Machine Intelligence & Robotic Control, Vol. 4, No. 3, 113–125 (2002)

Paper

Underactuated Manipulators: Control Properties and Techniques*

Alessandro De Luca†, Stefano Iannitti†, Raffaella Mattone†, and Giuseppe Oriolo†

Abstract: We consider planning and control problems for underactuated manipulators, a special instance of mechanical systems having fewer input commands than degrees of freedom. This class includes robots with passive joints, elastic joints, or flexible links. Structural control properties are investigated, showing that manipulators with passive joints in the absence of gravity are the most difficult to control. With reference to these, solutions are proposed for the typical problems of trajectory planning, trajectory tracking, and set-point stabilization. The relevance of nonlinear control techniques such as dynamic feedback linearization and iterative steering is clarified through illustrative examples.

Keywords: Underactuated Manipulators, Trajectory Planning, Trajectory Tracking, Set-Point Regulation, Nonlinear Control

1. Introduction

UNDERACTUATED controlled mechanical systems are characterized by the fact that the number of available independent commands is strictly less than the number of generalized coordinates. This class encompasses many interesting robotic devices, ranging from underwater vehicles to legged humanoids with some passive joints.

In recent years, a remarkable research effort has been devoted to the study of underactuated manipulators, i.e., fixed-base articulated chains of bodies whose dynamics is described by n nonlinear second-order equations actuated by $m < n$ control inputs (see [49]). The list of underactuated manipulators includes, among the others, rigid robots with transmission elasticity, lightweight robots with flexible links, and robots with passive joints. In the first two cases, underactuation is mainly a result of more accurate dynamic modeling of the system, with the available commands affecting directly only rigid-body motion. In the last case, underactuation is a consequence of special operative conditions: the failure of one or more actuators or an on-purpose 'minimalistic' design that avoids the use of full actuation.

The common definition of underactuation does not capture however the fact that mechanical systems within this class display different levels of difficulties from the control point of view. In particular, it has been recognized that manipulators with passive joints in the absence of gravity raise by far the most challenging theoretical problems, typically requiring non-classical feedback control approaches. To clarify this issue, we shall provide some inherent reasons for such differences, based on the analysis of structural control properties of a general dynamic model of underactuated robots.

Dynamic modeling, trajectory planning and feedback control of specific instances of underactuated mechanical systems have already been investigated. The basic motion tasks that are considered are the planning of (dynamically) feasible point-to-point trajectories, their asymptotic tracking by feedback control, and the regulation to a desired equilibrium configuration. Significant theoretical results can be found in [41] and [43]. Nevertheless, a general theory is not yet available and only case-by-case planning and/or control solutions have been obtained so far.

Motivated by this, we sketch a review of the most significant case studies found in the literature of underactuated robots with passive joints. Exploiting results from advanced nonlinear control theory, we describe in some detail two quite general approaches which have been proved to be effective in controlling these systems, namely dynamic linearization via feedback and iterative steering. For illustration, we also work out the application of these techniques to examples of planar manipulators with passive joints.

The paper is organized as follows. In Section 2, underactuated manipulators are described in a unified framework. The typical planning and control problems are defined in Section 3, while Section 4 presents the most relevant structural control properties. After providing a synopsis of the existing control approaches in Section 5, dynamic linearization and iterative steering techniques are used in Sections 6 and 7, respectively for planning and tracking rest-to-rest trajectories and for set-point stabilization of two different planar manipulators with passive joints.

2. Underactuated Manipulators

The dynamics of robot manipulators can be described in general by

$$B(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + e(\theta) = G(\theta)\,\tau \qquad (1)$$

where $\theta \in \mathbb{R}^n$ is the vector of generalized coordinates, $B(\theta)$ is the positive definite symmetric inertia matrix, $c(\theta, \dot{\theta})$ and $e(\theta)$ are respectively the vectors of velocity (Coriolis/centrifugal) and potential (gravitational/elastic) terms, and $G(\theta)$ is the matrix mapping the external forces/torques $\tau \in \mathbb{R}^m$ acting on the system to generalized forces performing work on θ.

When $m < n$, a manipulator is said to be *underactuated* (of degree $n - m$), i.e., it has less control inputs than

* Received December 8, 2002; accepted March 22, 2003.

† Dipartimento di Informatica e Sistemistica, Università degliStudi di Roma "La Sapienza", Via Eudossiana 18, 00184 Roma, Italy. E-mail: {deluca,oriolo}@dis.uniroma1.it, mattone@labrob.ing.uniroma1.it, iannitti@labrob1.ing.uniroma1.it

©2002 Cyber Scientific

generalized coordinates. If matrix G is full column rank, it is easy (e.g., see [18]) to show that, by performing an input transformation and a change of coordinates, the system dynamics takes on the partitioned structure

$$\begin{bmatrix} B_{aa}(q) & B_{ua}^T(q) \\ B_{ua}(q) & B_{uu}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_u \end{bmatrix} + \begin{bmatrix} c_a(q,\dot{q}) \\ c_u(q,\dot{q}) \end{bmatrix} + \begin{bmatrix} e_a(q) \\ e_u(q) \end{bmatrix} = \begin{bmatrix} \tau_a \\ 0 \end{bmatrix} \tag{2}$$

where $\tau_a \in \mathbb{R}^m$ and, with a slight abuse of notation, we kept the same symbols of Eq. (1) for the dynamic terms. The new vector of generalized coordinates q displays the partition in *actuated* (active) and *unactuated* (passive) degrees of freedom, respectively $q_a \in \mathbb{R}^m$ and $q_u \in \mathbb{R}^{n-m}$.

In particular, the last $n-m$ equations of the dynamics (2)

$$B_{ua}(q)\ddot{q}_a + B_{uu}(q)\ddot{q}_u + c_u(q,\dot{q}) + e_u(q) = 0 \tag{3}$$

represent a set of second-order differential constraints. In order to be *feasible*, a trajectory $q = q(t)$ should satisfy these constraints at all times t. Due to the intrinsic presence of these differential constraints as part of the system dynamics, these mechanisms have also been called *second-order nonholonomic* systems. Indeed, some useful ideas for trajectory planning and control of underactuated manipulators have been suggested by the analogy with kinematic systems subject to classical first-order nonholonomic constraints.

In principle, Eq. (2) includes the following types of N-joint manipulators:

a) rigid robots with n_a active and $n_u = N - n_a$ passive joints:

$$n = n_a + n_u, \qquad m = n_a;$$

b) robots with n_r rigid and $n_e = N - n_r$ elastic joints:

$$n = n_r + 2n_e, \qquad m = n_r + n_e;$$

c) robots with n_r rigid and $n_f = N - n_r$ flexible links, each modeled by η_f deformation modes:

$$n = n_r + (\eta_f + 1)n_f, \qquad m = n_r + n_f.$$

Another interesting example of underactuated manipulators is represented by kinematically redundant arms with all n joints passive and $m < n$ forces/torques applied to the end-effector as the only available input command [18]. From a modeling viewpoint, the dynamics of these mechanisms can be written as in Eq. (1) and thus can be equivalently expressed, up to kinematic singularities, in the form of Eq. (2).

Finally, we mention here that underactuated manipulators may be equipped with on/off brakes at the passive joints. Switching control strategies can be designed in this case; in particular, configuration control techniques have been addressed in [1], [2] and [5]. We shall not consider the presence of brakes in this paper.

In order to simplify the following analysis, as well as the control design, it is convenient to perform a preliminary *partial feedback linearization* of Eq. (2). Solving the second equation for $\ddot{q}_u$ and substituting in the first, one can check that the (globally defined) static feedback

$$\begin{aligned} \tau_a = {} & \left(B_{aa}(q) - B_{ua}^T(q)B_{uu}^{-1}(q)B_{ua}(q)\right) a + c_a(q,\dot{q}) \\ & + e_a(q) - B_{ua}^T(q)B_{uu}^{-1}(q)\ \left(c_u(q,\dot{q}) + e_u(q)\right) \end{aligned} \tag{4}$$

leads to a system in the form

$$\ddot{q}_a = a \tag{5}$$

$$B_{uu}(q)\,\ddot{q}_u = -B_{ua}(q)\,a - c_u(q,\dot{q}) - e_u(q) \tag{6}$$

where the actuated degrees of freedom are now directly controlled by the new acceleration input $a \in \mathbb{R}^m$.

Equations (5) and (6) can be seen as a canonical representation of underactuated manipulator dynamics. The controllability properties remains obviously the same as those of Eq. (2).

3. Formulation of Planning and Control Problems

In the control of mechanical systems, three basic problems arise:

P1 Trajectory planning
Given an initial state $(q^0, \dot{q}^0)$ and a final desired state $(q^d, \dot{q}^d)$, find a feasible trajectory $q(t)$ (i.e., satisfying Eq. (2) for some $\tau_a(t)$, with $t \in [0, T]$) that joins the initial and the final state. If $\dot{q}^0 = \dot{q}^d = 0$, this is a *rest-to-rest trajectory planning* problem. The motion time $T > 0$ may be assigned or not.

P2 Trajectory tracking
Given a feasible trajectory $q^d(t)$, with $t \in [0, \infty)$, find a feedback control law that asymptotically drives the tracking error $e(t) = q^d(t) - q(t)$ to zero, at least locally.

P3 Set-point regulation
Given a desired equilibrium configuration q^d, find a feedback control law that makes the state $(q, \dot{q}) = (q^d, 0)$ asymptotically stable, at least locally around the trajectory.

Note that by solving problem P1 for a finite time T, we implicitly obtain an input command $\tau_a(t)$ that drives the system between the two given states—a controllability result. Moreover, if there is no solution to the assigned trajectory planning problem P1, the corresponding trajectory tracking problem P2 will become meaningless. On the other hand, it may happen that a feasible trajectory $q(t)$ joining the two equilibrium states $(q^0, 0)$ and $(q^d, 0)$ exists, but we are not able to compute it in advance through a planning phase. Still, if problem P3 can be solved, and if $(q^0, \dot{q}^0)$ lies within the basin of attraction of the stabilizing controller, one obtains as a byproduct an asymptotic solution (i.e., for motion time T going to infinity) to problem P1.

