Analysis and Control of Nonlinear Systems
C.I. Byrnes, C.F. Martin and R.E. Saeks (editors)

C1-7

CONTROL PROPERTIES OF ROBOT ARMS WITH JOINT ELASTICITY

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Abstract. The control problem for robot manipulators with elasticity at the joints is considered. This class of robot models does not satisfy in general the necessary and sufficient conditions for linearization and for input-output decoupling using nonlinear static state-feedback. Instead, both these control problems can always be solved by using the more general class of nonlinear dynamic feedback. The model of robot arms with joint elasticity satisfies conditions that enable to construct such a decoupling and linearizing dynamic feedback. The procedure is applied to a planar two-link robot with one elastic joint.

1. Introduction

Experimental and simulation studies on industrial robot arms [1,2] have shown that elasticity of the transmission elements between actuators and links has a relevant influence on robot dynamics. In particular, robots that use transmission belts, long shafts or harmonic drives often show a vibrational behavior. In these situations, we say that joint elasticity is present in the structure. The main effect of joint elasticity is that the position of the actuator (i.e. the angle of the motor shaft) is not uniquely related to the position of the driven link. As a consequence, the conventional rigid arm dynamic model does not describe completely the relation between applied torques and links motion.

The modeling and control of robot arms with joint elasticity has recently become an active area of research. A dynamic model including the full nonlinear dynamic interactions among the inertial properties of links and actuators was first introduced in [3] and then used in [4-11]. The relevant feature of this model arises strictly from a control theoretical point of view. It is known that for robots with rigid links and transmission elements, linearization and noninteraction can always be achieved by means of a nonlinear static state-feedback (see e.g. [12]). These two properties may be lost in presence of joint elasticity, as for example in a planar two-link arm [4,5] which is the most common kinematic arrangement. Due to this, alternative approaches have been proposed to control robot arms with elastic joints: singular perturbation techniques [6], integral manifold design [7,8], model-reference adaptive control [9]. None of these methods is able to mimic the results obtained in the rigid robot, that is an exact linear and decoupled behavior in the closed-loop system.

There are some simple robots with elastic joints, like the single-link [10] and the two-link cylindric arm [11], for which the so-called inverse dynamics method can still be successfully applied. Thus, for this class of robotic systems the feedback linearization property, in the sense of [13,14], depends on the robot kinematics.

A different approach has been proposed in [15,16] for the control of a two-link and of a three-link anthropomorphic arms with joint elasticity. In both cases, the use of *dynamic* nonlinear state-feedback allows to obtain both input-output decoupling and exact state linearization in the closed loop. The results are based on sufficient conditions for full linearization via dynamic feedback which exploit the properties of the maximal controlled invariant distribution of the system [17].

In this paper, we give more generality to these results by showing that part of these sufficient conditions are always satisfied by the dynamic model of robots with joint elasticity. We briefly report also on the detailed analysis made in [18] of the most important kinematic structures of robots in this class; hybrid models, in which elasticity is present only at certain joints, are also considered. These results suggest that *any* robot with elastic joints can be fully linearized via a nonlinear state-feedback, which is either a static or a dynamic one.

2. Dynamic model of robot arms with joint elasticity

A robot arm with N elastic joints can be seen as a chain of 2N elastically coupled rigid bodies, N actuators and N links. The 2N mechanical degrees of freedom of such a structure, are controlled by only N independent inputs, the motor torques acting on the actuator side of the elastic joints.

To derive the dynamic equations of motion it is convenient to follow a Lagrangian approach. Two variables are associated to the i-th elastic joint (see Figure 1): q_{2i-1}, the position of the i-th actuator with respect to the (i-1)-th link, and q_{2i}, the position of the i-th link with respect to the previous one.

With this selection of 2N generalized joint coordinates $\mathbf{q}$, the description of the link kinematics is the same as in the rigid robot case. The potential energy $U(\mathbf{q})$ and the kinetic energy $T(\mathbf{q},\mathbf{q}')$ are computed in the usual way, considering the arm as an open chain of 2N rigid bodies, links and actuators. $U(\mathbf{q})$ contains, beside the gravitational contribution $U_g(\mathbf{q})$, also the elastic energy stored in the joints

$$U_e(\mathbf{q}) = \sum_{i=1}^{N} U_{e,i}(q_{2i-1}, q_{2i}) = \sum_{i=1}^{N} \frac{1}{2} K_i [q_{2i} - \frac{q_{2i-1}}{NT_i}]^2$$

where $K_i > 0$ is the elastic constant of joint i and $NT_i > 1$ is the transmission ratio. The Lagrangian $L = T - (U_g + U_e)$ obeys to the Euler-Lagrange equations which particularize as:

