Control Systems

Interconnected systems

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Outline

- General interconnected system state and interconnection equations
- Series
- Parallel
- Feedback

Consider a number of systems which influence each other through interconnections. We want to find a representation (state-space or transfer function) of the interconnected overall system

Let the single system be represented by $\mathcal{S}_i : \begin{cases} \dot{x}_i = A_i x_i + B_i u_i \\ y_i = C_i x_i + D_i u_i \end{cases} \qquad x_i \in \mathbb{R}^{n_i}$

The overall system has **state** x given by

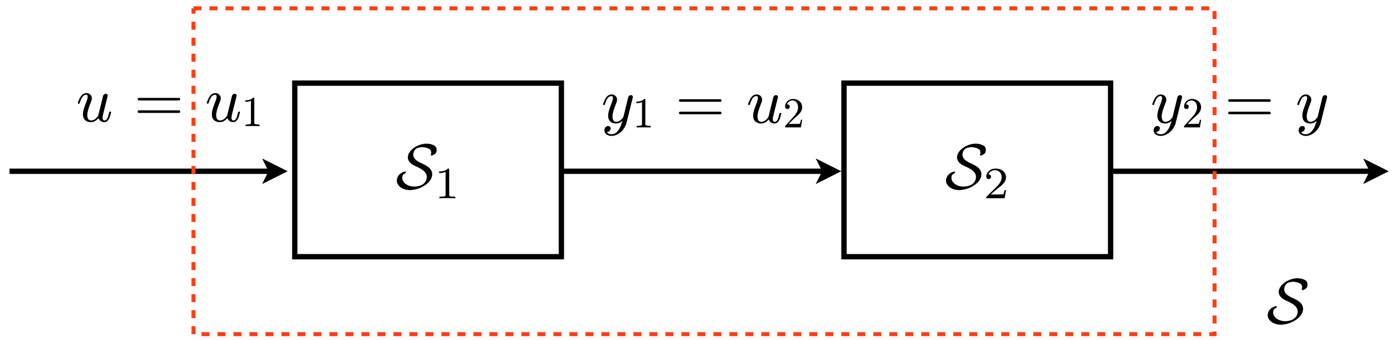
$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{pmatrix} \qquad x \in \mathbb{R}^n \qquad n = \sum_{i=1}^{m} n_i$$

and its representation (and behavior) depends upon how the subsystems are **interconnected**

3 different interconnections:

- **series**
- **parallel**
- **feedback**

series (state space)



$$\mathcal{S}_1 : \quad \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 + D_1 u_1 \end{cases}$$

$$\mathcal{S}_2 : \quad \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 + D_2 u_2 \end{cases}$$

$\mathcal{S}$ with state $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ input u and output y

- series system state space representation

interconnection equations

$$y_1 = u_2, \quad u = u_1, \quad y = y_2$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 x_1 + B_1 u_1 \\ A_2 x_2 + B_2 u_2 \end{pmatrix} = \begin{pmatrix} A_1 x_1 + B_1 u \\ A_2 x_2 + B_2 y_1 \end{pmatrix} = \begin{pmatrix} A_1 x_1 + B_1 u \\ A_2 x_2 + B_2 (C_1 x_1 + D_1 u) \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} u = Ax + Bu$$

$$y = y_2 = C_2 x_2 + D_2 u_2 = C_2 x_2 + D_2 (C_1 x_1 + D_1 u_1) = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2 D_1 u$$

$$= Cx + Du$$

series (state space)

series system has dynamics matrix

$$A = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \xrightarrow{\text{block triangular}} \mathrm{eig}\{A\} = \mathrm{eig}\{A_1\} \bigcup \mathrm{eig}\{A_2\}$$

$$B = \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} \quad C = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} \quad D = D_1 D_2$$

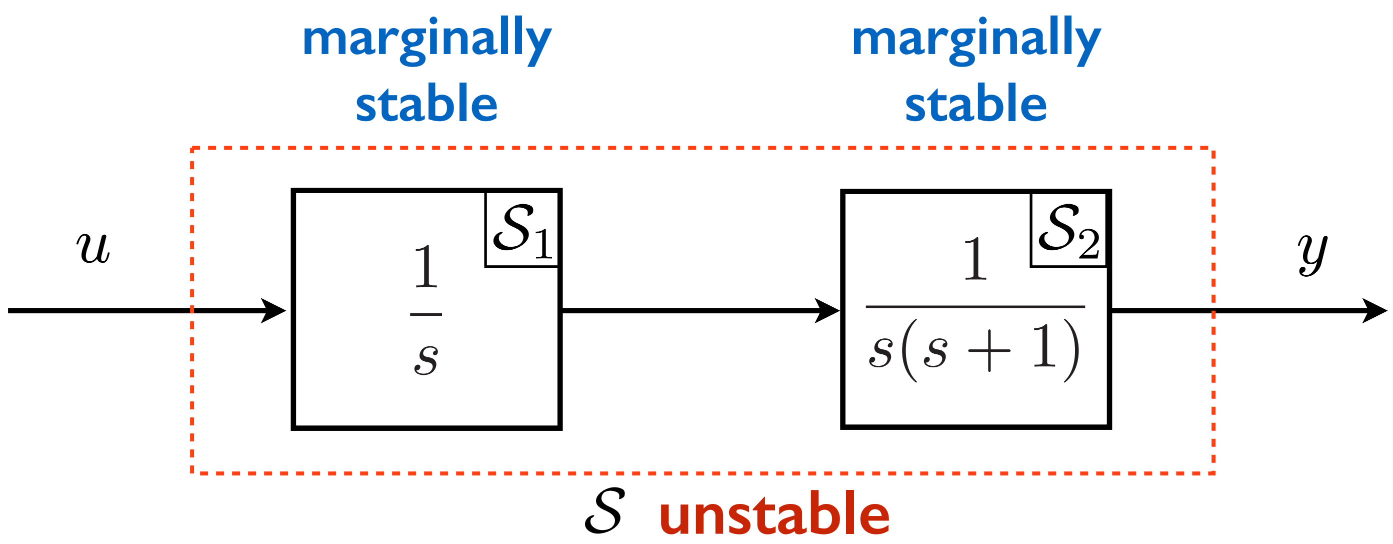
in general, the eigenvalues of the series of subsystems are given by the **union** of the single subsystem's eigenvalues

and therefore

- the series of asymptotically stable systems is also asymptotically stable
- if in a series a system is unstable, so is the interconnected system in series
- special care when interconnection two marginally stable systems

series

- but if $\mathcal{S}_1 : \dfrac{1}{s}$ and $\mathcal{S}_2 : \dfrac{1}{s(s+1)}$ each marginally stable, however when interconnected in series



$\mathcal{S}$: series system - transfer function

$$\frac{1}{s^2(s+1)}$$

unstable

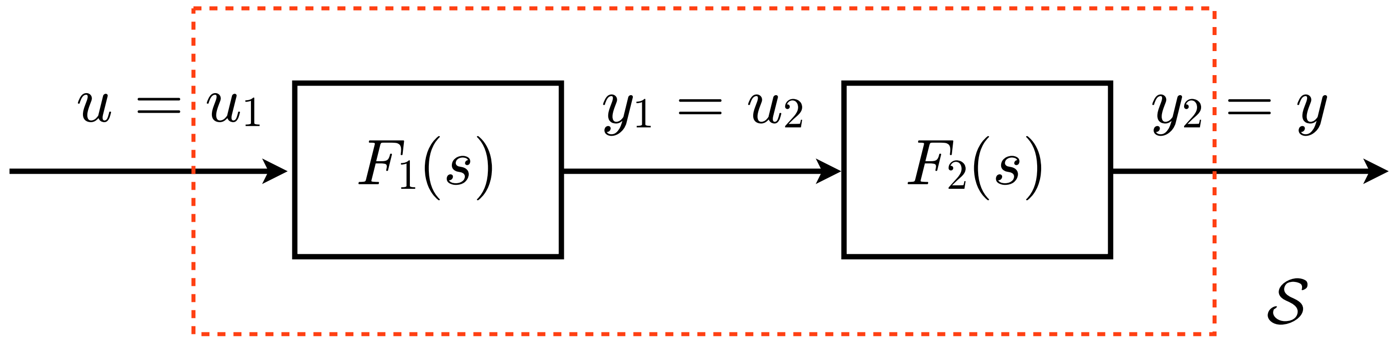
new behavior

- interconnection of marginally stable systems does not necessarily lead to instability

ex.: series of $\dfrac{s+1}{s(s+2)}$ (marginally stable) and $\dfrac{1}{s^2+10}$ (marginally stable) → $\dfrac{s+1}{s(s+2)} \cdot \dfrac{1}{s^2+10}$ (still marginally stable)

therefore in general there is no unique answer about stability when interconnecting in series two marginally stable systems

series (transfer function)



$$\mathcal{S}_1 : \quad F_1(s) = \frac{y_1(s)}{u_1(s)}$$

$$\mathcal{S}_2 : \quad F_2(s) = \frac{y_2(s)}{u_2(s)}$$

Hyp: for every subsystem $\mathcal{S}_i$ we assume coincidence of eigenvalues and poles (which does not imply that if we multiply two transfer functions there will be common factors)

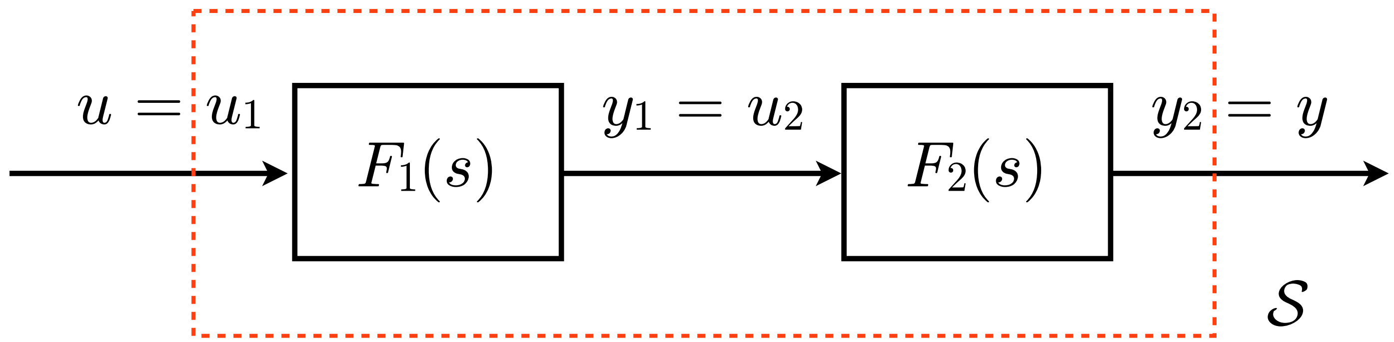
$$F(s) = \frac{y(s)}{u(s)} = \frac{y_2(s)}{u_1(s)}\frac{u_2(s)}{u_2(s)} = \frac{y_2(s)}{u_1(s)}\frac{y_1(s)}{u_2(s)} = \frac{y_2(s)}{u_2(s)}\frac{y_1(s)}{u_1(s)} = F_2(s)F_1(s) = F_1(s)F_2(s)$$

transfer functions of systems in **series multiply** together

- series can **alter** the **filtering capacity**

example (cancellations)