We recall that for fully actuated mechanical systems (i.e., with $m = n$) these three problems have always solution. As for P1, any trajectory $q(t)$ interpolating $q^0(0)$ and $q^d(T)$, for an arbitrary $T > 0$ and with any boundary velocities, is feasible provided that $q(t)$ is twice differentiable. Moreover, since there exists a nonlinear static state feedback that converts the system into a linear controllable form (the so-called *computed torque* method), P2 and P3 can be solved using standard control techniques. Indeed, even simpler feedforward plus linear feedback solutions are available (typically, based on Lyapunov/LaSalle analysis).

©2002 Cyber Scientific

4. Control Properties of Underactuated Manipulators

In this section we investigate some structural control properties of underactuated systems of the form (2) in order to gain a deeper insight into the solvability of problems P1–P3, and possibly to envisage appropriate design techniques.

We begin by noting that, when T is not assigned, the existence of a finite-time solution to P1 for any state $(q^d, \dot{q}^d)$ in a neighborhood of $(q^0, \dot{q}^0)$ is equivalent to the property of *local controllability* at $(q^0, \dot{q}^0)$. If local controllability holds at any state, then the system is *controllable* (in the natural sense) and P1 is solvable for any pair of initial and final states. On the other hand, the existence of a solution to problem P2 or P3 implies that the system is (locally or globally) *stabilizable* along the reference trajectory or, respectively, at the set-point.

Note that, in general, controllability does not imply stabilizability for nonlinear systems. As a matter of fact, as shown in [41], manipulators with passive joints in the absence of gravity cannot be stabilized at a point by smooth static feedback, as they violate the necessary condition due to Brockett [7]. As a consequence, any feedback law solving P3 must necessarily be discontinuous and/or time-varying.

Unfortunately, there are no general necessary and sufficient conditions for the study of controllability and stabilizability of nonlinear systems. In the following, we present a number of controllability-related properties, for which verifiable conditions exist. As structural control properties are invariant under regular feedback, we shall use either Eq. (2) or Eqs. (5) and (6) in our investigation.

4.1 Integrability

A first test on the controllability of an underactuated robot consists in checking whether the $(n-m)$-dimensional second-order differential constraint expressed by Eq. (3) is integrable in the sense of [41]. In particular, Eq. (3) may be partially integrable to a set of n_1 $(0 < n_1 \leq n-m)$ first-order differential constraints

$$h_1(q, \dot{q}) = 0 \tag{7}$$

or even completely integrable to a set of $n_2 \leq n_1$ holonomic constraints

$$h_2(q) = 0. \tag{8}$$

These integrability properties may be tested by the necessary and sufficient conditions given in [41]; if any of these holds, then full state controllability is lost. However, if Eq. (3) is only partially (but not completely) integrable, it is still possible to steer the mechanism between equilibrium points. In fact, this property (equilibrium controllability, see Section 4. 3) is easily established, essentially by noting that at equilibrium points the first-order differential constraint (7) becomes Pfaffian (see [41]), and recalling that the non-integrability of this kind of constraints implies the controllability of the associated kinematic system. One example of partial (but not complete) integrability is a planar $\mathrm{R}_u\,(n\mathrm{R}_a)$ manipulator[†] in the absence of gravity.

If Eq. (3) is instead completely integrable to $n-m$ holonomic constraints, then the system motion is confined to a particular m-dimensional submanifold of the configuration space, depending on the initial configuration $q^0 = (q_a^0, q_u^0)$. As a consequence, problems P1 and P3 admit no solution, except for special choices of the initial and final states. One example of complete integrability is a planar $\mathrm{R}_u\mathrm{R}_a$ robot in the absence of gravity [41].

4.2 First-order controllability

When the second-order differential constraint (3) is not partially or completely integrable, there is no kinematic or dynamic obstruction to the controllability of the system. However, the nature of this controllability can be very different, with relevant implications on the design of control methods.

For nonlinear systems, the simplest way to establish local controllability at an equilibrium point is to prove that the approximate linearization of the system around the point is controllable. Define the set of equilibrium states as $\mathcal{Q}^e = \{q = q^e : e_u(q^e) = 0,\ \dot{q} = 0\}$. Since c_u is quadratic in $\dot{q}$, the approximate linearization of the dynamics (5) and (6) at any point of $\mathcal{Q}^e$ is obtained as

$$\delta\ddot{q}_a = a$$
$$B_{uu}(q^e)\,\delta\ddot{q}_u + (\nabla_q^T e_u)(q^e)\,\delta q = -B_{ua}(q^e)\,a.$$

If this linear dynamics is controllable, local controllability of the original nonlinear system is guaranteed. Moreover, for each state $(q^d, 0) \in \mathcal{Q}^e$, problem P3 is solved by any control law that stabilizes the linear approximation; the corresponding trajectory represents an asymptotic solution to the associated P1 problem.

One can show that underactuated manipulators are controllable in the first approximation only if:

1. $m \geq n - m$;
2. $(\nabla_q e_u)(q^e)$ is full row rank.

In particular, underactuated manipulators are not linearly controllable in case of simultaneous absence of gravitational and flexibility/elasticity effects on the passive dof's $(e_u(q) \equiv 0, \forall q)$.

Underactuated systems that are controllable in the first approximation include robots with elastic joints and/or flexible links, as well as examples of robots with passive joints subject to gravity. In principle, the control of these mechanisms may be attempted using more conventional nonlinear techniques. In particular, robots with elastic joints are globally equivalent to linear controllable systems under the action of a nonlinear (static or dynamic) feedback: global solutions to problems P1, P2 and P3 can be found, e.g., in [16], [47] and, respectively, in [53]. As for robots with flexible links, the reader may refer to [12] for trajectory planning, to [25] for tracking and to [24] for set-point regulation techniques. Finally, linearly controllable underactuated manipulators under the action of gravity are the so-called Acrobot ($\mathrm{R}_u\,\mathrm{R}_a$) and Pendubot ($\mathrm{R}_a\,\mathrm{R}_u$), see, e.g., [20], [48]–[50] for specific stabilization strategies.

[†]We use in the following the notation $\mathrm{P}_a/\mathrm{P}_u$, $\mathrm{R}_a/\mathrm{R}_u$ for actuated/unactuated prismatic and, respectively, revolute joints. If present, a prefix n indicates the occurrence of $n \geq 2$ consecutive such joints.

©2002 Cyber Scientific

A similar approach can be followed in order to establish local controllability along a reference trajectory, using the approximate linearization of the system around a feasible trajectory $q^d(t)$. If the resulting linear (and, in general, time-varying) system is controllable, problem P2 can be locally solved using linear techniques.

4. 3 Nonlinear controllability

When linear controllability does not hold, it is necessary to investigate nonlinear controllability concepts. Among these, the most elementary is *accessibility*: a mechanical system is said to be accessible at $x^0 = (q^0, \dot{q}^0)$ if the set $\mathcal{R}(x^0)$ of states that are reachable from x^0 within any finite time includes some open subset of the state space. Such property may be easily tested through the well-known accessibility rank condition [40]. While for driftless systems accessibility is equivalent to controllability, this is not true for underactuated manipulators, because Eqs. (5) and (6) have a drift term.

A stronger concept is *small-time local controllability* (STLC), see [51]. A mechanical system is said to be small-time locally controllable at $x^0 = (q^0, \dot{q}^0)$ if, for any neighborhood $\mathcal{V}$ of x^0 and any time $T > 0$, the set $\mathcal{R}^T_{\mathcal{V}}(x^0)$ of states that are reachable from x^0 within time T along trajectories contained in $\mathcal{V}$ includes a neighborhood of x^0. STLC is a stronger property than controllability. In particular, a non-STLC but controllable system must in general perform finite-size maneuvers in order to achieve arbitrarily small reconfigurations. Therefore, while a feasible trajectory joining any two given states exists in this case by definition, its planning may be out of reach, at least with the available techniques. Interestingly, however, problem P3 (set-point regulation) for non-STLC systems may still be solvable; we will present an example in Section 7. 1.

Only sufficient conditions are available for testing STLC; see [6], [51] and, for the specific case of underactuated mechanical systems, also [18] and [13]. The planar $(\mathrm{RR})_a\mathrm{R}_u$ manipulator satisfies the sufficient conditions for STLC, as shown in [3] and [13]. In general, no conclusion about STLC can be drawn for a mechanism that violates these conditions; a notable exception is the class of manipulators with passive joints and a single actuator ($m = 1$) in zero gravity, which was explicitly proven to be non-STLC in [28].

The interest of the STLC notion is however limited, for there is no constructive control design based on such property; we only know [11] that STLC systems can be locally stabilized by time-varying feedback—an existence result. Therefore, new concepts of nonlinear controllability have been defined, in order to better reflect the nature of the control problems of interest for underactuated manipulators and, whenever possible, provide constructive design methods.

As a matter of fact, for a mechanical system the typical interest is solving problems P1 and P3 (i.e., proving controllability and stabilizability) for equilibrium states. On the other hand, second-order mechanical systems cannot be STLC at states with nonzero velocity[†] , showing that this property is indeed very restrictive. Motivated by this, the weaker concepts of *small-time local configuration controllability* (STLCC) and *equilibrium controllability* (EC) have been introduced (see [33], [34]).

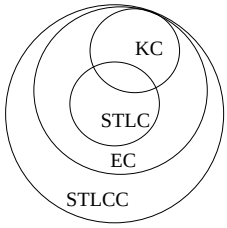

Fig. 1 Relations among nonlinear controllability concepts for mechanical systems at an equilibrium point

A system has the STLCC property at a *configuration* q^0 if, for any neighborhood $\mathcal{V}_q$ of q^0 in the configuration space, and any time $T > 0$, the set $\mathcal{R}^T_{\mathcal{V}_q}(q^0)$ of configurations that are reachable (with some final velocity $\dot{q}$) within T, starting from $(q^0, 0)$ and along configuration trajectories contained in $\mathcal{V}_q$, includes a neighborhood of q^0. By definition, an STLC system is also STLCC. Sufficient conditions for STLCC are given in [34].

A system has the EC property if it can be steered between any two equilibrium configurations in finite time. The sufficient conditions for STLCC, if verified at any configuration, guarantee that the system is equilibrium controllable. It can also be shown that an STLC system is also equilibrium controllable.

