate solution of the differential equation

$$\dot{\boldsymbol{\lambda}} = -\boldsymbol{A}^T\boldsymbol{\lambda}, \tag{11}$$

for suitable boundary conditions at $t = t_f$. While we could try to find the unknown constants of integration in order to determine $\boldsymbol{\lambda}^*(t)$, the fourth-order nature of the system makes this approach somewhat cumbersome. On the other hand, we have the following result on time-optimal control problems.

Theorem 1. *For a linear, controllable and single-input system, if all the eigenvalues of $\boldsymbol{A}$ have non-positive real parts, then there exists always a unique time-optimal solution that is nonsingular and the optimal control is of the* bang-bang *type. Moreover, if all the eigenvalues of $\boldsymbol{A}$ are real and non-positive, then there are at most $n-1$ switchings, being n the dimension of the system state.*

Proof. See [8], Coroll. 9.13 and Theo. 9.14, pp. 288-289. □

Thus, based on Theorem 1, one can seek the optimal control law in the form of a piecewise constant function over n intervals, having $n - 1$ switchings. For $r_f > 0$ and with a normalized input, this law has the general structure

$$u^*(t) = \begin{cases} +1 & [0, t_1] \cup \ldots \cup (t_{i-2}, t_{i-1}] \ldots \\ -1 & (t_1, t_2] \cup \ldots \cup (t_{i-1}, t_i] \ldots \end{cases} \cup (t_{n-1}, t_f]. \tag{12}$$

In the present case, the eigenvalues of $\boldsymbol{A}$ are listed in (9). Depending on the mass, stiffness and damping data, one may have $\zeta \geq \omega$, with all eigenvalues being real and non-positive (two always coincident at the origin, and the other two coincident only for $\zeta = \omega$), or $\zeta < \omega$, with a pair of complex conjugate eigenvalues with strictly negative real part. In the first case, we are guaranteed to have at most $n - 1 = 3$ switchings, while for the complex case we cannot infer anything a priori on the number of switchings.

Our solution approach will consider all cases with up to three switchings, computing the corresponding distinct switching instants t_i and the final time t_f for the CoM subsystem (3) in a parametrized form, and imposing then a coordinated motion, by verifying the effect of these solutions on the visco-elastic RM subsystem (5).

III. Switching Strategy for the Rigid Body

The CoM subsystem (3) is a scaled double integrator and its time-optimal rest-to-rest motion requires just one switching at $t_1 = t_f/2$ [7]. Integrating (3) from $r(0) = \dot{r}(0) = 0$ yields for the position

$$r(t) = \begin{cases} \dfrac{1}{2M}t^2 & 0 \leq t \leq t_1 \\ -\dfrac{1}{2M}\left(t^2 - 4t_1 t + 2t_1^2\right) & t_1 \leq t \leq t_f. \end{cases} \tag{13}$$

Setting $r(t_f) = r_f$ gives the optimal time $t_f = 2\sqrt{Mr_f}$.

Relaxing the time-optimality for the CoM subsystem, one can still reach the final goal r_f with zero velocity by considering multiple switchings in a bang-bang input profile. To generalize (13), we consider a generic number k of switchings at the time instants $t_1 < t_2 < \ldots < t_i < \ldots < t_k$. The expression of the CoM position during the last time interval $[t_k, t_f]$ will be given by

$$r(t) = \frac{(-1)^k}{2M}\left(t^2 + (-1)^{k-1}\sum_{i=1}^{k}(-1)^i t_i t + (-1)^k \sum_{i=1}^{k}(-1)^i t_i^2\right). \tag{14}$$

When increasing the number of switchings, one also obtains a longer motion time t_f. Thus, we should proceed incrementally in order to keep this time as short as possible. Further, all extra switchings other than the first one are redundant for the time-optimality of the CoM motion, but possibly needed for the coordinated motion of the visco-elastic subsystem. Note that for each extra switching we introduce a parameter that becomes a degree of freedom for the following coordination problem. It turns out that these parameters will be the key to solve our problem. Consider now more explicitly the two cases of $k = 2$ and $k = n - 1 = 3$ switchings.