$$\frac{d}{dt}[\frac{\partial T}{\partial \dot{q}_j}] - \frac{\partial T}{\partial q_j} + \frac{\partial U_g}{\partial q_j} - \frac{K_i}{NT_i}[q_{2i} - \frac{q_{2i-1}}{NT_i}] = \tau_i \qquad \text{for } j = 2i-1$$

$$\frac{d}{dt}[\frac{\partial T}{\partial \dot{q}_j}] - \frac{\partial T}{\partial q_j} + \frac{\partial U_g}{\partial q_j} + K_i[q_{2i} - \frac{q_{2i-1}}{NT_i}] = 0 \qquad \text{for } j = 2i$$

with i=1,...N; τ_i is the torque supplied by the actuator at the i-th joint. Performing the indicated derivatives, the dynamic model can be written as:

$$B_E(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{c}_E(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{e}_E(\mathbf{q}) + \mathbf{r}_E(\mathbf{q}) = \tau_E$$

In these 2N second order nonlinear differential equations, the forcing term τ_E has even components equal to zero and odd components equal to the N-dimensional vector τ of motor torques. $B_E(\mathbf{q})$ is the symmetric, positive definite inertia matrix, $\mathbf{c}_E(\mathbf{q},\mathbf{q}')$ collects the centrifugal and Coriolis terms, $\mathbf{e}_E(\mathbf{q})$ contains the gravitational forces while the elastic term $\mathbf{r}_E(\mathbf{q})$ is linear in the elastic joint displacements. The modeling of the hybrid case, with only some of the joints being elastic, follows in a similar way (see the case study in Section 5).

To derive the state and output equations associated to this system, it is convenient to define a 2NxN odd-columns selection matrix B_S = block diag { $[1\ 0]^T$ } and a Nx2N even-rows selection matrix C_S = block diag { [0 1] }. Defining the state as $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_v) = (\mathbf{q},\mathbf{q}')$ and the input as $\mathbf{u} = \tau$,

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_v \\ -B_E(\mathbf{x}_p)^{-1}\mathbf{n}_E(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ B_E(\mathbf{x}_p)^{-1}B_S \end{bmatrix}\mathbf{u} = f(\mathbf{x}) + g(\mathbf{x})\,\mathbf{u}$$

where for compactness $\mathbf{n}_E(\mathbf{x}) = \mathbf{c}_E(\mathbf{x}) + \mathbf{e}_E(\mathbf{x}_p) + \mathbf{r}_E(\mathbf{x}_p)$. For the definition of the outputs, the proper "joint" variables to be set under control are indeed the link positions

$$\mathbf{y} = C_s\,\mathbf{x}_p = h(\mathbf{x})$$

or, in scalar notation, $y_i = \mathbf{x}_{p,2i} = q_{2i}$, $i = 1,\ldots,N$.

3. Full linearization via nonlinear state-feedback

Consider a smooth nonlinear system of the square type (same number of inputs and outputs) in the form

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\,\mathbf{u} \qquad \mathbf{y} = h(\mathbf{x})$$

$\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$. A possible control strategy for this class of systems is to use state-feedback and state and input-space transfomations in order to obtain a closed-loop system which is linear, controllable and observable, and input-output decoupled (or noninteractive). The achievement of all these objectives is sometimes referred to as full linearization.

Necessary and sufficient conditions for the solution of various subsets of this problem are well-known, see e.g. [12-14,17,19]. Beside these, sufficient conditions exist for the solvability of the full linearization problem using either static or dynamic state feedback. The appealing feature of these conditions is that they are constructive in nature and relatively easy to check.

The full linearization problem using *static* feedback (FLS) consists of finding a pair (α,β), with β nonsingular, such that using $\mathbf{u} = \alpha(\mathbf{x})+\beta(\mathbf{x})\mathbf{v}$, the closed-loop system becomes diffeomorphic to a controllable and observable triple (A,B,C), with y_i depending only on v_i, for $i = 1,\ldots,m$. Everything may hold only locally around an equilibrium point $\mathbf{x}_0$. To state sufficient conditions of solvability, the *relative order* r_i of each output y_i is introduced as the least integer such that

$$L_g L_f^{r_i-1} h_i(\mathbf{x}) \not\equiv 0$$

These nonzero row vectors are by definition the rows of the mxm *decoupling matrix* $A(\mathbf{x})$.

Theorem 1 [18]. FLS can be solved if:

(S1) $A(\mathbf{x})$ is nonsingular (S2) $\sum_{i=1}^{m} r_i = n$

The full linearizing feedback and the diffeomorphism needed to display linearity are given in this case by the standard theory of decoupling via static state-feedback.