A special form of equilibrium controllability is the so-called *kinematic controllability* (KC) (see [8], [9]). A mechanical system has the KC property if every configuration is reachable via a sequence of *kinematic motions*, i.e., feasible paths in the configuration space that may be followed with any arbitrary timing law. A vector field whose flow generates a kinematic motion is called *decoupling*. Note that KC implies both EC and STLCC, while it has no implications on STLC. Examples of kineatically controllable underactuated manipulators are the planar $\mathrm{R}_u(\mathrm{RR})_a$, $\mathrm{R}_a\mathrm{R}_u\mathrm{R}_a$, and $(\mathrm{RR})_a\mathrm{R}_u$ robots (the latter being also STLC). If a mechanism is kinematic controllable, the trajectory planning problem P1 may be solved in an algorithmic fashion; however, stabilizing the system along the decoupling vector fields (problem P2) may be an issue and is currently a subject of research.

The relationships among the various nonlinear controllability concepts for mechanical systems are illustrated in **Fig. 1**.

4. 4 Feedback linearizability

Under appropriate conditions, controlled mechanical systems with nonlinear dynamics can be exactly transformed into linear controllable systems by means of a nonlinear state feedback and a change of coordinates. For underactuated robots, *static* feedback laws are never able to achieve this result, since they allow at most a partial feedback linearization such as the one presented in Eqs. (5) and (6).

[†]For example, consider the simple mechanical system described by $\ddot{q} = u$, $q \in \mathbb{R}$. From a state $x^0 = (q^0, \dot{q}^0)$ having $\dot{q}^0 > 0$, it is not possible to reach states $x^d = (q^d, \dot{q}^d)$ having $q^d < q^0$ with a trajectory arbitrarily close to x^0; in fact, any trajectory joining x^0 and x^d must intersect the axis $\dot{q} = 0$ in order to invert the velocity.

©2002 Cyber Scientific

More interestingly, the use of a nonlinear *dynamic feedback linearization* (DFL) approach can lead to an equivalent linear controllable system, provided that a set of m linearizing (also called *flat*) outputs exists. Unfortunately, only necessary *or* sufficient conditions (see, e.g., [32, Prop. 5.4.4], [10], [27]) are currently available for the existence of these linearizing outputs and no systematic procedure is defined in order to individuate these outputs[††] . Thus, the property of being exactly linearizable via dynamic feedback can only be established by studying the specific case at hand. When this result holds true, P1–P3 may then be solved by suitable control design on the linear side of the problem. However, in the absence of linear controllability, the original underactuation of the robot ultimately leads to the presence of control singularities, at least at the regulation point (otherwise smooth stabilization would be possible). Special care should be devoted to the handling of such singularities during motion (see Section 6. 1).

5. A Review of Control Techniques

The previous considerations on structural control properties should clarify the severe theoretical difficulties that arise when addressing control problems for underactuated manipulators and justify the variety of approaches taken by researchers in order to solve them on a case-by-case basis. An overview of case studies for planar underactuated manipulators found in the literature is given in **Table 1** and briefly discussed in this section.

In the absence of gravity, stabilization of a planar 2R manipulator with a passive elbow joint ($\mathrm{R}_a\mathrm{R}_u$ robot) has been obtained in [39] by means of a time-varying periodic feedback designed via Poincaré map analysis. For the same robot, iterative steering has been used in [19]. The lack of other successful solutions for underactuated manipulators with a single actuation command follows from the previous negative results on their controllability properties.

On the other hand, 2R planar manipulators with a single actuator in the presence of gravity ($\mathrm{R}_a\mathrm{R}_u$ or $\mathrm{R}_u\mathrm{R}_a$ robots) have been considered by several authors. Due to the gravitational drift, the region of the state space where the robot can be kept in equilibrium is reduced, and consists of two disjoint manifolds. Moving between these two requires appropriate swing-up maneuvers, whose synthesis has been tackled so far by energy-based [48], [50], passivity-based [26], [44], or iterative steering [20] control techniques. The trajectory tracking problem has been solved for these robots, based on nonlinear output regulation [4], [42]. Finally, also the control around a non-equilibrium configuration of a $\mathrm{R}_a\mathrm{R}_u$ robot in the vertical plane can be achieved using vibratory inputs [52].

Another planar underactuated manipulator that gained recently attention is the $(\mathrm{XY})_a\mathrm{R}_u$ robot, i.e., having two actuated joints of any kind (prismatic or rotational, and thus generically denoted by X and Y) and a third rotational passive joint. Most results are available for the no-gravity case. In [31], it is shown that the dynamic equations can be rewritten in terms of a so-called second-order chained form. Based on this result, a feedback control can be designed so as to stabilize the system on a (asymptotically vanishing) trajectory [55]. Set-point regulation is obtained in [36] using a deadbeat control scheme with variable period of application. In [3], a trajectory planning algorithm for this robotic system has been determined through the composition of (up to five) translational and rotational elementary motions of the last link. The key was recognizing the main role played by the motion of the *center of percussion* (CP) of the third link. Once a composed point-to-point trajectory is planned, a different controller should be used for each translational or rotational phase in order to achieve stable trajectory tracking. The CP point was also used for solving trajectory planning and control problems, both in the presence [21] and in the absence [22] of gravity, by means of dynamic feedback linearization. In this case, it is possible to determine a single smooth trajectory that joins any initial and desired robot configurations and, as a byproduct, a single (linear) feedback controller for tracking the whole trajectory. This approach has been recently extended to the class of $((n-1)\mathrm{X})_a\mathrm{R}_u$ planar manipulators (i.e., with only the last joint passive) in [23].

There are barely analysis and control results for manipulators with $n - m > 1$ passive joints. In [38], it has been shown that trajectory planning (problem P1) for a chain of n coupled planar rigid bodies subject to two Cartesian force inputs at one end can be performed, whenever each body is hinged at the CP of the previous one. As a matter of fact, the CP of the last body is a linearizing output. Interestingly, this can be seen as the dynamic counterpart of the nonholonomic N-trailer wheeled mobile robot (a kinematic system) with *zero hooking* [46]. In other terms, the above system is actually an $(\mathrm{XY})_a(n\mathrm{R})_u$ robot, for which there have been successful attempts to generalize the trajectory planning results holding for the $(\mathrm{XY})_a\mathrm{R}_u$ robot. In fact, the algorithm of [3] has been adapted to the $(\mathrm{XY})_a(n\mathrm{R})_u$ robot (without gravity) in [45]. This method, however, needs to decompose the global motion into a long sequence of translational and rotational phases for each passive link. Similarly, the dynamic feedback linerization approach presented in [21], [22] has been extended, under the same hinging hypothesis of [38], [45], to the trajectory planning and tracking of $(\mathrm{XY})_a(n\mathrm{R})_u$ robots moving in the absence or presence of gravity [14], [30].

In the following sections, we illustrate two methods that we have proposed (see again Table 1) and that have proven to be effective and generalizable, at least to some degree, to significant classes of manipulators with passive joints: exact linearization via dynamic feedback for trajectory planning/tracking (problems P1 and P2) and iterative steering for set-point regulation (problem P3).

6. Trajectory Planning and Tracking via Dynamic Feedback Linearization

The exact linearization technique via dynamic feedback [32] represents an effective solution to the P1 (trajectory planning) and P2 (trajectory tracking) problems. For this, a set of *linearizing* outputs

$$z = h(q), \qquad z \in \mathbb{R}^m \tag{9}$$

[††]The resort to dynamic feedback may become useful for linearization purposes only for systems with at least $m > 2$ control inputs.

©2002 Cyber Scientific

Table 1 Underactuated planar manipulators with passive joints: analysis, planning, and control results

Robot	Controllability	Trajectory Planning (P1) and Tracking (P2)	Set-point Regulation (P3)
$\mathrm{R}_u\mathrm{R}_a$	integrable [41]	–	–
$\mathrm{R}_a\mathrm{R}_u$	not STLC or STLCC [8]	open	periodic/Poincarè [39] iterative steering [19]
$\mathrm{R}_u\mathrm{R}_a$+gravity (Acrobot)	linearly controllable not STLC or STLCC [8]	output tracking [4]	energetic [48] iterative steering [20]
$\mathrm{R}_a\mathrm{R}_u$+gravity (Pendubot)	linearly controllable not STLC or STLCC [8]	output tracking [42]	energetic [50] passivity based [26], [44]
$\mathrm{P}_u\mathrm{R}_a$	integrable [15]	–	–
$\mathrm{P}_a\mathrm{R}_u$	not STLC or STLCC [8]	open	iterative steering [15]
$\mathrm{R}_u\mathrm{P}_a$	integrable [29]	–	–
$\mathrm{R}_a\mathrm{P}_u$	not STLC or STLCC [8]	open	iterative steering [29]
$\mathrm{R}_u\mathrm{R}_a\mathrm{R}_a$	KC, STLCC [9]	decoupling vectors [9]	open
$\mathrm{R}_a\mathrm{R}_u\mathrm{R}_a$	STLC [13] KC, STLCC [9]	decoupling vectors [9]	open
$(\mathrm{XY})_a\mathrm{R}_u$	STLC [3] KC [9], STLCC [34]	elementary maneuvers [3] DFL [21] decoupling vectors [9]	vanishing trajectory [55] variable deadbeat [36]
$(\mathrm{XY})_a\mathrm{R}_u$ +gravity	linearly controllable STLC [13], STLCC	DFL [22]	open
$[(n-1)\mathrm{X}]_a\mathrm{R}_u$	KC, STLC, STLCC	DFL [23]	open
$[(n-1)\mathrm{X}]_a\mathrm{R}_u$ +gravity	linearly controllable STLC, STLCC	DFL [23]	open
$(n_1\mathrm{R})_a\mathrm{R}_u(n_2\mathrm{R})_a$	STLC [13]	open	open
$(\mathrm{XY})_a(n\mathrm{R})_u$ (CP_i hinged)	open	DFL [30] elementary maneuvers [45]	open
$(\mathrm{XY})_a(n\mathrm{R})_u$ +gravity (CP_i hinged)	linearly controllable	DFL [30]	open

should be found, having the property that the whole state and the input of the system can be written in terms of z and its time derivatives. Then, it is possible to build a dynamic compensator of the form

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, q, \dot{q}) + \beta(\xi, q, \dot{q})\, v \\ a &= \gamma(\xi, q, \dot{q}) + \delta(\xi, q, \dot{q})\, v \end{aligned} \tag{10}$$

with state $\xi \in \mathbb{R}^\nu$ and new input $v \in \mathbb{R}^m$, such that the closed-loop system (5), (6), (9), (10) is input-state-output *linear and decoupled*, i.e., represented by m chains of integrators between v and z.