A. Two Switchings

We introduce a positive parameter α and define the instant of the first switching as a fraction of the final time t_f. Using geometric reasoning on the piecewise linear time profiles of $\dot{r}(t)$, the second switching is also identified as a function of the parameter α. One obtains:

$$t_1 = \frac{1}{\alpha}t_f \qquad t_2 = \frac{\alpha + 2}{2\alpha}t_f, \tag{15}$$

with $t_2 > t_1$ for any $\alpha > 0$. The corresponding final time is

$$t_f = 2\alpha \sqrt{\frac{1}{4\alpha - \alpha^2}} \sqrt{M r_f}. \qquad (16)$$

In order to have a real t_f and $t_2 < t_f$, the parameter should be limited as $\alpha \in (2, 4)$. Then, t_f in (16) will always be larger than the final time with one switching only, as expected.

B. Three Switchings

With three switches, we introduce two positive parameters α and β that characterize the first two switchings:

$$t_1 = \frac{1}{\alpha} t_f \qquad t_2 = \frac{1}{\beta} t_f, \qquad (17)$$

with $\alpha > \beta$. As before, with straightforward but tedious computations one obtains for the third switching

$$t_3 = \frac{2\alpha + \alpha\beta - 2\beta}{2\alpha\beta} t_f, \qquad (18)$$

and for the final time

$$t_f = 2\alpha \sqrt{\frac{\beta}{\alpha\beta(\alpha+4) + 4\alpha(2-\alpha) - 8\beta}} \sqrt{M r_f}. \qquad (19)$$

Also in this case, there will be limitations on the admissible values of α and β, as derived from the defining inequalities $0 < t_1 < t_2 < t_3 < t_f$. More on this later.

IV. Coordination of the Visco-Elastic Dynamics

Also for the RM subsystem, we can express the solution for the deformation $\phi(t)$ as a piecewise function over successive intervals during which the constant input command is $u = \pm 1$. In particular, for the case of two real and distinct negative eigenvalues η_1 and η_2 of the visco-elastic dynamics (5), i.e., when $\zeta > \omega$, we obtain in the first time interval $[0, t_1]$ (for $u = +1$)

$$\phi_1(t) = \frac{\eta_2 \bar{u}}{\eta_1 - \eta_2} e^{\eta_1 t} - \frac{\eta_1 \bar{u}}{\eta_1 - \eta_2} e^{\eta_2 t} + \bar{u}, \qquad (20)$$

where $\bar{u} = u/(b\omega^2)$. Similarly, in the generic k-th time interval ($k > 1$) one can show that the solution is

$$\phi_k(t) = (-1)^{k-1} \frac{2\bar{u}}{\eta_1 - \eta_2} \left(\eta_1 e^{\eta_2 (t - t_k)} - \eta_2 e^{\eta_1 (t - t_k)} \right) + (-1)^k 2\bar{u} + \phi_{k-1}(t). \qquad (21)$$

This recursive structure holds also for the case of real coincident or complex conjugate eigenvalues of (5). For details, see [9].

It can be shown that the phase diagram $(\phi(t), \dot{\phi}(t))$ for the visco-elastic system is made by spirals, due to the presence of exponential terms in (20),(21) and in their time derivatives. As a consequence, following a geometric approach to find the switching instants can be extremely challenging. On the other hand, the switching instants must be the same for both the CoM and the RM subsystems. Therefore, we can use the parametrized switching instants found in Sect. III and plug them in the above equations, verifying under which conditions the final state, undeformed and at rest, can be reached in the last interval:

$$\phi_n(t_f) = 0 \qquad \dot{\phi}_n(t_f) = 0. \qquad (22)$$

It can be verified analytically that the boundary conditions (22) are never satisfied neither using one switching nor using two switchings. In the first case, this is rather intuitive as the two-interval CoM solution is symmetric in that case, whereas the presence of damping suggests a larger control effort in the first acceleration phase and a reduced one in the second. Therefore, we turn to the three switching case, inserting the expressions (17) to (19) in (22) with $n = 3$. This yields a system of two (strongly) nonlinear equations in the two unknown parameters α and β.