The full linearization problem using *dynamic* feedback (FLD) consists of finding a quadruple (a,b,c,d) and an integer ν, such that using

$$\dot{\mathbf{z}} = a(\mathbf{x},\mathbf{z}) + b(\mathbf{x},\mathbf{z})\,\mathbf{v} \qquad \mathbf{u} = c(\mathbf{x},\mathbf{z}) + d(\mathbf{x},\mathbf{z})\,\mathbf{v}$$

with $\mathbf{z} \in \mathbb{R}^\nu$ and $\mathbf{v} \in \mathbb{R}^m$, the closed-loop system becomes diffeomorphic via $\eta = T(\mathbf{x},\mathbf{z})$ to

$$\dot{\eta} = A\eta + B\mathbf{v}, \qquad \mathbf{y} = C\eta$$

with (A,B) controllable, (C,A) observable, and y_i depending only on v_i, i = 1,...,m. The dimension ν of the dynamic compensator state **z** is not specified a priori. A result which guarantees the existence of a solution to FLD has been introduced in [16,20] and is based on the properties of the so-called Δ^*, the maximal controlled invariant distribution contained in the kernel of the output operator. In order to compute this distribution, the following dual algorithm is used:

Δ^* Algorithm [17]. Under regularity assumptions on the codistributions involved, the following sequence for k = 0,1,...

$$\Omega_{k+1}(\mathbf{x}) = \Omega_k(\mathbf{x}) \oplus L_f(\Omega_k \cap G^{\perp})(\mathbf{x}) \oplus \sum_{i=1}^{m} L_{g_i}(\Omega_k \cap G^{\perp})(\mathbf{x})$$

where

$$\Omega_0(\mathbf{x}) = \text{span}\{dh_i(\mathbf{x}),\ i = 1,\dots,m\} \qquad G(\mathbf{x}) = \text{span}\{g_i(\mathbf{x}),\ i = 1,\dots,m\}$$

converges at an iteration $k^* \leq n-1$ to the annihilator of Δ^*, in the sense that

$$\Omega_{k^*+1}(\mathbf{x}) = \Omega_{k^*}(\mathbf{x}) \qquad \text{and} \qquad \Delta^*(\mathbf{x}) = (\Omega_{k^*}(\mathbf{x}))^{\perp}.$$

The following two lemmas are used in order to restate geometrically the conditions of Theorem 1 and to find a generalization of it which guarantees the solvability of FLD.

Lemma 2 [17]. If A(**x**) is nonsingular, then:

a) $\Delta^* = (\text{span}\{dh_i, \dots, dL_f^{r_i-1} h_i;\ i=1,\dots,m\})^{\perp}$ b) $\Delta^* = 0 \iff \sum_{i=1}^{m} r_i = n$

Lemma 3 [18]. If $\sum_{i=1}^{m} L_{g_i}(\Omega_k \cap G^{\perp}) \subset \Omega_k$, k = 0,1,..., then there exists an $\alpha(\mathbf{x})$ such that setting $\tilde{f}(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})\alpha(\mathbf{x})$:

a) $\Delta^* = (\text{span}\{dh_i, \dots, dL_{\tilde{f}}^{\mu_i-1} h_i;\ i=1,\dots,m\})^{\perp}$ b) $\Delta^* = 0 \iff \sum_{i=1}^{m} \mu_i = n$

The assumption in Lemma 3, that we will refer to as *structural condition* , is a relaxation of the one in Lemma 2. Checking of this condition and computation of the indices μ_i are directly performed via the Δ^* algorithm. By construction, it can be shown that $\mu_i \geq r_i$ for all i, with strict equality holding for the smallest integers. Under the above structural condition, these indices μ_i are the multiplicities of the zeros at infinity associated to the given nonlinear system. We can state finally the following

Theorem 4 [16,18]. FLD can be solved if:

$$\text{(D1)} \quad \sum_{i=1}^{m} L_{g_i}(\Omega_k \cap G^{\perp}) \subset \Omega_k \quad \text{for all k} \qquad \text{(D2)} \quad \Delta^* = 0.$$

The following three-step procedure computes the full linearizing dynamic feedback [16,18]:

Full Linearization Procedure

Step 1. Apply a static state-feedback $\mathbf{u} = \alpha(\mathbf{x}) + \beta(\mathbf{x})\mathbf{w}$, with $\alpha(\mathbf{x})$ as in Lemma 3 computed directly through the Δ^* algorithm, and $\beta(\mathbf{x}) = Q(\mathbf{x})^{-1}$, with entries of $Q(\mathbf{x})$ given by

$$q_{i,j}(\mathbf{x}) = L_{g_j} L_{\tilde{f}}^{\mu_i-1} h_i(\mathbf{x})$$

Step 2. Apply a dynamic extension on **w**, that is add to each input channel w_i a number of integrators equal to μ-μ_i, where $\mu = \max \{\mu_i , i = 1,\ldots,m \}$

$$w_i = z_{i,1} \quad \dot{z}_{i,1} = z_{i,2} \quad \ldots \quad \dot{z}_{i,\mu-\mu_i} = \overline{w}_i$$