Assuming for the simplicity that the system has $m = 2$ inputs (as in the case study presented later), the linearizing algorithm proceeds qualitatively as follows. The two linearizing outputs z_1 and z_2 are differentiated repeatedly until at least an input is found in each of them. If the matrix (actually, the *decoupling matrix*) multiplying the inputs at this differentiation level is nonsingular, then static state feedback can be used in order to linearize the *input-output* behavior. However, if the sum of the orders of the output derivatives is strictly less than the dimension of the state space, full state linearization cannot be achieved. On the contrary, if the decoupling matrix is singular (thus, of rank one in the considered case), one can perform a state-dependent change of coordinates in the input space so as to let only one new input appear. On this input, a *dynamic extension* is performed, namely addition of one integrator (which becomes the first component ξ_1 of the state of the dynamic compensator (10)) driven by a new scalar input. Therefore, the derivatives of the two outputs at this level will not depend anymore on the newly defined inputs and we can proceed with their differentiation. This process is iterated as many times as needed so as to arrive at a final nonsingular (at least locally) decoupling matrix for the extended system. While doing this, the whole state ξ of the dynamic compensator (10) is built iteratively. At the end, the sum of the output derivative orders equals the dimension of the extended state space (robot + dynamic compensator) and full *input-state-output* linearization can be obtained by inversion.

Once the above construction has been carried out, the trajectory planning problem (P1) can be formulated and easily solved as a simple interpolation problem on the equivalent linear system. An interesting byproduct of this approach is that control techniques for linear single-input/single-output systems allow to exponentially stabilize the error dynamics, thus providing a straightforward solution also to the problem of tracking the planned trajectory (P2).

It should be mentioned that singularities may arise in the

resulting controller, essentially because dynamic feedback linearization is based on model inversion. Such singularities must be carefully kept into account and avoided when planning the trajectory via interpolation. This can be usually achieved by appropriately choosing the initialization of the dynamic compensator state ξ—actually an additional degree of freedom available in the design.

In [21] and [23], we have shown that planar three-link (or n-link) robots with passive rotational third (or last) joint can be exactly linearized via feedback, with or without gravity. The linearizing output is the Cartesian position of the center of percussion (CP) of the third (last) link. In the following, we show that the same procedure can be also applied in the presence of a double degree of underactuation, provided that a special mechanical condition is satisfied.

6.1 Example: An underactuated XYRR robot

An XYRR planar robot is a mechanism where the two distal joints are rotational, while the two proximal degrees-of-freedom may be any combination of prismatic and rotational joints. Assume that only the first two joints are actuated (thus, the robot is labeled $(\mathrm{XY})_a(\mathrm{RR})_u$), that the fourth link is hinged exactly at the center of percussion (CP_3) of the third link (see **Fig. 2**), and that the robot moves in the horizontal plane. Denote by l_i, d_i, and k_i ($i = 3, 4$) respectively the length of the i-th link, the distance between the i-th joint axis and the i-th link center of mass, and the distance between the i-th joint axis and the i-th link center of percussion CP_i. Under the special hinging condition of [38], [45], we have

$$k_3 = \frac{I_3 + m_3 d_3^2}{m_3 d_3} = l_3, \qquad k_4 = \frac{I_4 + m_4 d_4^2}{m_4 d_4}$$

where m_i and I_i are, respectively, the mass and the centroidal moment of inertia of the i-th link.

Choose the generalized coordinates as $q = (q_a, q_u) = (x, y, q_3, q_4)$, where (x, y) are the Cartesian coordinates of the base of the third link while q_3 and q_4 are the (absolute) orientation of the last two links w.r.t. the x-axis. After partial feedback linearization (see Eq. (4)), the robot dynamic equations take the form (5), (6):

$$\begin{aligned} \ddot{x} &= a_x \\ \ddot{y} &= a_y \\ l_3\ddot{q}_3 + \lambda_{34} c_{34} \ddot{q}_4 &= s_3 a_x - c_3 a_y - \lambda_{34} s_{34} \dot{q}_4^2 \\ l_3 c_{34} \ddot{q}_3 + k_4 \ddot{q}_4 &= s_4 a_x - c_4 a_y + l_3 s_{34} \dot{q}_3^2 \end{aligned} \tag{11}$$

where we have set for compactness $s_i = \sin q_i$, $c_i = \cos q_i$, $s_{ij} = \sin(q_i - q_j)$, $c_{ij} = \cos(q_i - q_j)$ $(i, j = 3, 4)$, and $\lambda_{34} = m_4 l_3 d_4 / (m_3 d_3 + m_4 l_3)$. Note that the last two equations in (11) have been conveniently scaled by constant factors (deforming the symmetry of matrix $B_u u(q)$). The inputs to the mechanism are the accelerations a_x and a_y.

The linearized outputs for system (11) are the Cartesian coordinates of CP_4, the center of percussion of the fourth link (see Fig. 2):

$$\begin{aligned} z_1 &= x + l_3 c_3 + k_4 c_4 \\ z_2 &= y + l_3 s_3 + k_4 s_4. \end{aligned} \tag{12}$$

Following dynamic linearization algorithm (see [30] for details), each of the two outputs in (12) will be differentiated six times. Starting from the second level of differentiation (acceleration level), we need to perform a dynamic extension at each step, being the intermediate decoupling matrices singular, which results in the total addition of four integrators. At the end, we obtain a linearized dynamic compensator of dimension $\nu = 4$, with state equations

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 + \dot{q}_4^2\, \xi_1 \\ \dot{\xi}_3 &= \xi_4 + 2\dot{q}_4^2\, \xi_2 - \mu\, t_{34}\, \dot{q}_4\, \xi_1 \\ \dot{\xi}_4 &= u_1 + \phi\, \dot{q}_4 - \psi(\dot{q}_3 - \dot{q}_4)\dot{q}_4 \end{aligned} \tag{13}$$

and output equation

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = R(q_3) \begin{bmatrix} \frac{1}{c_{34}} \left(\frac{k_4 - \lambda_{34}\, c_{34}}{k_4 - \lambda_{34}}\, \xi_1 + k_4\, \dot{q}_4^2 \right) + l_3\, \dot{q}_3^2 \\ u_2 \end{bmatrix} \tag{14}$$

where $R(q_3)$ is the planar rotation matrix defined by the angle q_3. In Eqs. (13) and (14), we have set

$$\begin{aligned} t_{34} &= s_{34}/c_{34} \\ \mu &= \frac{\xi_1}{k_4 - \lambda_{34}} + \dot{q}_4^2 \\ \psi &= \mu \xi_1 / c_{34}^2 \\ \phi &= 2\dot{q}_4^3\, \xi_1 - 3 t_{34}\, \mu\, \xi_2 + 3\dot{q}_4\, \xi_3 - t_{34}\, \xi_1\, \dot{\mu}. \end{aligned}$$

Finally, the auxiliary inputs u_1 and u_2 are obtained by inverting the expressions of $(d^6 z_1/dt^6, d^6 z_2/dt^6)$ in terms of the new input vector (v_1, v_2):

$$\begin{aligned} u_1 &= c_4 v_1 + s_4 v_2 \\ u_2 &= \frac{l_3}{\psi} \left[c_4 v_2 - s_4 v_1 - \dot{q}_4\, \xi_4 + (\dot{q}_3 - \dot{q}_4)\dot{\psi} - \dot{\phi} + \psi \delta \right] \end{aligned} \tag{15}$$

with

$$\delta = t_{34} \left[\frac{l_3 + \lambda_{34}\, c_{34}}{l_3 (k_4 - \lambda_{34})}\, \xi_1 + \dot{q}_4^2 \right].$$

Under the action of the dynamic compensator (13)–(15), the system is input-output decoupled and completely linearized in the proper coordinates, i.e., equivalent to two chains of six integrators from input to output:

$$\frac{d^6 z_1}{dt^6} = v_1, \qquad \frac{d^6 z_2}{dt^6} = v_2. \tag{16}$$

Note that the robot has $n = 8$ states $(q, \dot{q})$, while the compensator has $\nu = 4$ states ξ. Thus, the total number of output derivatives equals the dimension of the extended state space ($n + \nu = 12$). Furthermore, the linearizing algorithm defines in its intermediate steps also a transformation map between $(z_1, z_2, \dot{z}_1, \dot{z}_2, \ldots, d^5 z_1/dt^5, d^5 z_2/dt^5)$ and $(q, \dot{q}, \xi)$.

Planning a feasible trajectory on the equivalent representation (16) can be formulated as two separate smooth interpolation problems for the two outputs z_1 and z_2. For example, one could use two polynomial functions $z_{d,1}(t)$

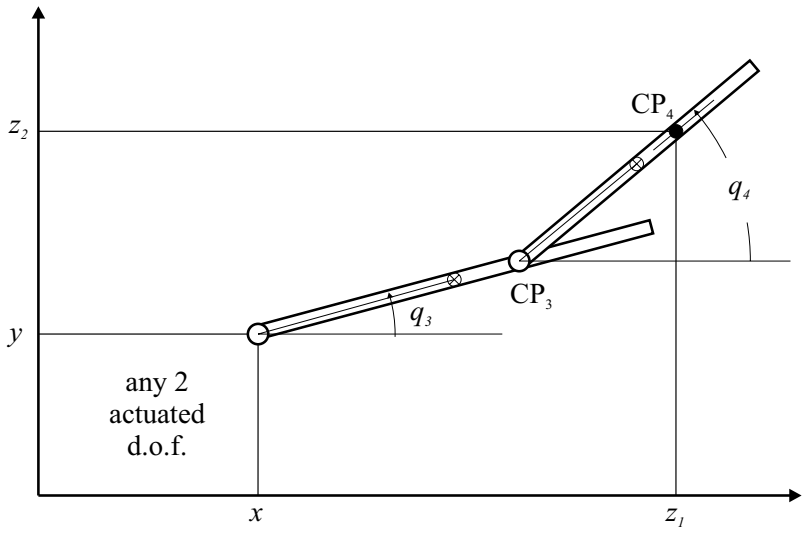

Fig. 2 An underactuated XYRR robot

and $z_{d,2}(t)$, for $t \in [0, T]$, to join the initial z^0 (corresponding to the start configuration q^0 at $t = 0$) with the final z^d (corresponding to the final configuration q^d at $t = T$), imposing appropriate boundary conditions at $t = 0$ and $t = T$ on the derivatives of z up to the fifth order. These will depend on both $\dot{q}(0)$ and $\dot{q}(T)$ (typically, zero for rest-to-rest maneuvers), as well as on the initial and final selected values for the compensator state ξ through the transformation map.