- $F_1(s)$ in series with $F_2(s)$



$$\mathcal{S}_1 \quad : \quad F_1(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1}$$

$$\mathcal{S}_2 \quad : \quad F_2(s) = \frac{1}{s-1}$$

$$F(s) = F_1(s)F_2(s) = \frac{(s-1)}{(s+1)}\frac{1}{(s-1)} = \frac{1}{s+1}$$

only 1 pole but 2 eigenvalues

- the interconnection has generated a hidden mode
- the interconnected system remains unstable since the eigenvalues have not changed and one is real positive.

recall the general PBH rank tests

$$\text{rank}\begin{pmatrix} A - \lambda_i I & | & B \end{pmatrix} = n \quad \rightarrow \quad \lambda_i \text{ controllable}$$

$$\text{rank}\begin{pmatrix} A - \lambda_i I \\ \hline C \end{pmatrix} = n \quad \rightarrow \quad \lambda_i \text{ observable}$$

- for the considered two systems we can find the following two realizations

$$\mathcal{S}_1: \quad A_1 = -1, \quad B_1 = 1, \quad C_1 = -2, \quad D_1 = 1 \qquad \lambda_1 = \text{-}1$$
$$\mathcal{S}_2: \quad A_2 = 1, \quad B_2 = 1, \quad C_2 = 1, \quad D_2 = 0 \qquad \lambda_2 = 1$$

- series state-space representation

$$A = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, \; B = \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & C_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \; D = 0$$

- PBH rank test

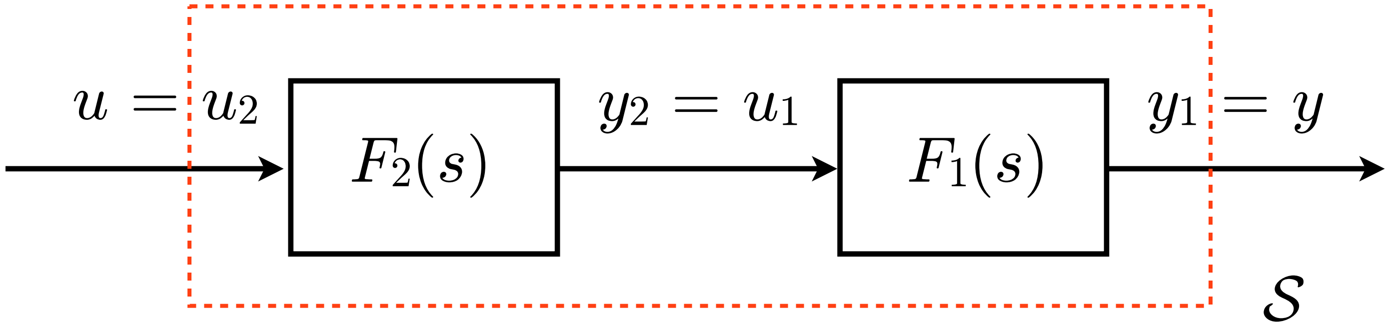
$$\mathrm{rk}\begin{pmatrix} A - \lambda_2 I & B \end{pmatrix} = \mathrm{rk}\begin{pmatrix} -2 & 0 & 1 \\ -2 & 0 & 1 \end{pmatrix} = 1 < n = 2, \quad \Rightarrow \lambda_2 \quad \text{uncontrollable}$$

$$\mathrm{rk}\begin{pmatrix} A - \lambda_2 I \\ C \end{pmatrix} = \mathrm{rk}\begin{pmatrix} -2 & 0 \\ -2 & 0 \\ 0 & 1 \end{pmatrix} = 2 = n, \quad \Rightarrow \lambda_2 \quad \text{observable}$$

- the series interconnection has generated, for the given example, an **uncontrollable** mode (the hidden dynamics characterized by the eigenvalue λ_2)

series interconnection but in different order

- $F_2(s)$ in series with $F_1(s)$



$$\mathcal{S}_1 \quad : \quad F_1(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1}$$

$$\mathcal{S}_2 \quad : \quad F_2(s) = \frac{1}{s-1}$$

$$F(s) = F_2(s)F_1(s) = \frac{1}{(s-1)}\frac{(s-1)}{(s+1)} = \frac{1}{s+1}$$

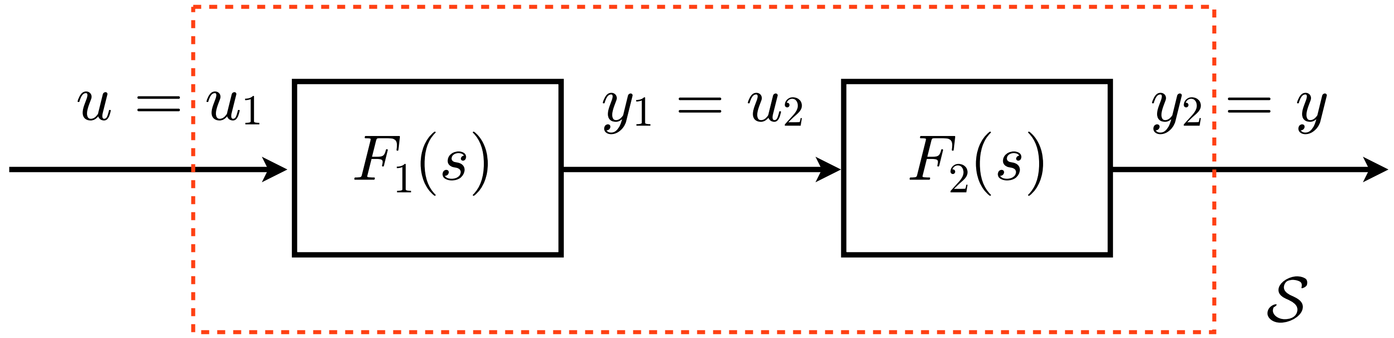
same transfer function as before, but

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} -2 & 1 \end{pmatrix} \quad D = 0$$

$$\text{rk} \begin{pmatrix} A - \lambda_2 I & B \end{pmatrix} = \text{rk} \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 = n, \quad \Rightarrow \lambda_2 \quad \text{controllable}$$

$$\text{rk} \begin{pmatrix} A - \lambda_2 I \\ C \end{pmatrix} = \text{rk} \begin{pmatrix} -2 & 1 \\ 0 & 0 \\ -2 & 1 \end{pmatrix} = 1 < n = 2, \quad \Rightarrow \lambda_2 \quad \text{unobservable}$$

- the series interconnection has generated, for the given example, an **unobservable** mode (the hidden dynamics characterized by the eigenvalue λ_2)



If, in the series of two systems $F_1(s) = N_1(s)/D_1(s)$ and $F_2(s) = N_2(s)/D_2(s)$ we have **cancellations** of common factors between $N_1(s)$ and $D_2(s)$ (zero/pole cancellation) or between $D_1(s)$ and $N_2(s)$ (pole/zero cancellation), we generate **hidden dynamics** which can either be uncontrollable or unobservable

For the system in figure (with the output of $F_1(s)$ being the input of $F_2(s)$)

- if a **zero** λ_c of $F_1(s)$ cancels out with a **pole** λ_c of $F_2(s)$ (zero/pole cancellation) we have generated **uncontrollable** hidden dynamics characterized by the eigenvalue λ_c
- if a **pole** λ_c of $F_1(s)$ cancels out with a **zero** λ_c of $F_2(s)$ (pole/zero cancellation) we have generated **unobservable** hidden dynamics characterized by the eigenvalue λ_c

example

natural modes when starting from non-zero initial conditions and applying an impulse

recall that

the zero state response to a generic input $u(t)$ can be computed as the convolution of impulsive response and $u(t)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

with impulse input →

$$x(t) = e^{At}x(0) + e^{At}B$$

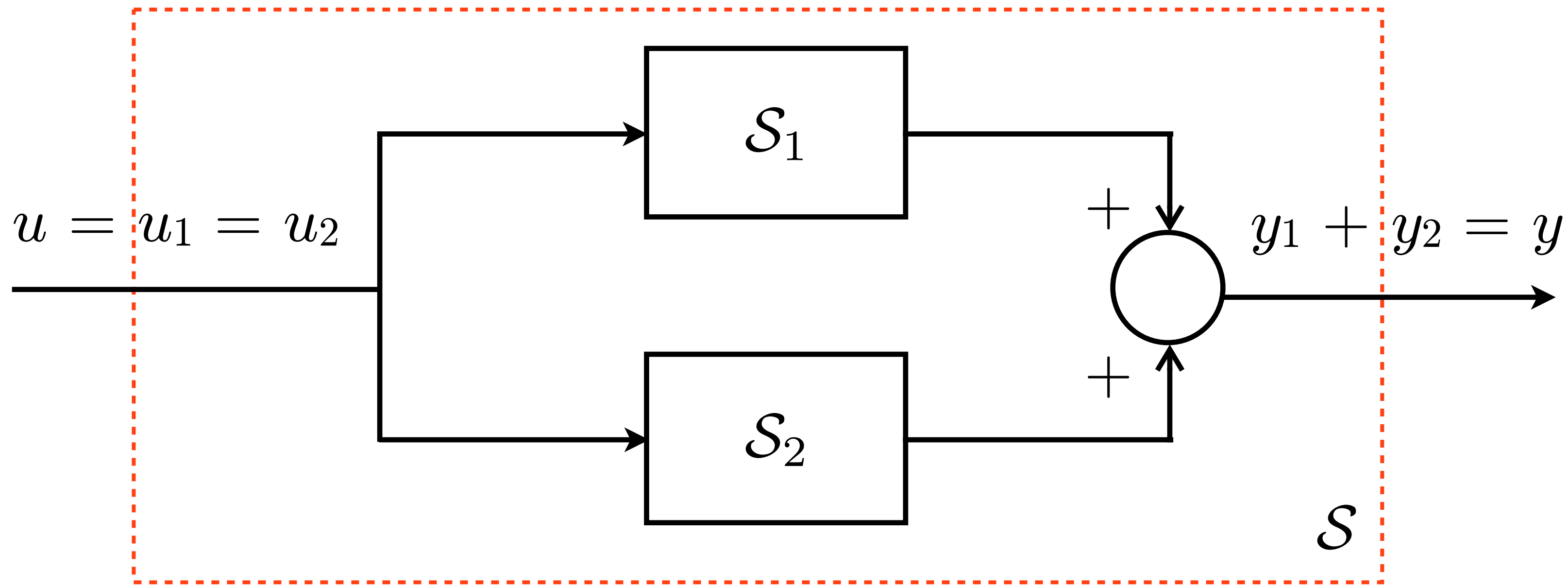
$$y(t) = Ce^{At}x(0) + Ce^{At}B$$

$e^{At}B$ displays all the controllable natural modes

Ce^{At} displays all the observable natural modes

$Ce^{At}B$ displays all the controllable and observable natural modes

parallel (state space)



$$\mathcal{S}_1 : \quad \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 + D_1 u_1 \end{cases}$$

$$\mathcal{S}_2 : \quad \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 + D_2 u_2 \end{cases}$$