The state **z** of the compensator is built up of dimension ν equal to the sum of the $(\mu-\mu_i)$. for

Step 3. Apply a static decoupling feedback:

$$\overline{\mathbf{w}} = \overline{\alpha}(\mathbf{x},\mathbf{z}) + \overline{\beta}(\mathbf{x},\mathbf{z})\, \mathbf{v}$$

on the extended system (denoted by overbars), where

$$\overline{\beta}(\mathbf{x},\mathbf{z}) = \overline{\beta}(\mathbf{x}) = \overline{A}(\mathbf{x})^{-1} = \{ L_{\overline{g}_j} L_{\overline{f}}^{\mu-1} h_i(\mathbf{x}) \}^{-1} \qquad \overline{\alpha}(\mathbf{x},\mathbf{z}) = -\overline{A}(\mathbf{x})^{-1} . \text{col} \{L_{\overline{f}}^{\ \mu} h_i (\mathbf{x},\mathbf{z})\}$$

The feedback law resulting from the composition of these three steps is the required dynamic compensator that solves FLD. The closed-loop system becomes equivalent to m chains of input-output integrators of *equal* length μ.

4. Full linearization of robot arms with elastic joints

In order to apply the above procedure for full linearization to robots with joint elasticity, we may prove first that the sufficient conditions of Theorem 4 are satisfied.

Theorem 5. The dynamics of robot arms with elastic joints satisfies the structural condition.

Proof (sketch of). It is convenient to perform a coordinate change in the state space from **x** = (**q**, **q'**) to **z** = (**q**, B_E(**q**) **q'**). In these coordinates, i.e. describing the system in terms of position and momentum, the state and output equations become

$$\dot{\mathbf{z}} = \begin{bmatrix} B_E(\mathbf{z}_1)^{-1}\mathbf{z}_2 \\ -\mathbf{e}_E(\mathbf{z}_1) + \dfrac{\partial T(\mathbf{z})}{\partial \mathbf{z}_1} \end{bmatrix} + \begin{bmatrix} 0 \\ B_S \end{bmatrix} \mathbf{u} \qquad \mathbf{y} = \begin{bmatrix} C_S & 0 \end{bmatrix} \mathbf{z}$$

In order to show that (D1) of Theorem 4 holds, induction is used on k. Starting from $\Omega_0 \cap G^{\perp} = \Omega_0 = \text{span}\{ [\, C_s\ 0 \,] \}$, since $L_g h(\mathbf{z}) = 0$ then $L_{gi} (\Omega_0) = 0 \subset \Omega_0$ for i =1,...,N and condition (D1) holds for k = 0. Assume that it holds true at the generic iteration k-1. By using the invariance under feedback of the Δ^* algorithm, Ω_k can be expressed as the span of exact one-forms $d\lambda$. The constant and simple structure of the input vector fields allows to characterize also the distribution $G^{\perp}$. Then, it is easy to show that

$$\sum_{i=1}^{N} L_{g_i} (\Omega_k \cap G^{\perp})(\mathbf{z}) = 0 \subset \Omega_k (\mathbf{z})$$

and (D1) holds also for iteration k. This completes the proof.∎

Together with the above result, the validity of condition (D2) of Theorem 4 for robot arms with joint elasticity would imply that all instances of this class can be fully linearized via dynamic state-feedback, when static feedback is not enough. Unfortunately, to prove in general that $\Delta^* = 0$ for these robots seems to be a hard task. However, a study done in [18] shows that for the most common and significative kinematic structures also the condition (D2) is satisfied. Table 1 contains a synopsis of the obtained results. It is worth noting that:

a) the uniform structure of relative orders of the rigid robot case is lost in general; however, if static feedback is enough for linearization, uniform r_i are found (but not viceversa);

b) the length of the closed-loop chains of integrators depends on the kinematic type of the arm and is always greater or equal to four, as opposed to constantly two for the rigid case.

As a consequence, the difficulty of a general proof of condition (D2) stands in the different possible ways through which the property $\Delta^* = 0$ is achieved.