However, it should be considered that the above linearization procedure is valid if and only if the following regularity conditions are satisfied

$$c_{34} \neq 0 \qquad \text{and} \qquad \psi \neq 0 \tag{17}$$

throughout the motion, since these quantities appear in the denominator of the dynamic compensator expressions. The conditions (17) can be easily given as an interesting physical interpretation. In particular, $c_{34} \neq 0$ means that the fourth link should never become orthogonal to the third, while $\psi \neq 0$ holds as long as the acceleration ξ_1 of the center of percussion CP_4 along the fourth link axis does not vanish during the motion. Besides, being $\xi_1^2 = \ddot{z}_1^2 + \ddot{z}_2^2$, such a regularity condition can be checked directly from the linearized outputs trajectory, without actually computing ξ_1. In any case, one way to avoid the singularity during the motion is to reset the component ξ_1 of the dynamic compensator state whenever it approaches zero.

For illustration, the trajectory planning technique outlined above has been applied to generate a feasible trajectory from $q^0 = (1, 1, 0, \pi/8)$ to $q^d = (1, 2, 0, \pi/4)$ [m, m, rad, rad], with $T = 10$ s. The underactuated manipulator has $l_3 = k_3 = 1$, $l_4 = 1$, $k_4 = 2/3$ and $\lambda_{34} = 1/3$ [m]. The resulting trajectory for the center of percussion CP_4 of the fourth link is shown in **Fig. 3**, while the corresponding cartesian motion of the last two links is depicted in **Fig. 4** and **Fig. 5** (stroboscopic view)† . The last two links undergo a counterclockwise rotation of $360°$. Assuming that also the first two joints are rotational, the motion of the whole $(RR)_a(RR)_u$ manipulator appears as in **Fig. 6**.

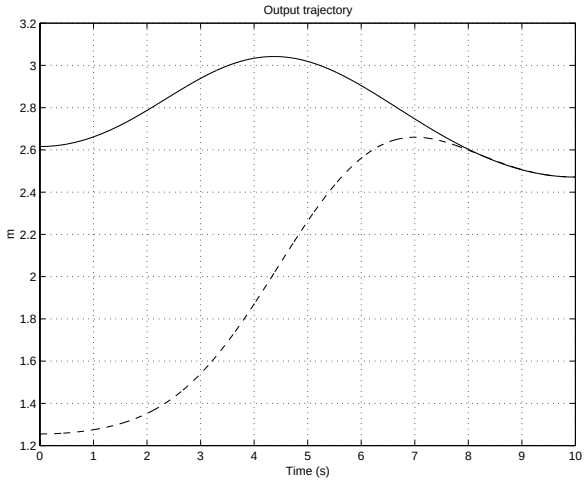

Fig. 3 Trajectory planning: linearizing outputs z_1 (—), z_2 (- -)

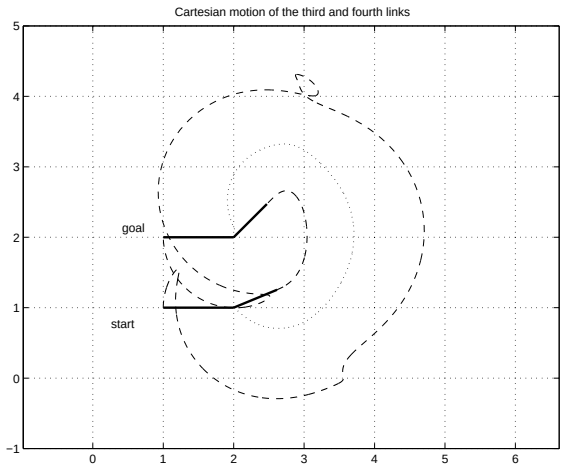

Fig. 4 Trajectory planning: Cartesian motion of the last two links

As already mentioned, this dynamic linearization approach also yields a straightforward solution to the trajectory tracking problem. The simple linear control laws (two

† In order to gain clarity, the last link is represented only until its center of percussion (k_4) and not with its full length (l_4).

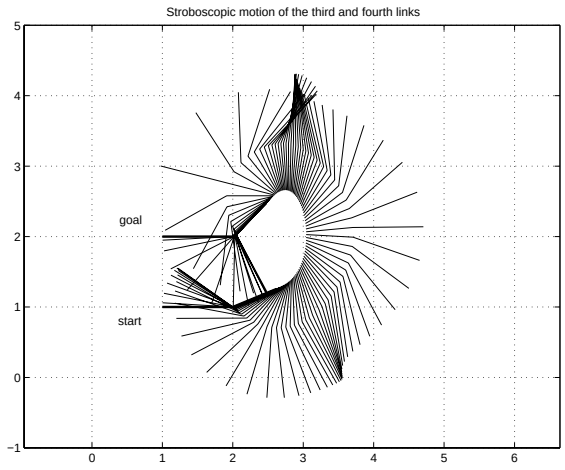

Fig. 5 Trajectory planning: stroboscopic motion of the last two links

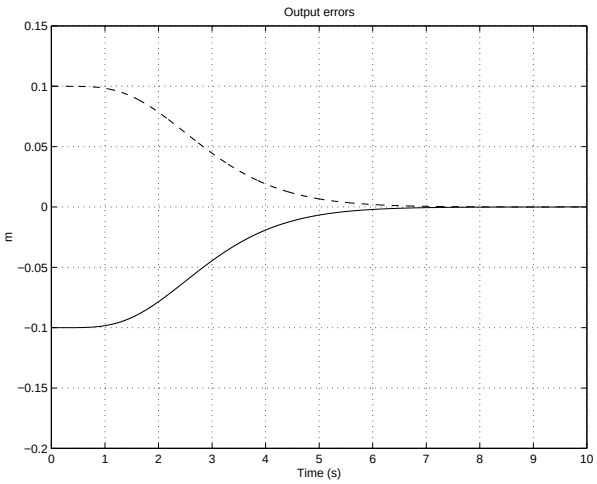

Fig. 7 Trajectory tracking: output errors e_1 (—), e_2 (- -)

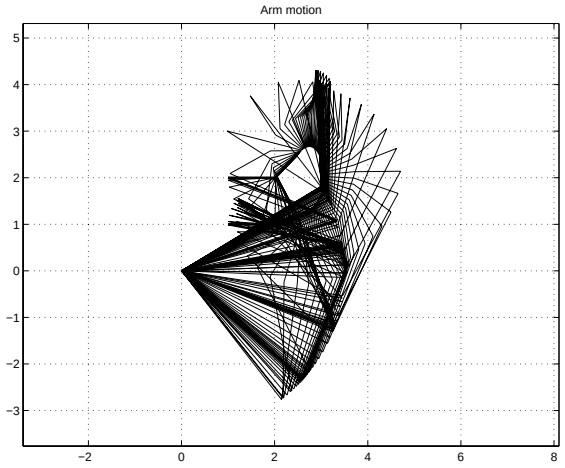

Fig. 6 Trajectory planning: stroboscopic motion of the $(\mathrm{RR})_a(\mathrm{RR})_u$ manipulator

independent generalized PD^5 controller) designed on the linear dynamics (16)

$$v_i = \frac{d^{\,6} z_{d,i}}{dt^{\,6}} + \sum_{j=0}^{5} f_{j,i} \left(\frac{d^{\,j} z_{d,i}}{dt^{\,j}} - \frac{d^{\,j} z_i}{dt^{\,j}} \right), \qquad i = 1, 2 \tag{18}$$

where $s^6 + f_{5,i}s^5 + f_{4,i}s^4 + f_{3,i}s^3 + f_{2,i}s^2 + f_{1,i}s + f_{0,i}$, $i = 1, 2$, are Hurwitz polynomials, will drive the tracking error exponentially to zero. **Figure 7** shows this convergence for the case of the third link base initially placed at a Cartesian position corresponding to an off-trajectory start (i.e., with initial output errors $e_i = z_{d,i} - z_i \neq 0$, for $i = 1, 2$).

7. Set-Point Regulation via Iterative Steering

The iterative steering technique for solving problem P3 is based on the general stabilization framework proposed in [35]. For systems in the form (5) and (6), it requires in principle the application of two control phases [19]. In the first, called *alignment*, the active joints q_a are brought to the desired equilibrium $(q_a^d, 0)$ using, e.g., a terminal controller or a fast PD controller. At the end of this phase, the passive joints q_u will be drifted to some position and velocity, and must be driven to the desired state $(q_u^d, 0)$. This is achieved in the *contraction* phase by the iterative application of an appropriate steering control (an open-loop finite-time command), whose task is to decrease (of possibly varying duration T) the passive joint state error $(q_u^d - q_u(kT), -\dot{q}_u(kT))$ at each iteration while guaranteeing $q_a(kT) = q_a^d$ and $\dot{q}_a(kT) = 0$, with $k = 1, 2, \ldots$. At the end of each iteration, the state of the system is measured and the parameters of the steering controller are updated accordingly, resulting in a sampled feedback action. The general results of [35] indicate how to choose the open-loop controller so as to ensure the asymptotic stability of the desired equilibrium, with exponential rate of convergence: essentially, the open-loop control law must be a Hölder-continuous function of the desired reconfiguration (see [35]). Moreover, a certain degree of robustness is obtained: ultimate boundedness is guaranteed in the presence of persistent perturbations, whereas small non-persistent perturbations are rejected.