$\mathcal{S}$ with state $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ input u and output y

interconnection equations

$$y = y_1 + y_2, \quad u = u_1 = u_2$$

- parallel system state **space representation**

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 x_1 + B_1 u_1 \\ A_2 x_2 + B_2 u_2 \end{pmatrix} = \begin{pmatrix} A_1 x_1 + B_1 u \\ A_2 x_2 + B_2 u \end{pmatrix} \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = Ax + Bu \\ y &= y_1 + y_2 = C_1 x_1 + D_1 u_1 + C_2 x_2 + D_2 u_2 = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (D_1 + D_2) u \\ &= Cx + Du \end{aligned}$$

parallel (state space)

parallel system has dynamic matrix

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \xrightarrow{\text{block diagonal}} \text{eig}\{A\} = \text{eig}\{A_1\} \bigcup \text{eig}\{A_2\}$$

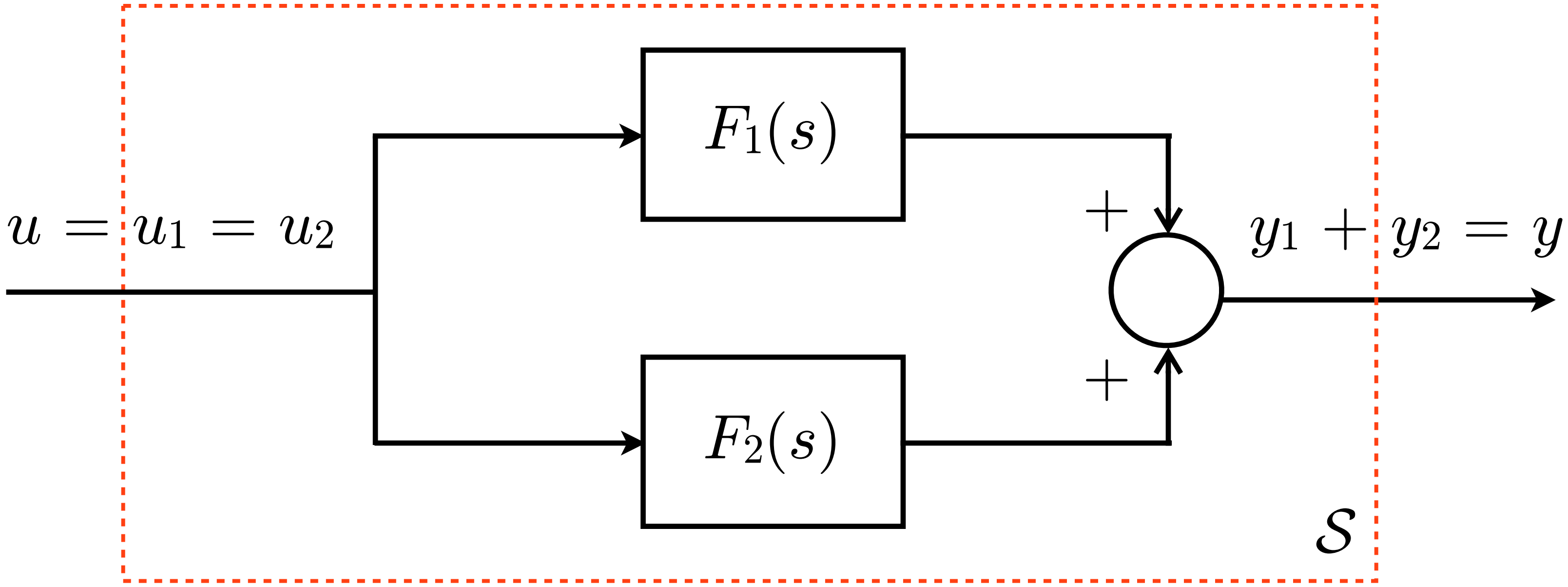
$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \quad D = D_1 + D_2$$

in general, the eigenvalues of the parallel of subsystems are given by the **union** of the single subsystem's eigenvalues

no new time behaviors can appear

- the parallel of asymptotically stable systems is asymptotically stable
- if one of the system in the parallel interconnection is unstable, so is the whole system
- the parallel of a marginally stable system and an asymptotically stable system is marginally stable
- the parallel of two marginally stable systems is marginally stable

parallel (transfer function)



$$\mathcal{S}_1: \quad F_1(s) = \frac{y_1(s)}{u_1(s)}$$

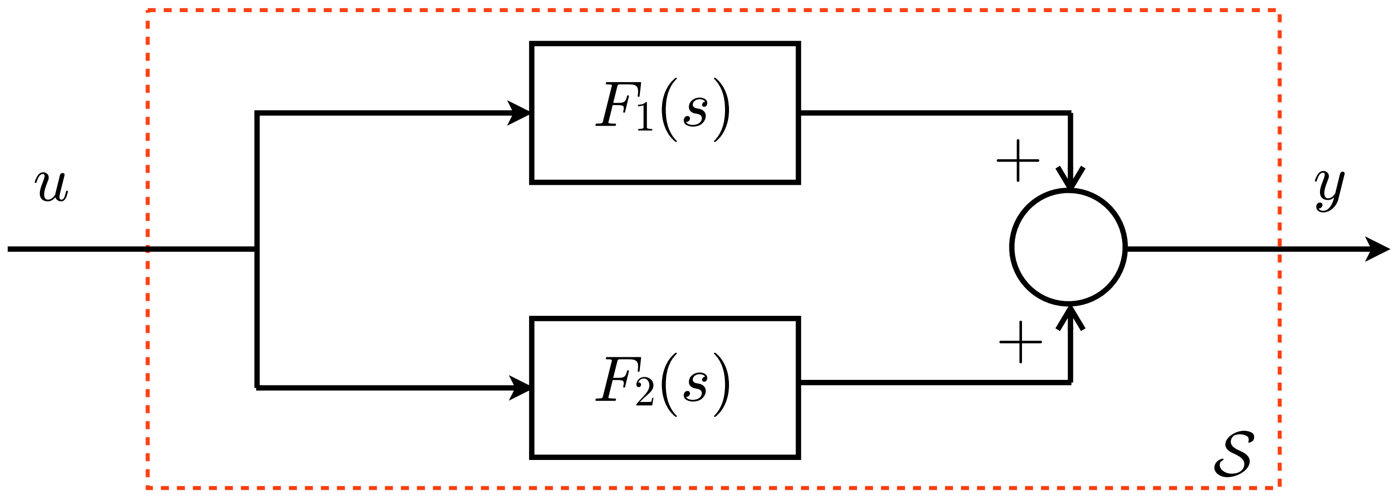
$$\mathcal{S}_2: \quad F_2(s) = \frac{y_2(s)}{u_2(s)}$$

for every subsystem $\mathcal{S}_i$ we assume coincidence of eigenvalues and poles

$$F(s) = \frac{y(s)}{u(s)} = \frac{y_1(s) + y_2(s)}{u(s)} = \frac{y_1(s)}{u(s)} + \frac{y_2(s)}{u(s)} = \frac{y_1(s)}{u_1(s)} + \frac{y_2(s)}{u_2(s)} = F_1(s) + F_2(s)$$

transfer function of systems in **parallel** **add** together

example (cancellations)



$$\mathcal{S}_1 : \quad F_1(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1}$$

$$\mathcal{S}_2 : \quad F_2(s) = \frac{1}{s+1}$$

$$F(s) = F_1(s) + F_2(s) = \frac{s-1}{s+1} + \frac{1}{s+1} = \frac{s}{s+1} = 1 - \frac{1}{s+1}$$

only 1 pole but 2 eigenvalues

since there is a cancellation (creation of a hidden dynamics) we need to look at the state-space representation to understand if it's a loss of controllability or observability

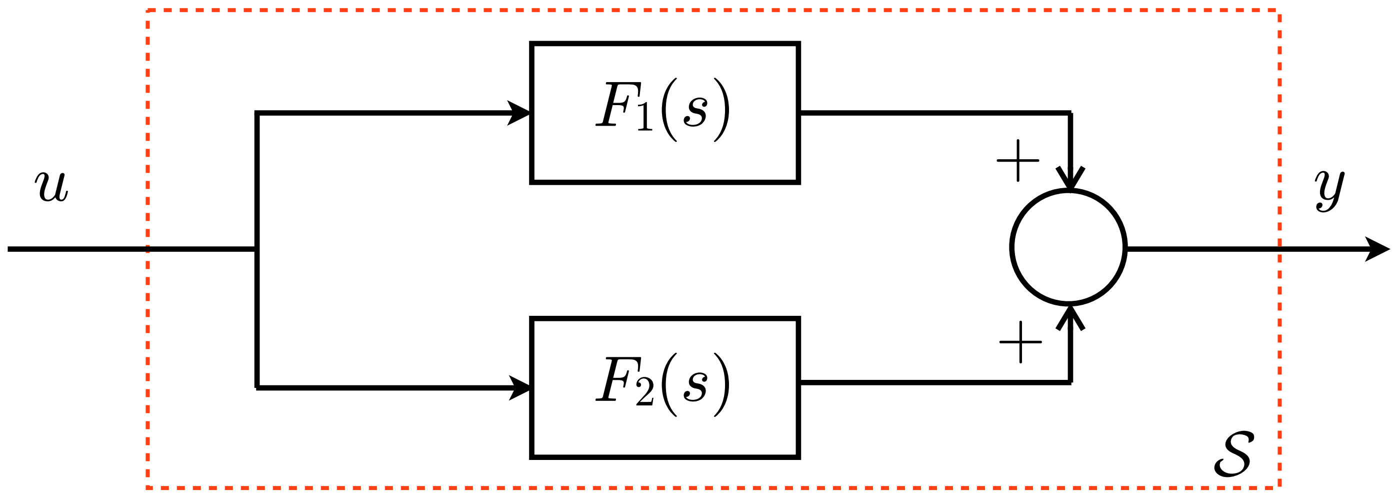
we first realize each subsytem and the interconnect them

$$\mathcal{S}_1 : \quad A_1 = -1, \quad B_1 = 1, \quad C_1 = -2, \quad D_1 = 1$$

$$\mathcal{S}_2 : \quad A_2 = -1, \quad B_2 = 1, \quad C_2 = 1, \quad D_2 = 0$$

$$\mathcal{S} : \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 \end{pmatrix}, \quad D = D_1 = 1$$

example (cancellations)



$$\mathcal{S}_1: \qquad F_1(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1}$$

$$\mathcal{S}_2: \qquad F_2(s) = \frac{1}{s+1}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 \end{pmatrix}, \quad D = D_1 = 1$$

PBH test for controllability and observability for $\lambda =$ -1

$$\text{rk}\begin{pmatrix} A - \lambda I & B \end{pmatrix} = \text{rk}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1 < n = 2 \quad \Rightarrow \quad \lambda = -1$$ **uncontrollable**

$$\text{rk}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \text{rk}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 1 \end{pmatrix} = 1 < n = 2 \quad \Rightarrow \quad \lambda = -1$$ **unobservable**

the **parallel** interconnection has generated, for the given example, an **unobservable** and **uncontrollable** eigenvalue and corresponding natural mode mode e^{-t}