A closed-form solution of (22) for the optimal parameters α and β is not at all immediate, and we have to resort to a numerical method. Sufficient conditions for the existence of a solution are provided in [10]. Thus, the problem is transformed into a constrained optimization in which we minimize the loss function

$$f(\alpha, \beta) = |\phi_3(t_f)| + |\dot{\phi}_3(t_f)|. \qquad (23)$$

subject to the constraints on the admissibility of the parameters

$$\alpha - 2 > 0 \qquad (24)$$

$$\beta - 1 > 0 \qquad (25)$$

$$\alpha\beta - 2(\alpha - \beta) > 0 \qquad (26)$$

$$\alpha\beta(\alpha + 4) + 4\alpha(2 - \alpha) - 8\beta > 0. \qquad (27)$$

Once the solution (α^*, β^*) has been found, substituting the two optimal values in eqs. (17) to (19) yields the optimal switching times and the final time, with $t_1^* < t_2^* < t_3^* < t_f^*$. The optimal control law is the associated bang-bang sequence $u^*(t) = \{+1, -1, +1, -1\}$ (for $r_f > 0$).

V. Numerical Results

Using the data $m = b = 0.5$ [kg], $d = 1$ [N· s/m], and $r_f = 10$ [m], we have considered three cases:

A) $k = 0.5$ [N/m] (real distinct eigenvalues): $t_f^* = 9.2474$ [s], obtained for $\alpha^* = 2.7705$ and $\beta^* = 1.1913$;
B) $k = 1$ [N/m] (coincident eigenvalues): $t_f^* = 7.8747$ [s], obtained for $\alpha^* = 2.4626$ and $\beta^* = 1.1383$;
C) $k = 2$ [N/m] (complex eigenvalues): $t_f^* = 7.1407$ [s], obtained for $\alpha^* = 2.2591$ and $\beta^* = 1.0959$.

For solving (23)–(27), we have used the *fmincon* routine of MATLAB. In all cases, convergence was achieved within 20 iterations. The results in Figs. 2–4 show the phase plane diagrams of both the CoM and the RM subsystems, as well as the optimal control $u^*(t)$ plotted against the switching function $\boldsymbol{\lambda}^{*T}(t)\boldsymbol{B}$. The phase plots of the rigid body start at $(0, 0)$ and arrive at $(r_f, 0)$ as desired, whereas the phase plots of the visco-elastic dynamics start and return at $(0, 0)$, achieving thus coordination. The boundary conditions for determining the evolution of the four components of the costate $\lambda^*(t)$ were found imposing four linear equations based on the obtained switching times: $\boldsymbol{\lambda}^{*T}(t_i^*)\boldsymbol{B} = 0$, for $i = 1, 2, 3$, together with the condition on the Hamiltonian $H^*(t_f^*) = 0$ (see [7]).

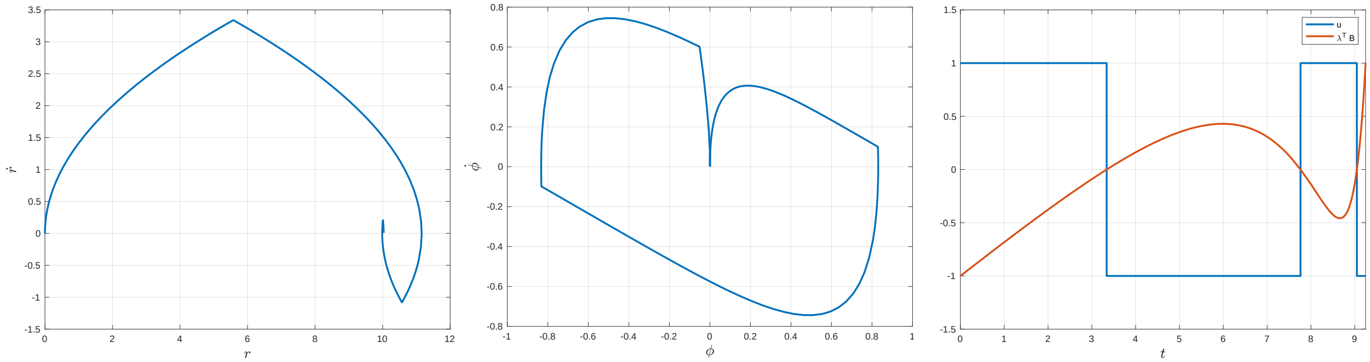

Fig. 2. Case A (real distinct eigenvalues): phase diagrams for CoM [left] and RM [center]; time-optimal control (blue) and switching function (red) [right].