One difficulty in applying the conceptual approach outlined above to system (5), (6) lies in the computation of a steering control that enforces a suitable contraction; this is mainly due to the presence of a drift term in the dynamic equations. As proposed in [19], a useful tool is the *nilpotent approximation* of the system, which is by construction polynomial and strictly triangular (and hence forward-integrable). For an underactuated manipulator in the absence of gravity, the nilpotent approximation preserves as much as possible the controllability properties of the original dynamics (this does not happen with the linear approximation). Based on the approximate system, a contracting steering control can then be computed which satisfies the Hölder-continuity conditions. However, one may find that this controller works only from certain *contraction regions* of the passive joint state space. In this case, it may be necessary to perform an intermediate phase (called *transition*) between alignment and contraction, so as to bring $(q_u, \dot{q}_u)$ from the value attained at the end of the first phase to a state belonging to one of the contraction regions. The design of the transition phase depends on the specific mechanism under consideration.

In [19], we presented a complete solution to the P3 problem for a 2R planar robot with passive second joint. Here-

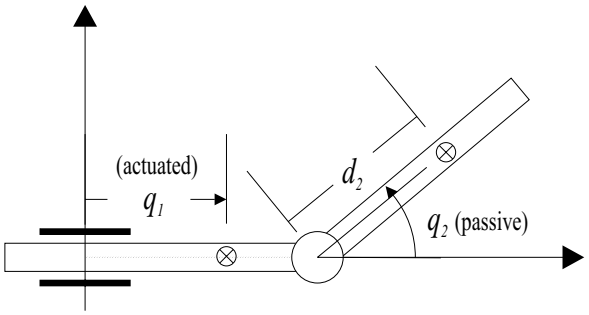

Fig. 8 An underactuated PR robot

after, we sketch the application of the same technique to an underactuated PR robot. This system is not STLC, according to the results in [28].

7.1 Example: An underactuated PR robot

A planar PR robot with a passive second joint† is shown in **Fig. 8**. After the partial feedback linearizing law (4), the system dynamics takes the form:

$$\ddot{q}_1 = a$$
$$\ddot{q}_2 = \frac{1}{k_2}\,\sin q_2\, a$$

where q_1, q_2 are the generalized coordinates and we have set $k_2 = (I_2 + m_2 d_2^2)/m_2 d_2$, with m_2, I_2 respectively the mass and the centroidal moment of inertia of the second link, and d_2 the distance between the second joint axis and the center of mass of the second link.

For the alignment phase, we can use a simple PD controller

$$a = k_p(q_1^d - q_1) - k_d\dot{q}_1, \qquad k_p, k_d > 0 \tag{19}$$

to bring the first joint to the desired position. Denoting by $(q_{2k}, \dot{q}_{2k})$, the passive joint state at the beginning of the k-th iteration ($k = 1, 2, \ldots$), the contraction phase is obtained by the iterated application of the polynomial steering control for a period T_k

$$a(t) = \frac{A_k}{T_k^2}\left(42\lambda^5 - 105\lambda^4 + 90\lambda^3 - 30\lambda^2 + 3\lambda\right) \tag{20}$$

where $\lambda = t/T_k$ and

$$T_k = (1-\eta_1)\,\frac{q_{2d} - q_{2k}}{\dot{q}_{2k}} \tag{21}$$

$$A_k = \sqrt{\frac{T_k(1-\eta_2)\dot{q}_{2k}}{k_2^2\beta\,\sin 2q_{2k}}} \tag{22}$$

being $\beta = 3/80080$ and $1-\eta_1$, $1-\eta_2 \in (0,1)$ the chosen contraction rates for the passive joint position and velocity errors, respectively.

The following arguments are used in the control design.

- The steering control (20)–(22) has been designed (see [15] for details) on the basis of the following nilpotent approximation of the system computed at $(q_k, \dot{q}_k)$:

$$\dot{\zeta}_1 = 1$$
$$\dot{\zeta}_2 = a$$
$$\dot{\zeta}_3 = -\zeta_2$$
$$\dot{\zeta}_4 = -\frac{k_2\dot{q}_{2k}^2}{4\cos q_{2k}}\,\zeta_1^2 - \frac{1}{2}\,\zeta_3.$$

  This local approximation is expressed in terms of a new state ζ, related to the original state $(q, \dot{q})$ through a change of coordinates based on the structure of the system Lie Algebra.
- The parameter choice (21), (22) meets the requirements of the iterative steering paradigm (essentially, Hölder-continuity of the steering control with respect to the desired reconfiguration), provided that $\eta_1 < \eta_2$; in fact, this will guarantee that the passive joint position error converges to zero faster than the velocity error, so that T_k is always finite.
- Other contraction conditions come from the requirements that T_k should be positive and A_k real and finite. In particular, one finds that contraction is guaranteed if the state belongs to one of the following regions:

$$\begin{cases} \dot{q}_2(0) > 0 \\ q_{2d} > q_2(0) \\ q_2(0) \in \text{I or III} \end{cases} \quad \text{or} \quad \begin{cases} \dot{q}_2(0) < 0 \\ q_{2d} < q_2(0) \\ q_2(0) \in \text{II or IV}, \end{cases}$$

  where roman numbers define the four (open) quadrants of the 2π angle. If at the end of the alignment phase the robot is not in a contraction region, a *transition* phase is required (see [15] for the detail on transition maneuvers). However, once the proper contraction region is reached, it is never left during the iterated application of the steering control.

For illustration, the set-point regulation technique described above has been applied to an underactuated $\mathrm{P}_a\mathrm{R}_u$ robot having $m_1 = m_2 = 1$ [kg], $I_2 = 1$ [kg m^2], $d_2 = 0.5$ [m], and thus $k_2 = 2.5$ [m]. The desired configuration to be stabilized is $q^d = (0, \pi/4)$, starting from an initial configuration $q^0 = (1, -\pi/4)$ [m, rad]. **Figures 9** and **10** show the joint evolutions during the alignment, transition, and contraction phases. Note how the second joint velocity is kept constant at the end of the alignment phase (by setting $a = 0$) until q_2 enters the appropriate contraction region. The acceleration command a and the actual robot input τ_a (a scalar force) on the active prismatic joint are reported in **Fig. 11**.

In order to test the robustness of the stabilizing strategy, we have applied the same previous control law in the presence of a model perturbation due to viscous friction at both joints, with friction coefficients $b_1 = b_2 = 0.02$ [N·s/m, Nm·s/rad]. The results are shown in **Figs. 12–14**: the alignment phase is still achieved by the PD law (19) in about the same time, while the longer transition phase ends at $t = 18.3$ [s]. Sufficient error contraction is preserved but, after each contraction phase, a re-alignment of the first joint is needed due to the incomplete cancellation

†Based on the results in [41], the same manipulator with a passive *first* joint is integrable, in the sense that the second-order differential constraint (2) turns out to be holonomic. Therefore, such a mechanism would not be controllable.

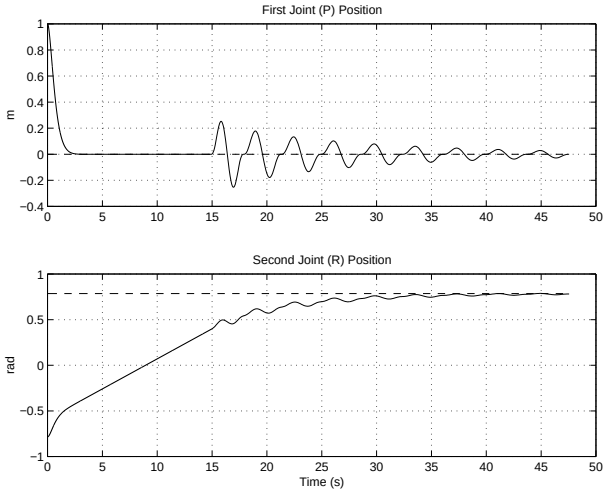

Fig. 9 Set-point regulation: joint positions (–) and their reference (- -)

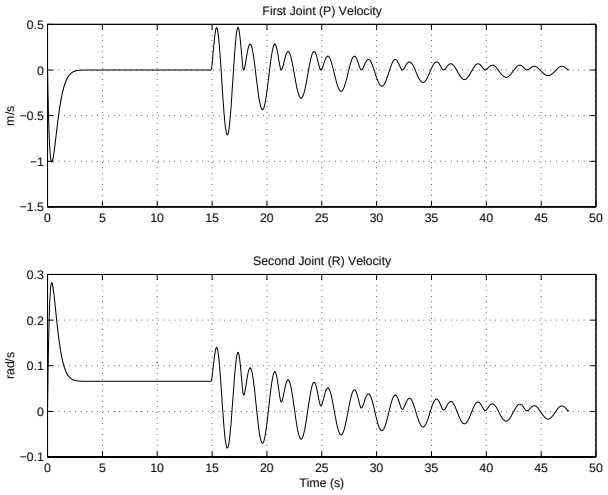

Fig. 10 Set-point regulation: joint velocities

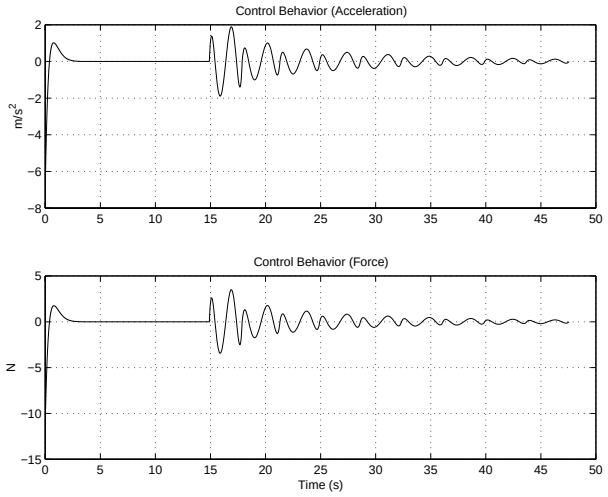

Fig. 11 Set-point regulation: first joint acceleration a and force τ_a

of the actuated joint dynamics achieved by the partial feedback linearizing law. In any case, this modification is still consistent with the iterative steering paradigm. Note finally that the control effort is slightly reduced in the presence of friction (compare Figs. 11 and 14).