Let two systems $F_1(s)$ and $F_2(s)$ have a common pole p_i

$$F_1(s) = \frac{N_1(s)}{D_1(s)} = \frac{N_1(s)}{(s-p_i)D_1'(s)} \qquad F_2(s) = \frac{N_2(s)}{D_2(s)} = \frac{N_2(s)}{(s-p_i)D_2'(s)}$$

put in evidence the common pole

parallel $\Rightarrow$

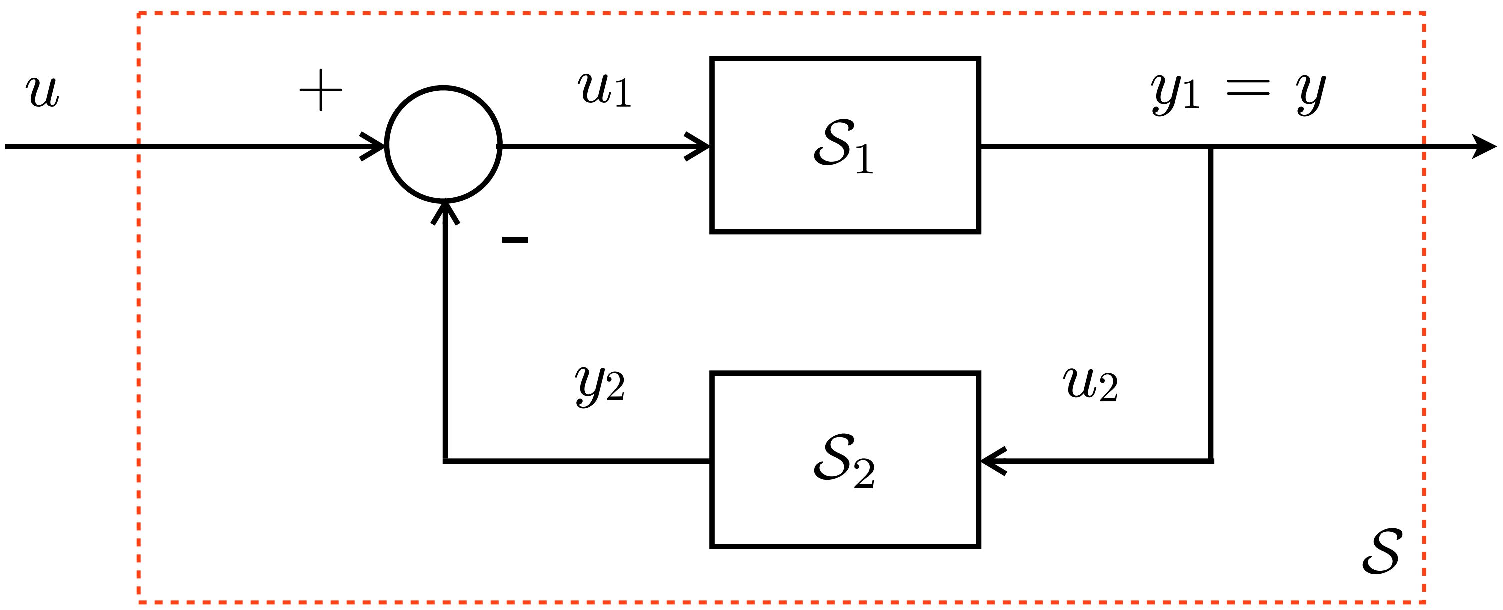
$$F(s) = F_1(s) + F_2(s) = \frac{N_1(s)}{(s-p_i)D_1'(s)} + \frac{N_2(s)}{(s-p_i)D_2'(s)}$$

$$= \frac{N_1(s)D_2'(s) + N_2(s)D_1'(s)}{(s-p_i)D_1'(s)D_2'(s)}$$

degree has been lowered by 1

In general if two systems have **eigenvalues** (poles) in **common** then in the parallel interconnection we generate an **unobservable** and **uncontrollable** hidden dynamics (here with dynamics characterized by the eigenvalue p_i)

feedback (state space)



$$\mathcal{S}_1 : \begin{cases} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1 + D_1 u_1 \end{cases}$$

$$\mathcal{S}_2 : \begin{cases} \dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y_2 &= C_2 x_2 + D_2 u_2 \end{cases}$$

assume D_1 and D_2 equal to 0
(special case, other cases as exercises)

$\mathcal{S}$ with state $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ input u and output y

interconnection equations

$$u_1 = u - y_2, \quad y = y_1 = u_2$$

- **state space representation** of the feedback interconnection of the two system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 x_1 + B_1 (u - y_2) \\ A_2 x_2 + B_2 y_1 \end{pmatrix} = \begin{pmatrix} A_1 x_1 - B_1 C_2 x_2 + B_1 u \\ A_2 x_2 + B_2 C_1 x_1 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u = Ax + Bu \\ y &= y_1 = C_1 x_1 = \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx \end{aligned}$$

since we are feeding back the output (measured variable) it is also called an **output feedback**

feedback (state space)

feedback system has dynamics matrix

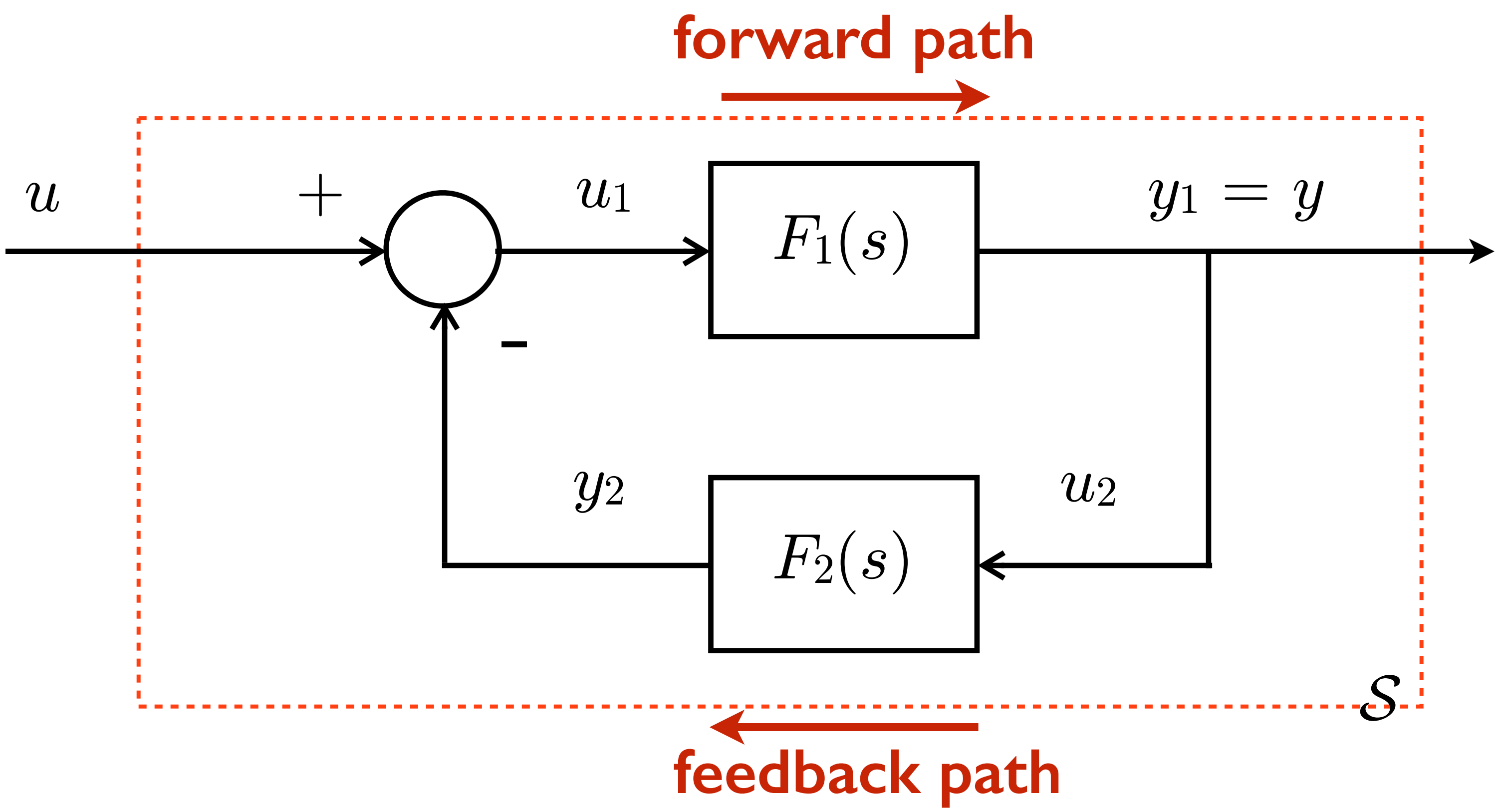
$$A = \begin{pmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix} \xrightarrow{\text{no special structure}} \mathrm{eig}\{A\} \neq \mathrm{eig}\{A_1\} \bigcup \mathrm{eig}\{A_2\}$$

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} C_1 & 0 \end{pmatrix} \quad D = 0$$

in general, the **eigenvalues** of the feedback of two subsystems **differ** from those of the single subsystems

new time behaviors usually appear

feedback (transfer function)



$$\mathcal{S}_1 : \quad F_1(s) = \frac{y_1(s)}{u_1(s)}$$

$$\mathcal{S}_2 : \quad F_2(s) = \frac{y_2(s)}{u_2(s)}$$

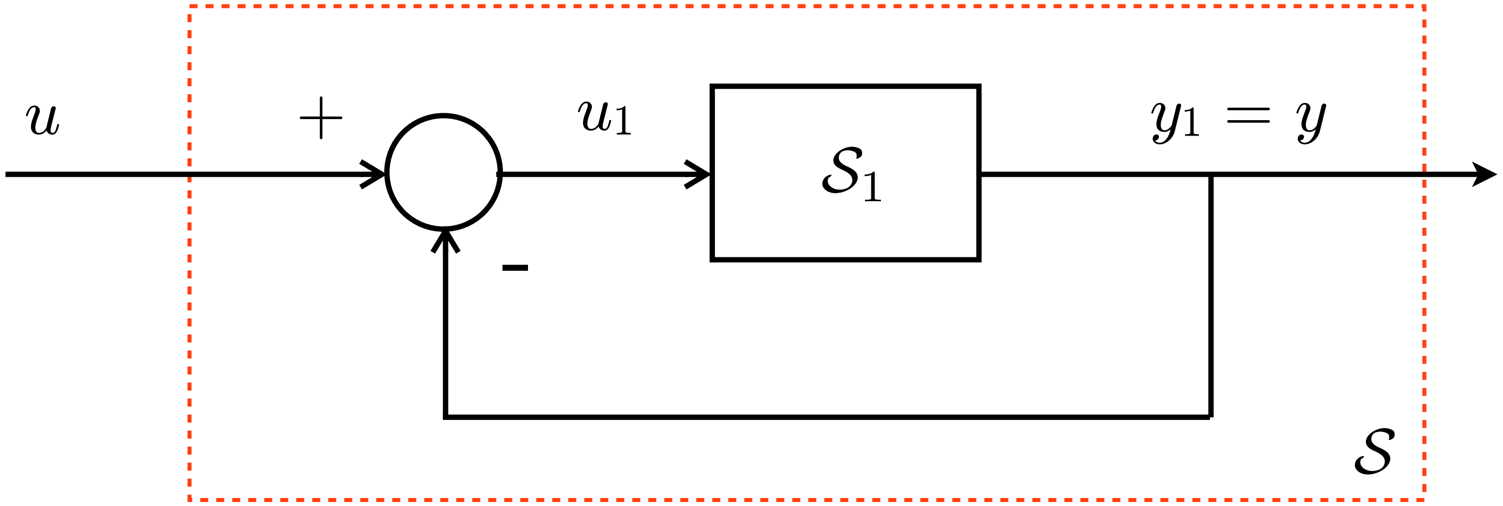
for every subsystem $\mathcal{S}_i$ we assume coincidence of eigenvalues and poles (**no hidden dynamics**)

$$\begin{aligned} y(s) &= y_1(s) = F_1(s)[u(s) - y_2(s)] = F_1(s)[u(s) - F_2(s)u_2(s)] \\ &= F_1(s)[u(s) - F_2(s)y(s)] \\ &\Rightarrow [1 + F_1(s)F_2(s)]y(s) = F_1(s)u(s) \end{aligned}$$