5. A case study

Consider a two-link robot arm moving on a horizontal plane.The first link is actuated through a direct-drive motor. The second joint shows a significant elasticity. For simplicity, center of masses are assumed to be located at the joints (the motors) and at the tip (a load). For the various definitions refer to Figure 2. In this hybrid structure **q** will have three components: q_1 = rotation of the first link w.r.t. the base frame; q_2 = rotation of the motor at the second joint; q_3 = rotation of the second link w.r.t. the first link. Since $U_g = 0$, the potential energy U is given by

$$U = U_e = \frac{1}{2} K \left(q_3 - \frac{q_2}{NT} \right)^2$$

The kinetic energy T of the system is the sum of the three terms T_{mot1}, T_{mot2}, T_{load} and can be rewritten as a quadratic form in the velocity **q'**

$$T = \frac{1}{2} \dot{\mathbf{q}}\, B_E(\mathbf{q})\, \dot{\mathbf{q}}$$

The inertia matrix $B_E(\mathbf{q})$ of the elastic system is

$$B_E(\mathbf{q}) = \begin{bmatrix} A_1 + 2A_3 \cos q_3 & JR_1 & A_2 + A_3 \cos q_3 \\ JR_1 & JR_1 & 0 \\ A_2 + A_3 \cos q_3 & 0 & A_2 \end{bmatrix}$$

while the Coriolis, centrifugal and elastic forces are collected in

$$\mathbf{n}_E(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -A_3 \sin q_3 (2\dot{q}_1 \dot{q}_3 + \dot{q}_3^2) \\ 0 \\ A_3 \sin q_3\, \dot{q}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{K}{NT} \left(q_3 - \frac{q_2}{NT}\right) \\ K \left(q_3 - \frac{q_2}{NT}\right) \end{bmatrix}$$

The following constants are used:

$$A_1 = JR_0 + JR_1 + JR_p + m_1 l_1^2 + m_p (l_1^2 + l_2^2) \qquad A_2 = JR_p + m_p l_2^2$$

$$A_3 = m_p l_1 l_2 \qquad A_4 = (m_1 + m_p)\, l_1 l_2 \qquad A_5 = (m_1 + m_p)\, l_1^2$$

For this robot with mixed joints the state, input and output are defined as $\mathbf{x} = (q_1, q_2, q_3, q_1', q_2', q_3')$, $\mathbf{u} = (\tau_1, \tau_2)$, $\mathbf{y} = (q_1, q_3)$; thus, n = 6 and m = 2. The drift and input vector fields, $f(\mathbf{x})$ and $g(\mathbf{x})$, and the output function $h(\mathbf{x})$ are respectively:

$$f(\mathbf{x}) = \begin{bmatrix} x_4 & x_5 & x_6 & f_4(x_2,x_3,x_4,x_6) & f_5(x_2,x_3,x_4,x_6) & f_6(x_2,x_3,x_4,x_6) \end{bmatrix}^T$$

$$g(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & g_{41}(x_3) & -g_{41}(x_3) & g_{61}(x_3) \\ 0 & 0 & 0 & -g_{41}(x_3) & g_{52}(x_3) & -g_{61}(x_3) \end{bmatrix}^T$$

$$h(\mathbf{x}) = \begin{bmatrix} x_1 & x_3 \end{bmatrix}^T$$

where

$$g_{41} = \frac{A_2}{A_3 (A_4 - A_3 \cos^2 x_3)} \qquad g_{52} = g_{41}(x_3) + \frac{1}{JR_1} \qquad g_{61} = - g_{41}(x_3) - \frac{\cos x_3}{A_4 - A_3 \cos^2 x_3}$$

$$f_4 = \frac{A_2 \sin x_3 (x_4 + x_6)^2 + A_3 \sin x_3 \cos x_3 \; x_4^2}{A_4 - A_3 \cos^2 x_3} + \frac{K (x_3 - \frac{x_2}{NT}) [\frac{A_2}{A_3} (\frac{NT - 1}{NT}) + \cos x_3]}{A_4 - A_3 \cos^2 x_3}$$

$$f_5 = - f_4 + \frac{K}{NT} (x_3 - \frac{x_2}{NT}) (\frac{1}{JR_1} - 2\, g_{41}(x_3))$$

$$f_6 = - f_4 - \frac{K (x_3 - \frac{x_2}{NT}) [\frac{A_5}{A_3} - (\frac{3NT + 1}{NT}) \cos x_3]}{A_4 - A_3 \cos^2 x_3} - \frac{A_5 \sin x_3 \; x_4^2 + A_3 \sin x_3 \cos x_3 (x_4 + x_6)^2}{A_4 - A_3 \cos^2 x_3}$$