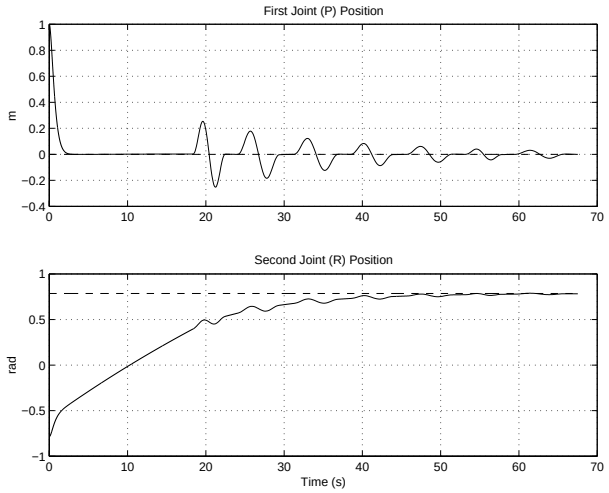

Fig. 12 Set-point regulation: joint positions (–) and their reference (- -) in the presence of viscous friction

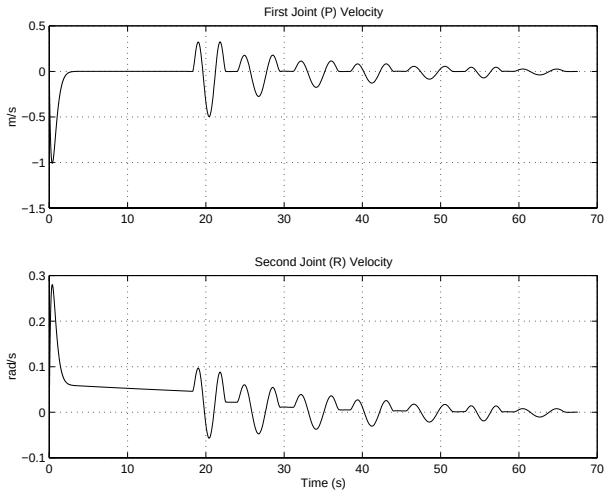

Fig. 13 Set-point regulation: joint velocities in the presence of viscous friction

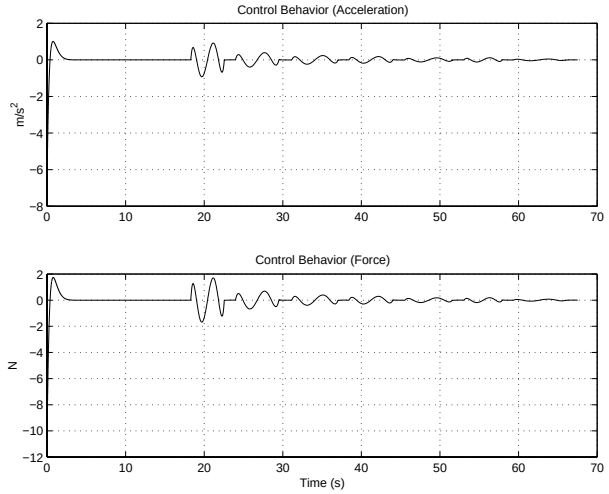

Fig. 14 Set-point regulation: first joint acceleration a and force τ_a in the presence of viscous friction

8. Conclusions

Underactuated manipulators are attracting considerable scientific interest in the robotics and control communities.

Unfortunately, they suffer from the lack of general results on feasible trajectory planning and on set-point regulation or trajectory tracking control design. Several interesting 'case studies' have been solved so far, but only with ad-hoc solutions.

In this paper, we have discussed some relevant structural control properties of underactuated mechanical systems, such as integrability of the second-order differential constraints, linear controllability, small-time local controllability, or kinematic controllability, as well as the relations among them.

This analysis helps in assessing more carefully the difficulties to be encountered when addressing motion problems for specific instances within the broad class of underactuated mechanical systems. In particular, the class of manipulators with passive joints in the absence of gravity (or any other potential terms) was found to be considerably more difficult to control. Also, the availability of a single actuation command imposes severe constraints to the definition of feasible trajectories for achieving a desired reconfiguration.

From the point of view of trajectory planning methods and feedback control design, we have then illustrated the use of two advanced techniques: dynamic feedback linearization and iterative steering. The former allows in particular to command a planar manipulator with double degree of underactuation along a rest-to-rest trajectory, while the latter has been used to stabilize to a desired configuration a simple planar underactuated manipulator having only one actuator. Supported by simulation results, we have also pointed out the ability of avoiding control singularities inherent to the dynamic linearization (flatness) approach, as well as some robustness to dynamic perturbations for the iterative steering method.

The integrated mechanical and control design of on-purpose underactuated robotic systems that use a reduced number of actuators has still to face rather difficult theoretical problems. Among the open issues in control research for underactuated second-order mechanical systems we mention: the need for a global theory capable of an automatic handling of dynamic singularities, technical developments in the nilpotent approximation of systems with drift, a deeper understanding of cases with degree of underactuation $n - m \geq 2$, and the consideration of non-planar structures.

Whenever robot underactuation is instead the result of an actuator failure, in order to preserve robot operation one should be able to detect and isolate the occurred fault online and without extra measurements and then immediately switch in a safe and stable way from a conventional controller to one designed assuming the absence of the faulted actuator. Preliminary results on such a fault tolerant control architecture are promising (see, e.g., [17], [54]).

Acknowledgments

This work was partially supported by MURST within the *MISTRAL* project.

References

[1] H. Arai and S. Tachi, "Position control of a manipulator with passive joints using dynamic coupling," *IEEE Trans. on Robotics and Automation*, vol. 7, no. 4, pp. 528–534, 1991.

[2] H. Arai, K. Tanie, and S. Tachi, "Dynamic control of a manipulator with passive joints in operational space," *IEEE Trans. on Robotics and Automation*, vol. 9, no. 1, pp. 85–93, 1993.

[3] H. Arai, K. Tanie, and N. Shiroma, "Nonholonomic control of a three-dof planar underactuated manipulator," *IEEE Trans. on Robotics and Automation*, vol. 14, no. 5, pp. 681–695, 1998.

[4] M. D. Berkemeier and R. S. Fearing, "Tracking fast inverted trajectories of the underactuated Acrobot," *IEEE Trans. on Robotics and Automation*, vol. 15, no. 4, pp. 740–750, 1999.

[5] M. Bergerman and Y. Xu, "Optimal control sequence for underactuated manipulators," in *Proc. of 1996 IEEE Int. Conf. on Robotics and Automation*, Minneapolis, MN, 1996, pp. 3714–3719.

[6] R. M. Bianchini and G. Stefani, "Controllability along a trajectory: A variational approach," *SIAM J. on Control and Optimization*, vol. 31, no. 4, pp. 900–927, 1993.

[7] R. W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*, R. W. Brockett, R. S. Millman, and H. J. Sussmann, Eds. Boston, MA: Birkhäuser, 1983, pp. 181–191.

[8] F. Bullo, A. D. Lewis, and K. M. Lynch, "Controllable kinematic reductions for mechanical systems: Concepts, computational tools, and examples," in *Proc. of 2002 Int. Symp. on Mathematical Theory of Networks and Systems*, Notre Dame, IL, 2002.

[9] F. Bullo and K. M. Lynch, "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems," *IEEE Trans. on Robotics and Automation*, vol. 17, no. 4, pp. 402–412, 2001.

[10] B. Charlet, J. Lévine, and R. Marino, "On dynamic feedback linearization," *Systems and Control Lett.*, vol. 13, pp. 143–152, 1989.

[11] J.-M. Coron, "Links between local controllability and local continuous stabilization," in *Proc. of 2nd IFAC Symp. on Nonlinear Control Systems*, Bordeaux, F, 1992, pp. 477–482.

[12] A. De Luca and G. Di Giovanni, "Rest-to-rest motion of a two-link robot with a flexible forearm," in *Proc. of 2001 IEEE/ASME Int. Conf. on Advanced Intelligent Mechatronics*, Como, I, 2001, pp. 929–935.

[13] A. De Luca and S. Iannitti, "A simple STLC test for mechanical systems underactuated by one control," in *Proc. of 2002 IEEE Int. Conf. on Robotics and Automation*, Washington, DC, 2002, pp. 1735–1740.

[14] A. De Luca and S. Iannitti, "Smooth motion planning for XY$n\overline{\text{R}}$ planar underactuated robots," in *Proc. of IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, Lausanne, CH, 2002, pp. 1651–1656.

[15] A. De Luca, S. Iannitti, and G. Oriolo, "Stabilization of a PR planar underactuated robot," in *Proc. of 2001 IEEE Int. Conf. on Robotics and Automation*, Seoul, KR, 2001, pp. 2090–2095.

[16] A. De Luca and P. Lucibello, "A general algorithm for dynamic feedback linearization of robots with elastic joints," in *Proc. of 1998 IEEE Int. Conf. on Robotics and Automation*, Leuven, B, 1998, pp. 504–510.

[17] A. De Luca and R. Mattone, "Actuator fault detection and isolation using generalized momenta," in *Proc. of 2003 IEEE Int. Conf. on Robotics and Automation*, Taipei, 2003, to be published.

[18] A. De Luca, R. Mattone, and G. Oriolo, "Steering a class of redundant mechanisms through end-effector generalized forces," *IEEE Trans. on Robotics and Automation*, vol. 14, no. 2, pp. 329–335, 1998.

[19] A. De Luca, R. Mattone, and G. Oriolo, "Stabilization of an underactuated planar 2R manipulator," *Int. J. Robust and Nonlinear Control*, vol. 10, no. 4, pp. 181–198, 2000.

[20] A. De Luca and G. Oriolo, "Stabilization of the Acrobot via iterative state steering," in *Proc. of 1998 IEEE Int. Conf. on Robotics and Automation*, Leuven, B, 1998, pp. 3581–3587.

[21] A. De Luca and G. Oriolo, "Motion planning and trajectory control of an underactuated three-link robot via dynamic feedback linearization," in *Proc. of 2000 IEEE Int. Conf. on Robotics and Automation*, San Francisco, CA, 2000, pp. 2789–2795.