$F_1(s)F_2(s)$ is called **loop function**

$$F(s) = \frac{y(s)}{u(s)} = \frac{F_1(s)}{1 + F_1(s)F_2(s)}$$

unit (negative) feedback



$$\mathcal{S}_1 : \quad \begin{cases} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1 \end{cases}$$

$$\mathcal{S}_1 : \quad F_1(s) = \frac{y_1(s)}{u_1(s)}$$

$\mathcal{S}_1$ **open-loop** system

$\mathcal{S}$ **closed-loop** system

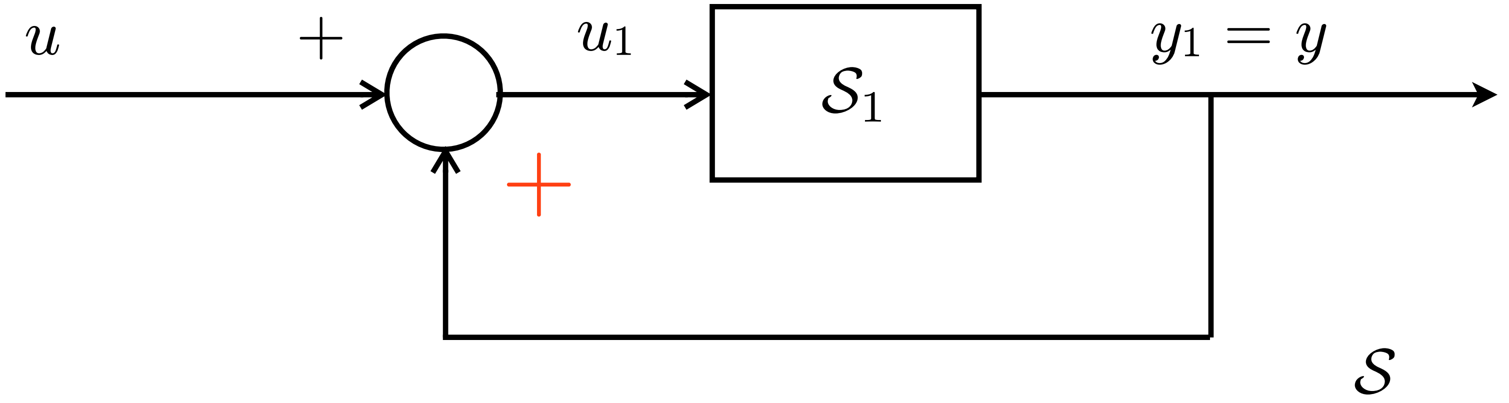
$$u_1 = u - y_1, \quad y = y_1$$

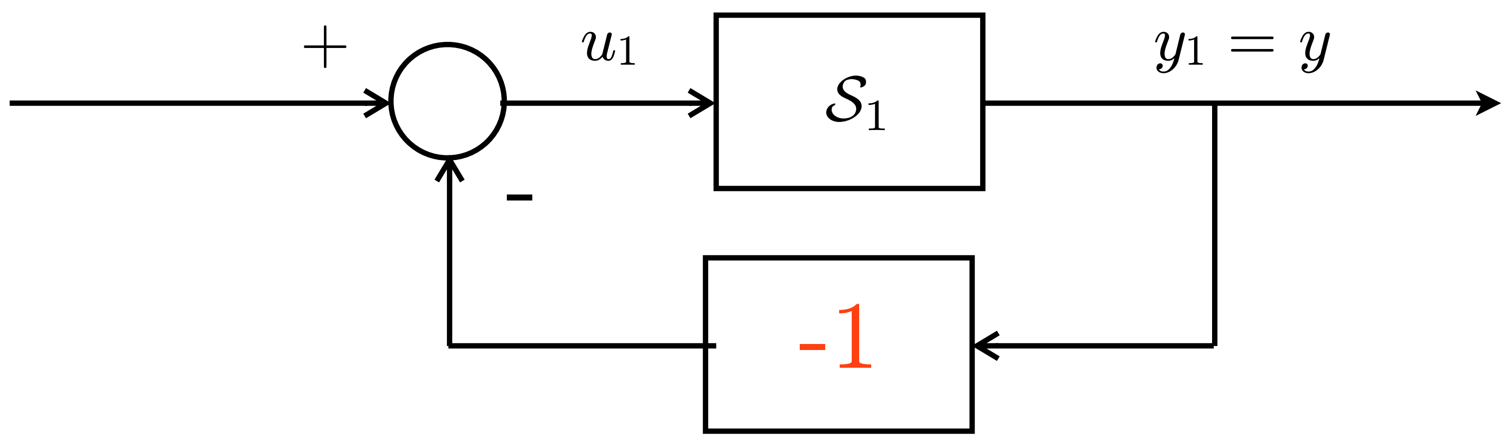
$$\begin{aligned} \dot{x} &= \dot{x}_1 = A_1 x_1 + B_1(u - y_1) = A_1 x_1 - B_1 C_1 x_1 + B_1 u \\ &= (A_1 - B_1 C_1)x + B_1 u = Ax + Bu \\ y &= y_1 = C_1 x_1 = Cx \end{aligned}$$

$$\text{eig}\{A\} \neq \text{eig}\{A_1\} \longleftarrow A = A_1 - B_1 C_1 \qquad B = B_1 \qquad C = C_1$$

$$F(s) = \frac{y(s)}{u(s)} = \frac{F_1(s)}{1 + F_1(s)}$$

unit positive feedback





positive feedback system

$$F(s) = \frac{y(s)}{u(s)} = \frac{F_1(s)}{1 - F_1(s)}$$

change

stability of the closed loop system

example (unit feedback):

$$F_1(s) = \frac{2}{s-1} \quad \Rightarrow \quad F(s) = \frac{F_1(s)}{1+F_1(s)} = \frac{2/(s-1)}{1+2/(s-1)} = \frac{2}{s-1+2} = \frac{2}{s+1}$$

open-loop unstable → closed-loop asymptotically stable

$$F_2(s) = \frac{s-3}{s^2+s+1} \qquad F(s) = \frac{F_2(s)}{1+F_2(s)} = \frac{s-3}{s^2+2s-2}$$

open-loop asymptotically stable → closed-loop unstable

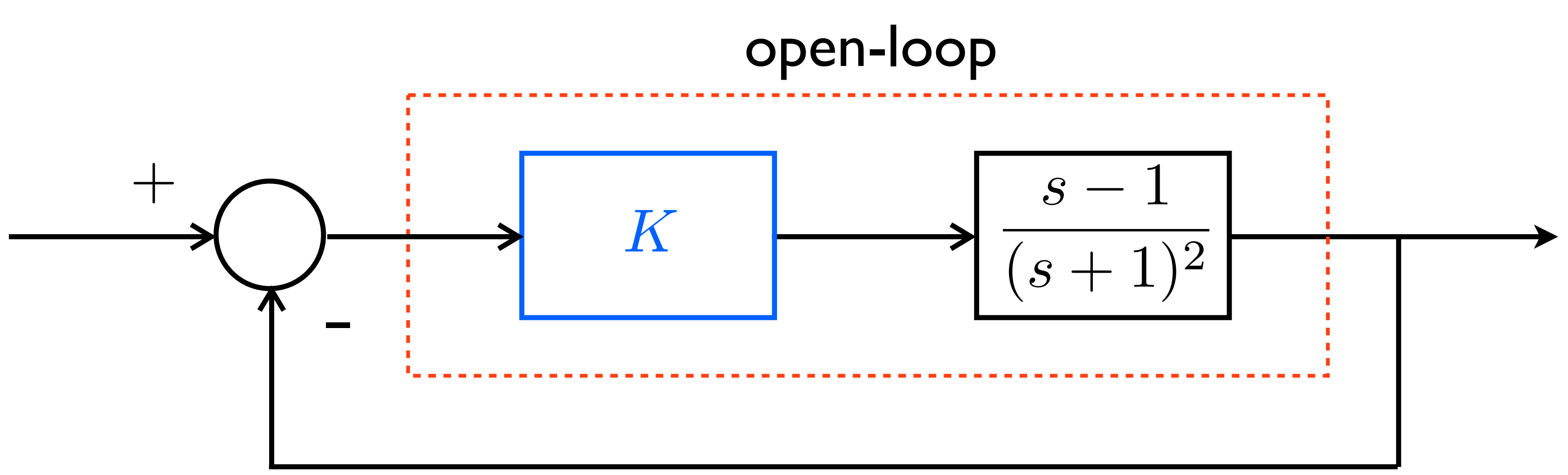
stability of the closed loop system

example (unit feedback):

$$F_3(s) = \frac{K(s-1)}{(s+1)^2} \qquad F(s) = \frac{F_3(s)}{1+F_3(s)} = \frac{K(s-1)}{s^2 + s(2+K) + 1 - K}$$