It is easy to verify that the relative orders of the outputs are $r_1 = r_2 = 2$ and the associated decoupling matrix

$$A(\mathbf{x}) = L_g L_f h(\mathbf{x}) = \begin{bmatrix} g_{41}(x_3) & - g_{41}(x_3) \\ g_{61}(x_3) & - g_{61}(x_3) \end{bmatrix}$$

is structurally singular. Thus, the results of Theorem 1 cannot be used. Instead one can apply the Δ^* algorithm, keeping into account the result of Theorem 5 in order to reduce the computational burden. From this algorithm, a feedback modification by

$$\alpha(\mathbf{x}) = \begin{bmatrix} - \dfrac{f_4(x_2, x_3, x_4, x_6)}{g_{41}(x_3)} \\ 0 \end{bmatrix}$$

of the drift vector field $f(\mathbf{x})$ yields a new

$$\tilde{f}(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}) \alpha(\mathbf{x}) = \begin{bmatrix} x_4 & x_5 & x_6 & 0 & \tilde{f}_5(x_2, x_3) & \tilde{f}_6(x_2, x_3, x_4) \end{bmatrix}^T$$

with

$$\tilde{f}_5(x_2, x_3) = f_4(\mathbf{x}) + f_5(\mathbf{x}) \qquad \tilde{f}_6(x_2, x_3, x_4) = f_6(\mathbf{x}) - \frac{g_{61}(\mathbf{x})\, f_4(\mathbf{x})}{g_{41}(\mathbf{x})}$$

where the functional dependencies of the lhs terms can be checked directly from the expressions of the model. Then

$$\Delta^* = (\text{span} \{ dh_1 , dL_{\tilde{f}} h_1 , dh_2 , dL_{\tilde{f}} h_2 , dL_{\tilde{f}}^2 h_2 , dL_{\tilde{f}}^3 h_2 \})^{\perp} = 0$$

and $\mu_1 = 2$, $\mu = \mu_2 = 4$. Since

$$L_{\tilde{f}}^2 h_2(\mathbf{x}) = \tilde{f}_6(x_2, x_3, x_4) \qquad L_{\tilde{f}}^3 h_2(\mathbf{x}) = \frac{\partial \tilde{f}_6}{\partial x_2} x_5 + \frac{\partial \tilde{f}_6}{\partial x_3} x_6$$

the following matrix will be nonsingular

$$\beta(\mathbf{x}) = \begin{bmatrix} L_{g_1} L_{\tilde{f}} h_1(\mathbf{x}) & L_{g_2} L_{\tilde{f}} h_1(\mathbf{x}) \\ L_{g_1} L_{\tilde{f}}^3 h_2(\mathbf{x}) & L_{g_2} L_{\tilde{f}}^3 h_2(\mathbf{x}) \end{bmatrix}^{-1}$$

and the first step of the full linearization procedure, $\mathbf{u} = \alpha(\mathbf{x}) + \beta(\mathbf{x})\,\mathbf{w}$, is completely defined. Next, a dynamic extension of order $\mu - \mu_1 = 2$ is performed on the new input w_1

$$w_1 = z_1 \qquad \dot{z}_1 = z_2 \qquad \dot{z}_2 = \overline{w}_1 \qquad\qquad w_2 = \overline{w}_2$$

Using the overbar to denote the extended state $(\mathbf{x},\mathbf{z}) = (\mathbf{x}, z_1, z_2) \in \mathbb{R}^8$, the state and output equations of the extended system become

$$\dot{\overline{\mathbf{x}}} = \overline{f}(\overline{\mathbf{x}}) + \overline{g}(\overline{\mathbf{x}})\,\overline{\mathbf{w}} = \begin{bmatrix} x_4 & x_5 & x_6 & z_1 & \overline{f}_5(\overline{\mathbf{x}}) & \overline{f}_6(\overline{\mathbf{x}}) & z_2 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \overline{g}_{52}(\mathbf{x}) & 0 & 0 & 0 \end{bmatrix} \overline{\mathbf{w}}$$

$$\mathbf{y} = \overline{h}(\overline{\mathbf{x}}) = h(\mathbf{x})$$

Finally, the system above can be decoupled and linearized by means of a static decoupling feedback from the extended state. The decoupling matrix for this system has the form

$$\overline{A}(\mathbf{x}) = L_{\overline{g}} L_{\overline{f}}^3 \overline{h}(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$$

globally nonsingular and with relative orders both equal to 4. Thus

$$\overline{r}_1 + \overline{r}_2 = 8 = \overline{n}$$

and both conditions of Theorem 1 are fullfilled. The static decoupling feedback of step 3 is given by

$$\overline{\mathbf{w}} = \overline{\alpha}(\overline{\mathbf{x}}) + \overline{\beta}(\mathbf{x})\,\mathbf{v}$$

$$\overline{\beta}(\mathbf{x}) = \overline{A}(\mathbf{x})^{-1} \qquad \overline{\alpha}(\overline{\mathbf{x}}) = -\overline{A}(\mathbf{x})^{-1} L_{\overline{f}}^4 \overline{h}(\overline{\mathbf{x}}) = -\overline{A}(\mathbf{x})^{-1} \begin{bmatrix} 0 \\ L_{\overline{f}}^4 \overline{h}_2(\overline{\mathbf{x}}) \end{bmatrix}$$