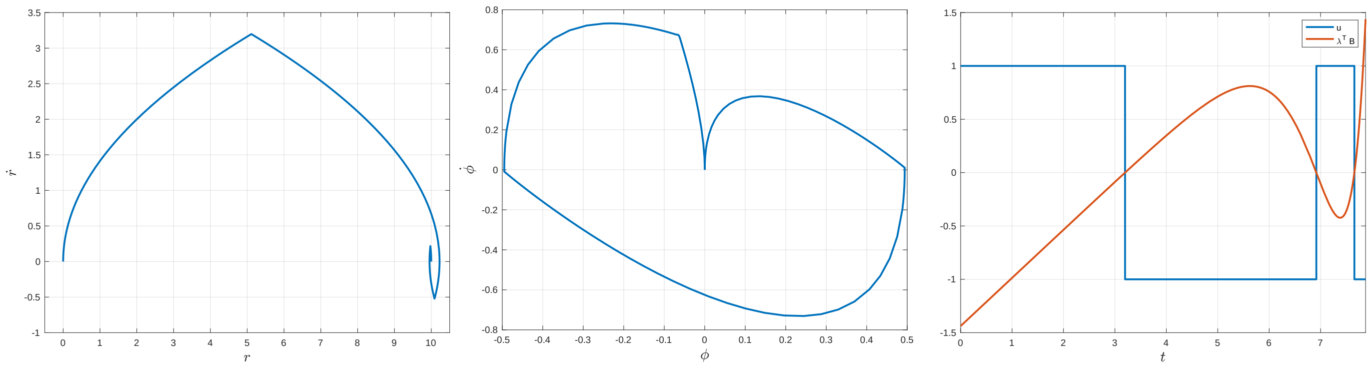

Fig. 3. Case B (coincident eigenvalues): phase diagrams for CoM [left] and RM [center]; time-optimal control (blue) and switching function (red) [right].

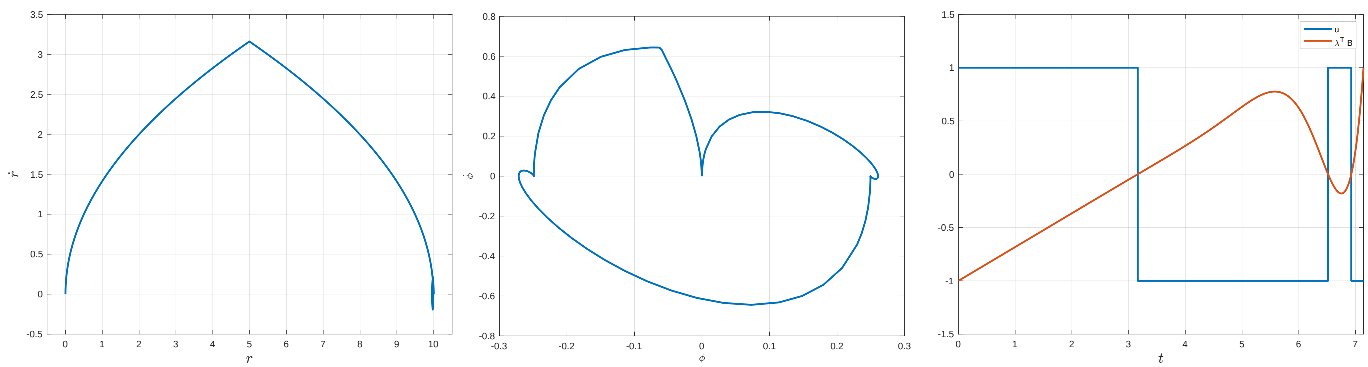

Fig. 4. Case C (complex eigenvalues): phase diagrams for CoM [left] and RM [center]; time-optimal control (blue) and switching function (red) [right].

VI. Conclusions

We have provided a semi-analytic solution to the time-optimal rest-to-rest motion of a two-mass system interconnected by a visco-elastic joint. The optimal control law is always bang-bang with three switchings, as determined by decoupling the problem in a rigid and a visco-elastic dynamics, solving the first in a parametrized way and substituting the results in the second so as to achieve coordination.

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