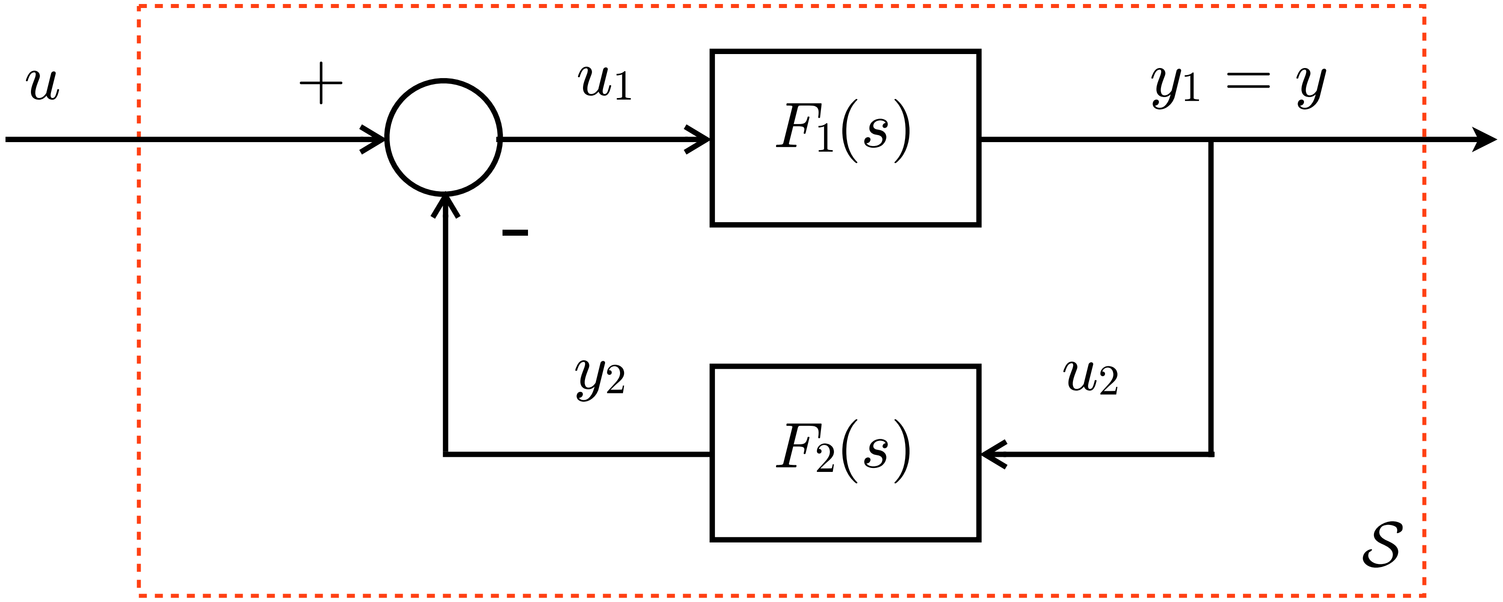
open-loop → closed-loop

- asymptotically stable for $-2 < K < 1$
- marginally stable for $K = 1$ or $K = -2$
- unstable in all other cases



K could be seen as a design parameter (controller)

feedback (cancellations)



$$F(s) = \frac{y(s)}{u(s)} = \frac{F_1(s)}{1 + F_1(s)F_2(s)}$$

we may have cancellations here

- if a **zero** of $F_1(s)$ cancels out with a **pole** of $F_2(s)$ (zero/pole cancellation)

$$F_1(s) = \frac{(s+a)N_1'(s)}{D_1(s)} \qquad F_2(s) = \frac{N_2(s)}{(s+a)D_2'(s)}$$

$F_1(s)$: n_1 poles

$F_2(s)$: n_2 poles

$$F(s) = \frac{(s+a)^2 N_1'(s)D_2'(s)}{(s+a)[D_1(s)D_2'(s) + N_1'(s)N_2(s)]}$$

$$= \frac{(s+a)N_1'(s)D_2'(s)}{D_1(s)D_2'(s) + N_1'(s)N_2(s)}$$

$\rightarrow$ $F(s)$: $n_1 + n_2$ - 1 poles

the cancelled pole of F_2 becomes a hidden eigenvalue

feedback (cancellations)

- if a **pole** of $F_1(s)$ cancels out with a **zero** of $F_2(s)$ (pole/zero cancellation)

$$F_1(s) = \frac{N_1(s)}{(s+a)D_1'(s)} \qquad F_2(s) = \frac{(s+a)N_2'(s)}{D_2(s)}$$

$F_1(s)$: n_1 poles
$F_2(s)$: n_2 poles

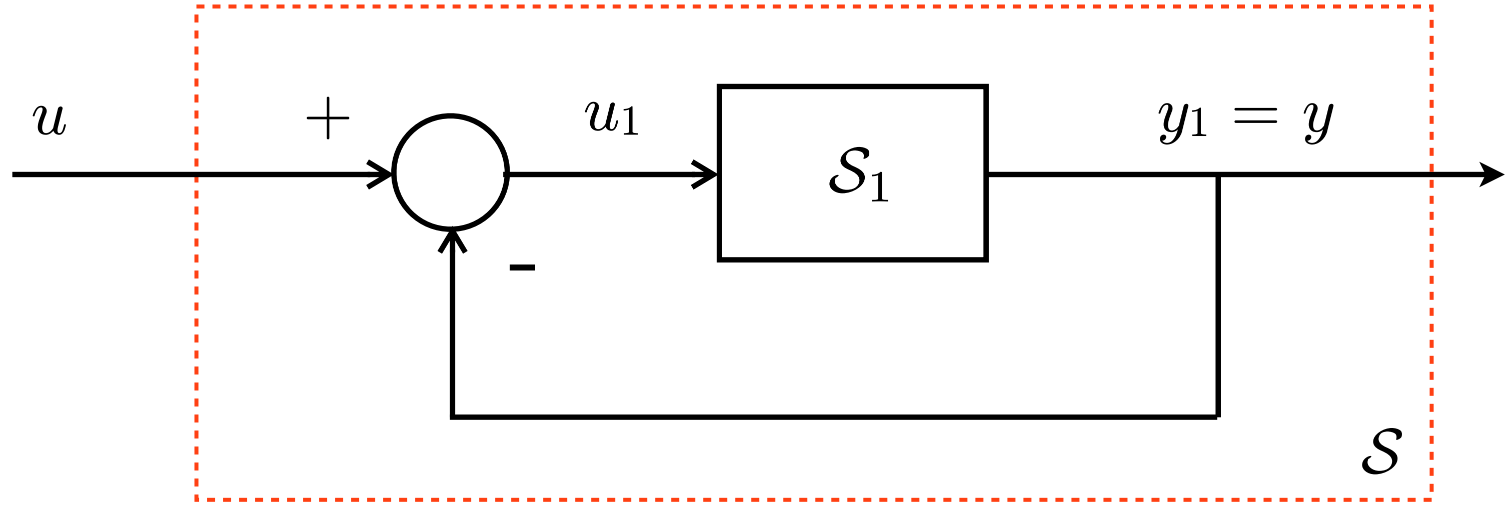
||

$$F(s) = \frac{N_1(s)D_2(s)}{(s+a)[D_1'(s)D_2(s) + N_1(s)N_2'(s)]}$$

$F(s)$: $n_1 + n_2$ poles

- if a **zero** λ_c of $F_1(s)$ cancels out with a **pole** λ_c of $F_2(s)$ (zero/pole cancellation) we have generated **uncontrollable** and **unobservable** hidden dynamics characterized by the eigenvalue λ_c
- if a **pole** λ_c of $F_1(s)$ cancels out with a **zero** λ_c of $F_2(s)$ (pole/zero cancellation) there are **no hidden dynamics** but the pole λ_c remains unchanged at closed-loop

feedback (cancellations)



what happens if the open-loop system $\mathcal{S}_1$ has hidden modes?

(i.e., for the open-loop system, not all the eigenvalues become poles)

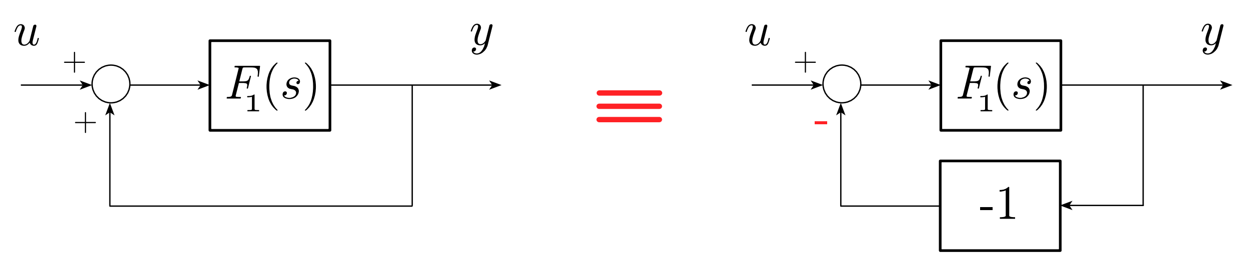
$$F_1(s) = \frac{(s+a)N_1'(s)}{(s+a)D_1'(s)}$$

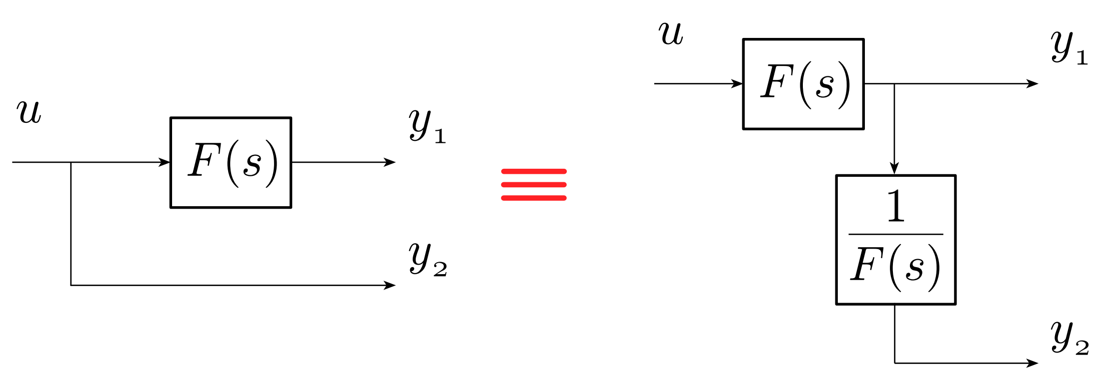
$$\Rightarrow \quad F(s) = \frac{(s+a)N_1'(s)}{(s+a)N_1'(s)+(s+a)D_1'(s)} = \frac{(s+a)N_1'(s)}{(s+a)[N_1'(s)+D_1'(s)]}$$

$$= \frac{N_1'(s)}{N_1'(s)+D_1'(s)}$$

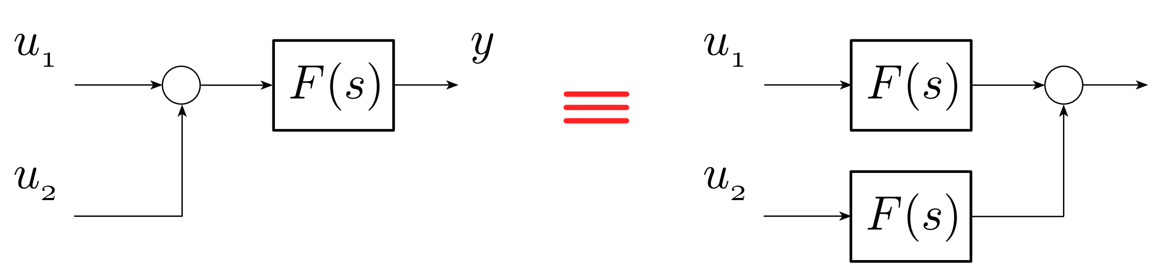
- in a unit feedback system, the **closed-loop** system has **hidden modes** if and only if the open-loop has them
- the open-loop hidden modes are inherited **unchanged** by the closed-loop

some useful block manipulations

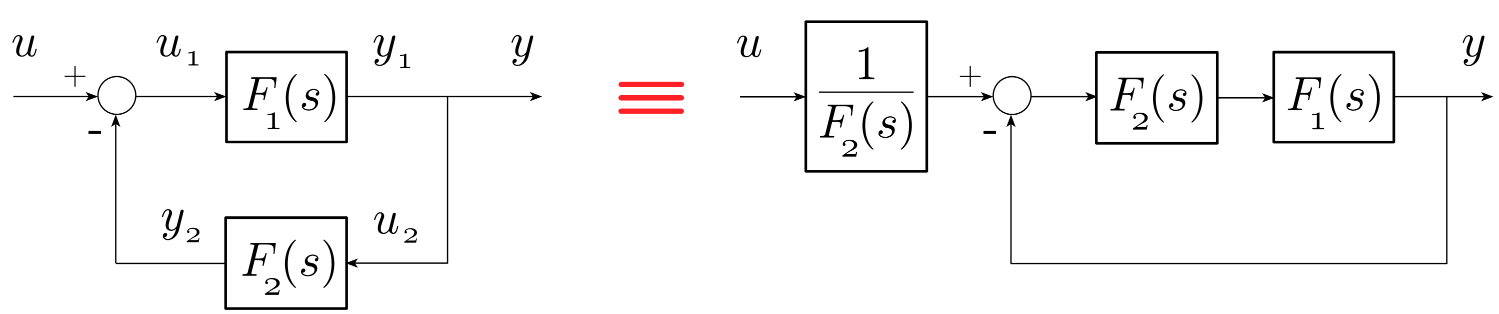




Att.:
these are purely algebraic block manipulations and do not correspond to real systems manipulation (compare, for example, systems dimension)