[22] A. De Luca and G. Oriolo, "Motion planning under gravity for underactuated three-link robots," in *Proc. of IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, Takamatsu, J, 2000, pp. 139–144.

[23] A. De Luca and G. Oriolo, "Trajectory planning and control for planar robots with passive last joint," *Int. J. of Robotics Research*, vol. 21, no. 5-6, pp. 575–590, 2002.

[24] A. De Luca and B. Siciliano, "Regulation of flexible arms under gravity," *IEEE Trans. on Robotics and Automation*, vol. 9, no. 4, pp. 463–467, 1993.

[25] A. De Luca and B. Siciliano, "Inversion-based nonlinear control of robot arms with flexible links," *AIAA J. of Guidance, Control, and*

Dynamics, vol. 16, no. 6, pp. 1169–1176, 1993.
[26] I. Fantoni, R. Lozano, and M. W. Spong, "Energy based control of the Pendubot," *IEEE Trans. on Automatic Control*, vol. 45, no. 4, pp. 725–729, 2000.
[27] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon, "Flatness and defect of nonlinear systems: Introductory theory and examples," *Int. J. of Control*, vol. 61, no. 6, pp. 1327–1361, 1995.
[28] R. M. Hirschorn and A. D. Lewis, "Geometric first-order controllability conditions for affine connection control systems," in *Proc. of 40th IEEE Conf. on Decision and Control*, Orlando, FL, 2001, pp. 4216–4211.
[29] S. Iannitti, "Motion planning and control of a class of underactuated robots," Ph.D. thesis, Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", Dec. 2001.
[30] S. Iannitti and A. De Luca, "Dynamic feedback control of XY$n\overline{\text{R}}$ planar robots with n rotational passive joints," *J. of Robotic Systems*, to be published, 2003.
[31] J. Imura, K. Kobayashi and T. Yoshikawa, "Nonholonomic control of a 3 link planar manipulator with a free joint," in *Proc. of 35th IEEE Conf. on Decision and Control*, Kobe, J, 1996, pp. 1435–1436.
[32] A. Isidori, *Nonlinear Control Systems*,3rd ed. Berlin, Germany: Springer Verlag, 1995.
[33] A. D. Lewis, "Local configuration controllability for a class of mechanical systems with a single input," in *Proc. of 4th European Control Conf.*, Brussels, B, 1997.
[34] A. D. Lewis and R. M. Murray, "Configuration controllability of simple mechanical control systems," *SIAM J. on Control and Optimization*, vol. 35, no. 3, pp. 766–790, 1997.
[35] P. Lucibello and G. Oriolo, "Robust stabilization via iterative state steering with an application to chained-form systems," *Automatica*, vol. 37, no. 1, pp. 71–79, 2001.
[36] T. Mita and T. K. Nam, "Control of underactuated manipulators using variable period deadbeat control," in *Proc. of 2001 IEEE Int. Conf. on Robotics and Automation*, Seoul, KR, 2001, pp. 2735–2740.
[37] P. Mullhaupt, B. Srinivasan, and D. Bonvin, "Analysis of exclusively kinetic two-link underactuated mechanical systems," *Automatica*, vol. 38, no. 9, pp. 1565–1573, 2002.
[38] R. M. Murray, M. Rathinam, and W. Sluis, "Differential flatness of mechanical control systems: A catalog of prototype systems," in *Proc. of 1995 ASME Int. Mechanical Engineering Congress and Exposition*, San Francisco, CA, 1995.
[39] Y. Nakamura, T. Suzuki, and M. Koinuma, "Nonlinear behavior and control of nonholonomic free-joint manipulator," *IEEE Trans. on Robotics and Automation*, vol. 13, no. 6, pp. 853–862, 1997.
[40] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. Berlin, Germany: Springer Verlag, 1990.
[41] G. Oriolo and Y. Nakamura, "Control of mechanical systems with second-order nonholonomic constraints: Underactuated manipulators," in *Proc. of 30th IEEE Conf. on Decision and Control*, Brighton, UK, 1991, pp. 2398–2403.
[42] L. E. Ramos, B. Castillo-Toledo, and J. Alvares, "Nonlinear regulation of an underactuated system," in *Proc. of 1997 IEEE Int. Conf. on Robotics and Automation*, Albuquerque, NM, 1997, pp. 3288–3293.
[43] M. Rathinam and R. M. Murray, "Configuration flatness of Lagrangian systems underactuated by one control," *SIAM J. of Control and Optimization*, vol. 36, no. 1, p. 164–179, 1998.
[44] A. S. Shiriaev, O. Kolesnichenko, "On passivity based control for partial stabilization of underactuated systems," in *Proc. of 39th IEEE Conf. on Decision and Control*, Sydney, AUS, 2000, pp. 2174–2179.
[45] N. Shiroma, H. Arai, and K. Tanie, "Nonholonomic motion planning for coupled planar rigid bodies with passive revolute joints," *Int. J. of Robotics Research*, vol. 21, no. 5-6, pp. 563–574, 2002.
[46] O. J. Sørdalen, "Conversion of the kinematics of a car with n trailers into a chained form," in *Proc. of 1993 IEEE Int. Conf. on Robotics and Automation*, Atlanta, GA, vol. 1, 1993, pp. 382–387.
[47] M. W. Spong, "Modeling and control of flexible joint robots," *ASME J. of Dynamic Systems, Measurement, and Control*, vol. 109, pp. 310–319, 1987.
[48] M. W. Spong, "The swing up control problem for the Acrobot," *IEEE Control Systems*, vol. 15, no. 1, pp. 49–55, 1995.
[49] M. W. Spong, "Underactuated mechanical systems," in *Control Problems in Robotics and Automation*, LNCIS, B. Siciliano and K. P. Valavanis, Eds. Berlin, Germany: Springer Verlag, 1998, vol. 230, pp. 135–150.
[50] M. W. Spong and D. Block, "The Pendubot: A mechatronic system for control research and education," in *Proc. of 34th IEEE Conf. on Decision and Control*, New Orleans, LA, 1995, pp. 555–557.
[51] H. J. Sussmann, "A general theorem on local controllability," *SIAM J. on Control and Optimization*, vol. 25, no. 1, pp. 158–194, 1987.
[52] T. Suzuki, "Anti-gravity control of free-joint manipulators by vibratory input," in *2002 Proc. of IEEE Int. Conf. on Robotics and Automation*, Washington, DC, 2002, pp. 1315–1320.
[53] P. Tomei, "A simple PD controller for robots with elastic joints," *IEEE Trans. on Automatic Control*, vol. 36, no. 10, pp. 1208–1213, 1991.
[54] M. L. Visinsky, J. R. Cavallaro, and I. D. Walker, "A dynamic fault tolerance framework for remote robots," *IEEE Trans. on Robotics and Automation*, vol. 11, no. 4, pp. 477–490, 1995.
[55] T. Yoshikawa, K. Kobayashi, and T. Watanabe, "Design of a desirable trajectory and convergent control for 3-d.o.f. manipulator with a nonholonomic constraint," in *Proc. of 2000 IEEE Int. Conf. on Robotics and Automation*, San Francisco, CA, 2000, pp. 1805–1810.

Biographies

Alessandro De Luca received the Ph.D. from the University of Rome "La Sapienza" in 1987. Since 2000, he is *Full Professor* in the Department of Computer and Systems Science, teaching Industrial Robotics and Automatic Control. He has published over 100 papers in journals, books, and conferences on modeling, planning, and control of different robotic systems, including nonholonomic wheeled mobile robots, manipulators with elastic joints or flexible links, kinematically redundant arms, underactuated robots. Dr. De Luca has been *Associate Editor* from 1994 to 1998 and since then is *Editor* of the *IEEE Transactions on Robotics and Automation*. He is a Member of IEEE.

Stefano Iannitti received the Ph.D. in Systems Engineering in 2001 from the University of Rome "La Sapienza." During 2002, he has been a post-doc at the Department of Mechanical Engineering, Northwestern University (Evanston, USA). He joined the Italian Space Agency (ASI) in March 2003. His research interests are in the modeling and control of underactuated robotic systems.

Raffaella Mattone received the Ph.D. in Systems Engineering in 1997 from the University of Rome "La Sapienza." In 1997-98, she was a guest scientist at the Fraunhofer IPA in Stuttgart, where she used fuzzy/neural techniques for the localization and classification of unstructured objects in a robotic cell. In 1998–2000, she was a Research Engineer at Ascom Systec AG in Switzerland, working on automated video surveillance and pattern recognition. Since then, she is a research associate at the Department of Computer and Systems Science in Rome. Her research interests include underactuated manipulator control and fault detection and isolation in robotic systems.

Giuseppe Oriolo received the *Laurea* degree in Electrical Engineering in 1987 and the Ph.D. degree in Systems Engineering in 1992, both from the University of Rome "La Sapienza." Since 1998 he is *Associate Professor* of Automatic Control at the Department of Computer and System Science of the same university. He has published over 80 papers in international journals, books, and conferences. His research interests are in the area of nonlinear control and robotics. Dr. Oriolo is *Associate Editor* of the *IEEE Transactions on Robotics and Automation* since 2001. He is a Senior Member of IEEE.

©2002 Cyber Scientific

Volume 4, No. 3 / September 2002

Machine Intelligence and Robotic Control

Special Issue on
Underactuated Robot Manipulators

Guest Editorial:
Special Issue on Underactuated Robot Manipulators ***75***
Keigo Watanabe

Papers:
Underactuation in Robotic Grasping Hands ***77***
Thierry Laliberté, Lionel Birglen, and Clément M. Gosselin

Discrete-Time Path Planning and Control of a Highly Underactuated Manipulator ***89***
Dirk Weidemann, Norbert Scherm, and Bodo Heimann

Control of Free-Joint Manipulators with Only One Actuator upon Dynamic Behavioral Analysis ***99***
Takahiro Suzuki and Yoshihiko Nakamura

Underactuated Manipulators: Control Properties and Techniques ***113***
Alessandro De Luca, Stefano Iannitti, Raffaella Mattone, and Giuseppe Oriolo

Evolution of Underactuated Robots and Control Techniques ***127***
Lanka Udawatta, Keigo Watanabe, Kiyotaka Izumi, and Kazuo Kiguchi