The final system is equivalent to two chains of four input-output integrators between **v** and **y**. Two of these eight states are due to dynamic compensation. The structure of the nonlinear dynamic compensator which acts as linearizing controller is summarized in Figure 3.

6. Conclusion

The kinematic class of robot arms with elastic joints which requires nonlinear dynamic compensation for linearization and decoupling is quite large. It includes all structures having at least two rotary elastic joints with parallel axes of rotation, e.g. a planar two-link arm or the first three links of an anthropomorphic robot arm. In any case, full linearization can be achieved. For the whole class of robot arms with joint elasticity, the obtained linear closed-loop system is represented by input-output integration paths which are of order equal or greater than four. This length may be considered as a rough measure for the difficulty of controlling the dynamic motion of the arm and has direct implications on the definition of the class of desired trajectories that can be reproduced as system outputs.

The results obtained with the present exact design may be used also to gain some better understanding on the implementation of different and possibly simpler control strategies. For instance, the uniform length μ of the closed-loop chains can be used to select the dimension of a linear reference model in an adaptive approach.

A physical interpretation of the role of dynamic state-feedback in robots with joint elasticity can be given. In these manipulators dynamic compensation *loosen* the couplings among the elastic phenomena at different joints in the structure. In fact, whenever two elastic joints do not interact (e.g. when their axes are orthogonal) only the rigid degrees of freedom of the arm need to be decoupled and static feedback serves this purpose. For hybrid structures, with only some of the joints being elastic, dynamic compensation is used instead to *balance* the length of the input-output paths from joint torques to links motion; by slowing down the response of the rigid joints, the torques acting through the elastic ones will have enough time to come into effective play and counteract the dynamic cross-effects.

Acknowledgements

I would like to thank Professor Alberto Isidori for his constant advice during my Doctorate and for many illuminating suggestions.

References

[1] **Good, M.C., Sweet, L.M., Strobel, K.L.,** Dynamic Models for Control System Design of Integrated Robot and Drive Systems,*Trans. ASME J. Dyn. Systems, Meas., and Control,* **107**,1985.
[2] **Sweet, L.M., Good, M.C.,** Redefinition of the Robot Motion Control Problem, *IEEE Control Systems Mag.*, **5**, 3, 1985.
[3] **Nicosia, S., Nicolò, F., Lentini, D.,** Dynamical Control of Industrial Robots with Elastic and Dissipative Joints, *8th IFAC World Congress*, Kyoto, 1981.
[4] **Cesareo, G., Marino, R.,** On the Controllability Properties of Elastic Robots, *6th Int. Conf. Analysis and Optimization of Systems*, Nice, 1984.
[5] **De Simone, C., Nicolò, F.,** On the Control of Elastic Robots by Feedback Decoupling, *IASTED Int. J. Robotics and Automation*, **1**, 2, 1986.
[6] **Marino, R, Nicosia, S.** On the Feedback Control of Industrial Robots with Elastic Joints: A Singular Perturbation Approach, *Università di Roma "Tor Vergata"*, Rap.84.01, 1984.
[7] **Khorasani, K., Spong, M.W.,** Invariant Manifolds and Their Application to Robot Manipulators with Flexible Joints,*IEEE Conf. Robotics and Automation*, St.Louis, 1985.
[8] **Khorasani, K., Kokotovic, P.V.,** Feedback Linearization of a Flexible Manipulator near its Rigid Body Manifold, *Systems & Control Lett.*, **6**, 1985.
[9] **Tomei, P., Nicosia, S., Ficola, A.,** An Approach to the Adaptive Control of Elastic at Joints Robots, *IEEE Conf. Robotics and Automation*, S.Francisco, 1986.
[10] **Marino, R., Spong, M.W.,** Nonlinear Control Techniques for Flexible Joint Manipulators: A Single Link Case Study, *IEEE Conf. Robotics and Automation*, S.Francisco, 1986.
[11] **Forrest-Barlach, M.G., Babcock, S.M.,** Inverse Dynamics Position Control of a Compliant Manipulator, *IEEE Conf. Robotics and Automation*, S.Francisco, 1986.
[12] **Tarn,T.J., Bejczy, A.K., Isidori, A., Chen, Y.,** Nonlinear Feedback in Robot Arm Control, *23rd IEEE Conf. Decision and Control*, Las Vegas, 1984.
[13] **Jakubczyk, B., Respondek, W.,** On Linearization of Control Systems, *Bull. Acad. Pol. Sci., Ser. Math. Astr. Phys.*, **28**, 1980.
[14] **Hunt, L.R., Su, R., Meyer, G.,** Design for Multi-Input Nonlinear Systems, in: *Differential Geometric Control Theory* (R.W.Brockett, R.S.Millman, H.Sussmann Eds.), *Birkhauser*, 1983.
[15] **De Luca, A., Isidori, A., Nicolò, F.,** An Application of Nonlinear Model Matching to the Control of Robot Arm with Elastic Joints, *IFAC Symp. Robot Control*, Barcelona, 1985.
[16] **De Luca, A., Isidori, A., Nicolò, F.,** Control of Robot Arms with Elastic Joints Via Nonlinear Dynamic Feedback, *24th IEEE Conf. Decision and Control*, Ft. Lauderdale, 1985.
[17] **Isidori, A.,** *Nonlinear Control Systems: An Introduction* , Lecture Notes in Control and Information Sciences, **72**, *Springer* , 1985.
[18] **De Luca A.,** Control of Robot Arms with Joint Elasticity: A Differential-Geometric Approach, Ph.D.Thesis (in Italian), University of Rome, December 1986.
[19] **Cheng, D., Isidori, A., Respondek, W., Tarn, T.J.,** On the Linearization of Nonlinear Systems with Outputs, in preparation.
[20] **Isidori, A.,** Control of Nonlinear Systems Via Dynamic State-Feedback, in: *Algebraic and Geometric Methods in Nonlinear Control Theory* (M.Fliess, M.Hazewinkel Eds.), *Reidel* , 1985.