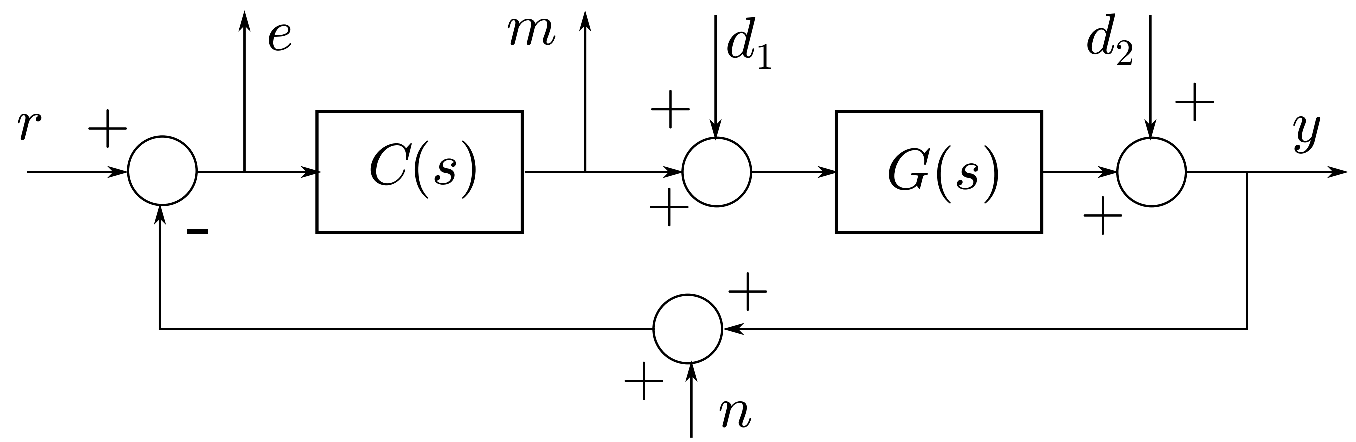


$$
\begin{aligned}
y(s) &= F(s)(u_1(s) + u_2(s)) \\
&= F(s)u_1(s) + F(s)u_2(s)
\end{aligned}
$$



equivalence can be easily shown comparing the signals (their Laplace transforms)

fundamental transfer functions



the superposition principle allows us to compute separately each contribution to the chosen output

$$\begin{aligned} y(s) &= T(s)r(s) + P(s)S(s)d_1(s) + S(s)d_2(s) - T(s)n(s) \\ e(s) &= S(s)r(s) - P(s)S(s)d_1(s) - S(s)d_2(s) - S(s)n(s) \\ m(s) &= S_u(s)r(s) - T(s)d_1(s) - S_u(s)d_2(s) - S_u(s)n(s) \end{aligned}$$

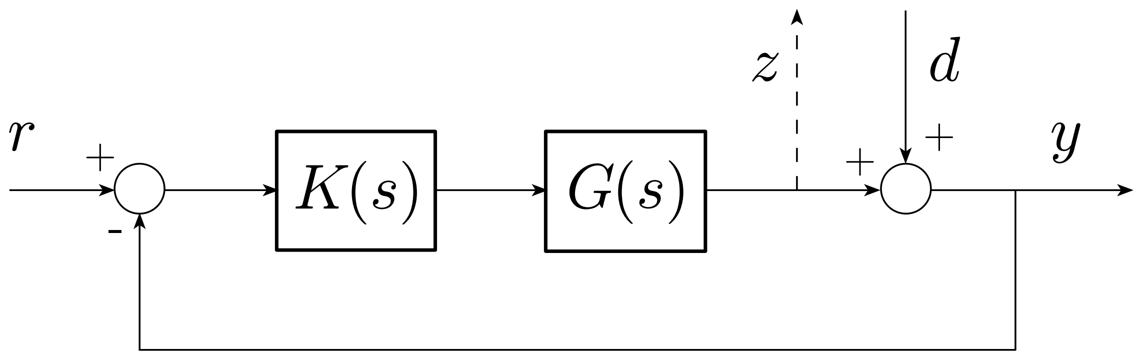
where

$$S(s) = \frac{1}{1+G(s)C(s)} = \frac{1}{1+L(s)}$$ **sensitivity function**

$$T(s) = \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{L(s)}{1+L(s)}$$ **complementary sensitivity function**

$$S_u(s) = \frac{C(s)}{1+G(s)C(s)} = \frac{C(s)}{1+L(s)}$$ **control sensitivity function**

example I



imagine that, for the feedback system shown in figure, we are interested in analyzing the effect of the input d (disturbance) on the output of the system $G(s)$, that is on z

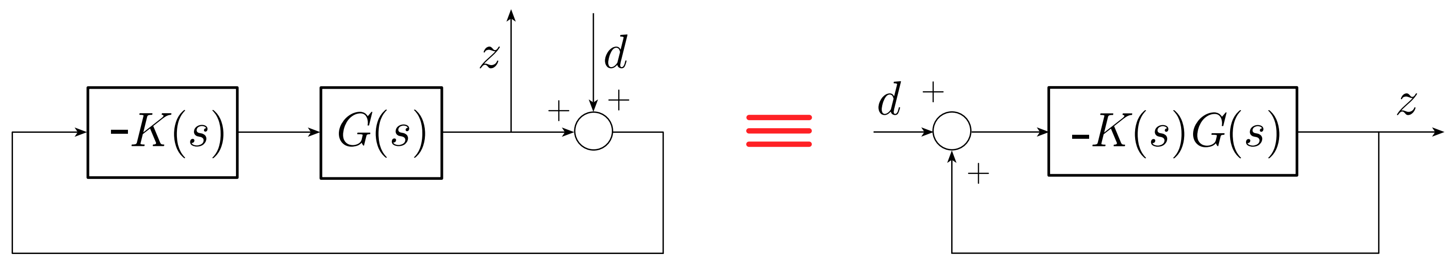
the **superposition principle** allows us to compute separately the contribution to $z(s)$ of $d(s)$ and the contribution of $r(s)$

$$z(s) = W_{dz}(s)d(s) + W_{rz}(s)r(s)$$

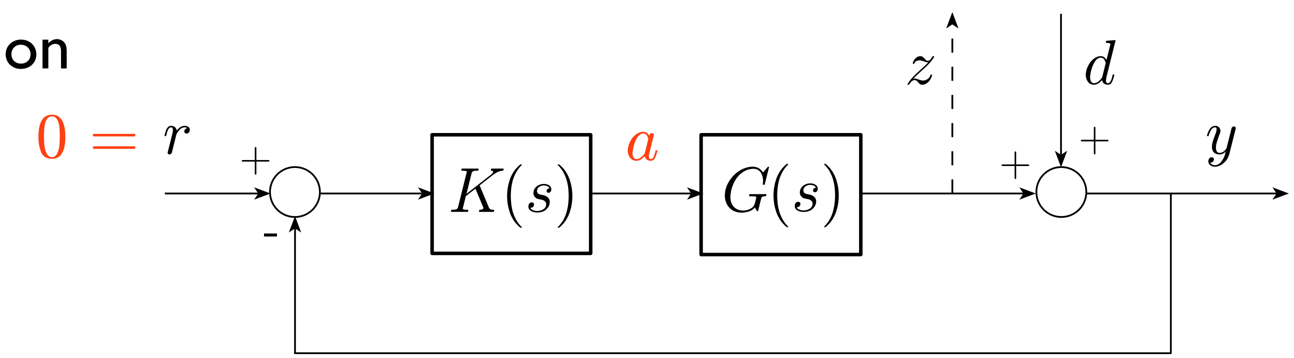
- we can isolate the effect of d on z by setting the other inputs (here only r) to zero and derive the transfer function $W_{dz}(s)$

- in order to obtain $W_{dz}(s)$ we can either manipulate, using the previous blocks manipulation rules, the feedback system or proceed algebraically

- block manipulation (with a little imagination)



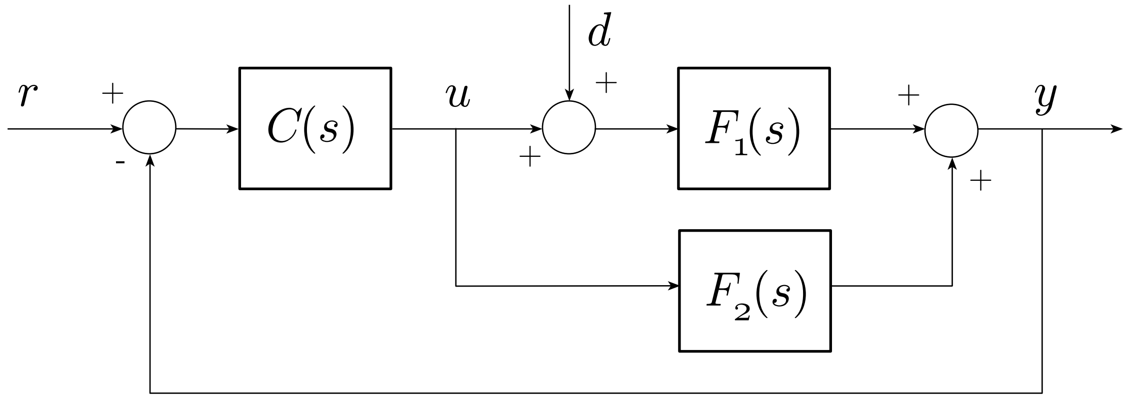
- algebraic solution



$$z(s) = G(s)a(s) = -\, G(s)K(s)y(s) = -\, G(s)K(s)[d(s) + z(s)]$$

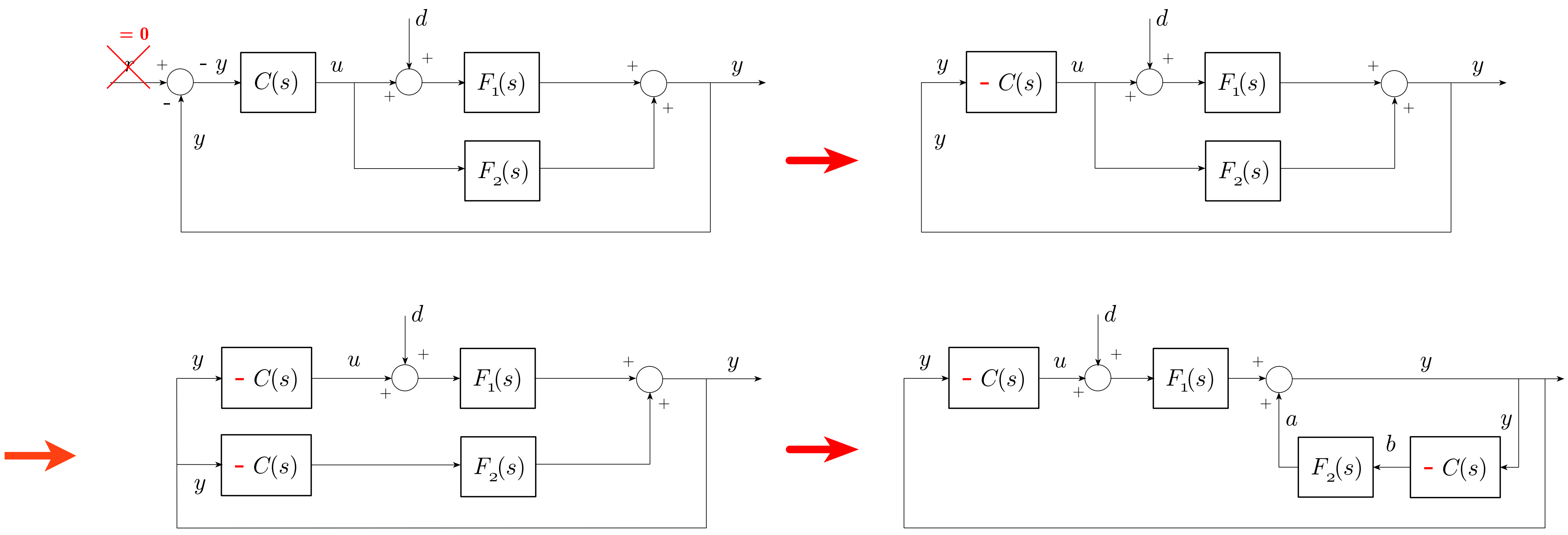
$$W_{dz}(s) = \frac{z(s)}{d(s)} = \frac{-K(s)G(s)}{1 - [-K(s)G(s)]} = -\frac{K(s)G(s)}{1 + K(s)G(s)}$$

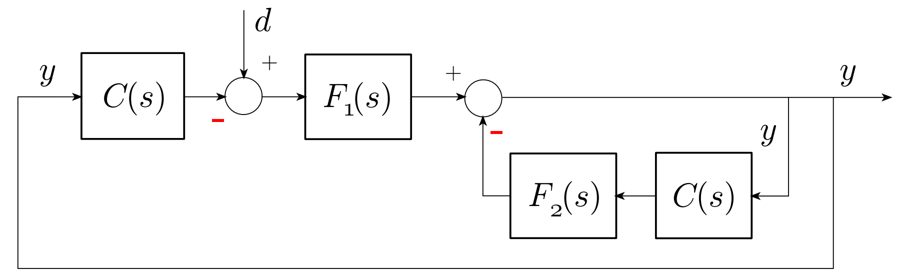
example II

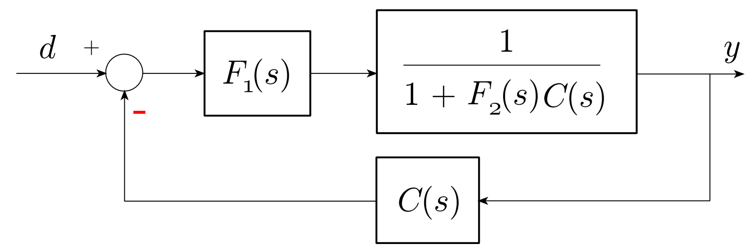


find the two transfer functions $W_{dy}(s)$ and $W_{ry}(s)$

- block manipulations







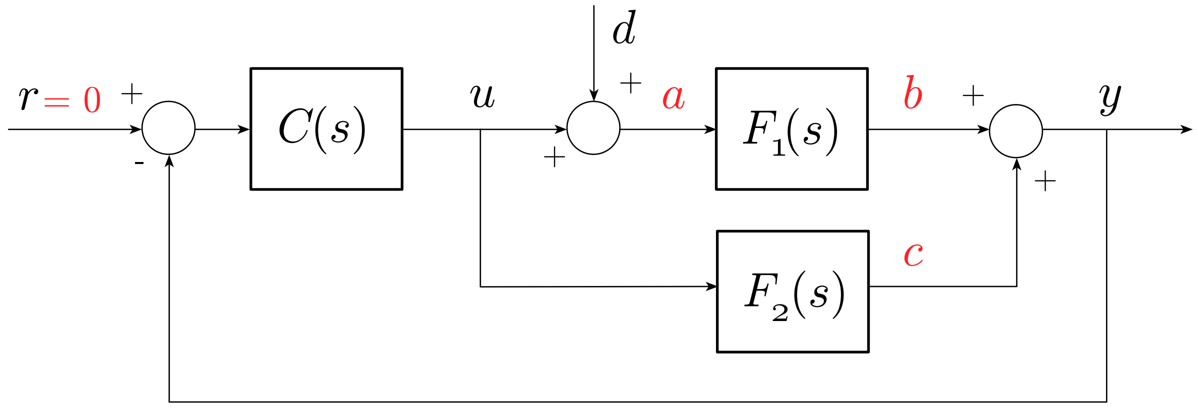
we know the formula for this scheme

$$W_{dy}(s) = \frac{F_1(s)}{1 + [F_1(s) + F_2(s)]\, C(s)}$$

and analogously

$$W_{ry}(s) = \frac{[F_1(s) + F_2(s)]\, C(s)}{1 + [F_1(s) + F_2(s)]\, C(s)}$$

- algebraic solution
  - identify all the signals which appear in the interconnected system
  - write down the relationships between these signals in the s domain (we are considering only forced responses so we use the simple relationship between the input, the output and the transfer function)
  - solve for the ratio output/input which characterizes the sought transfer function



$$
\begin{aligned}
y(s) &= b(s) + c(s) \\
b(s) &= F_1(s)a(s) \\
c(s) &= F_2(s)u(s) \\
a(s) &= u(s) + d(s) \\
u(s) &= -C(s)y(s)
\end{aligned}
$$

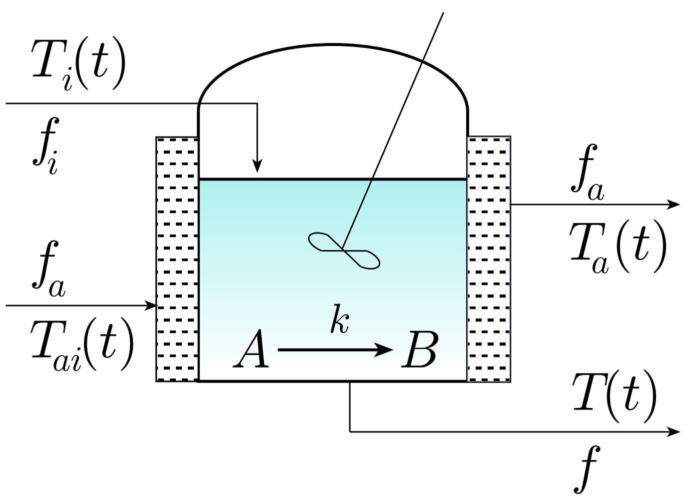
$$
\begin{aligned}
y &= b + c = F_2 u + F_1(u + d) \\
&= F_1 d - (F_1 + F_2)Cy
\end{aligned}
$$

solve for y/d

example III

Consider the chemical reactor modeled as a continuously stirred tank (CSTR) where an exothermic reaction $A \rightharpoonup B$ occurs.

In order to remove the heat of the reaction, the reactor is surrounded by a jacket in which a cooling liquid flows with flow f_a



T reactor temperature

T_{ai} T_a jacket input and output temperatures

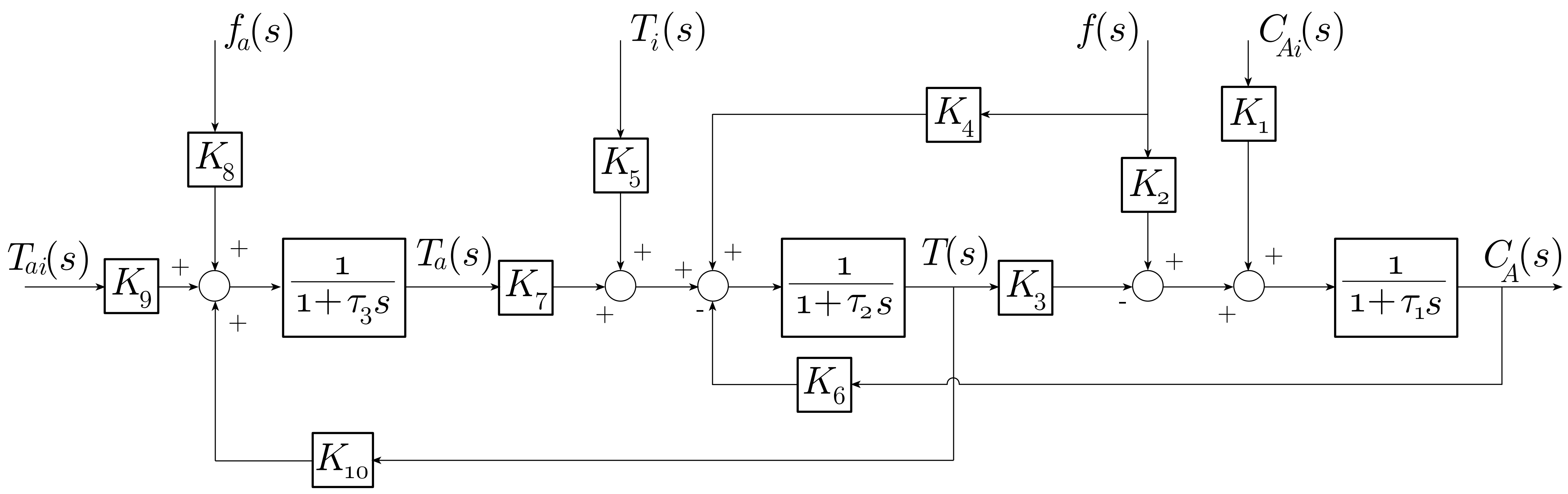
f_i f reactor inlet and outlet flow

C_{Ai} C_A inlet and outlet concentrations of A

simplified model

$$
\begin{aligned}
C_A(s) &= \frac{1}{1+\tau_1 s}(K_1 C_{Ai}(s) + K_2 f(s) - K_3 T(s)) \\
T(s) &= \frac{1}{1+\tau_2 s}(K_4 f(s) + K_5 T_i(s) - K_6 C_A(s) + K_7 T_a(s)) \\
T_a(s) &= \frac{1}{1+\tau_3}(K_8 f_a(s) + K_9 T_{ai}(s) + K_{10} T(s))
\end{aligned}
$$

example III



Finding how the different inputs contribute to the output could be a useful exercise