# of joints	type of robot	relative orders r_i	multiplicities μ_i	dynamic feedback needed ?	closed-loop integrators
1	single link	4	4	no	1 x 4
2	cylindric	4, 4	4, 4	no	2 x 4
2	polar	4, 4	4, 4	no	2 x 4
2	planar	2, 2	2, 6	YES	2 x 6
2	hybrid polar a)	3, 2	4, 2	YES	2 x 4
2	hybrid planar b)	2, 2	2, 4	YES	2 x 4
3	anthropomorphic	3, 2, 2	4, 2, 6	YES	3 x 6

a) first joint elastic b) second joint elastic

Table 1 - Control properties of robot arms with joint elasticity

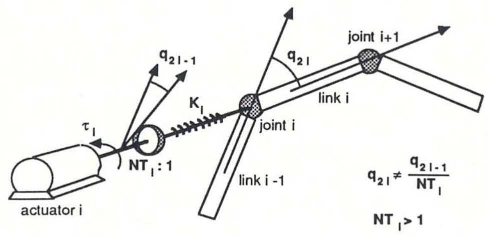

Figure 1 - Definition of variables for elastic joint i

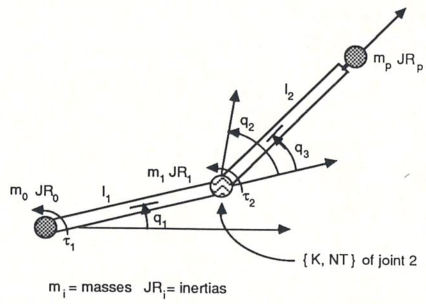

Figure 2 - A two-link planar robot arm with the second joint elastic

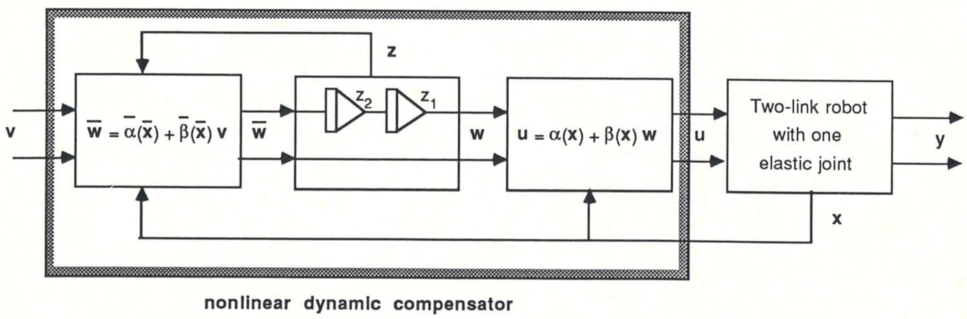

Figure 3 - Linearizing controller for the two-link arm with one elastic joint