Control Systems - January 7, 2019

Text & short solution. (There may be typos, and the solution is just a sketch.)

1) Consider the following plant

$$\dot{x}_1 = 2x_1 - 3x_2 + 3u$$
$$\dot{x}_2 = x_1 - 2x_2 + u$$
$$y = 2x_1 - 2x_2$$

1. Determine which natural modes are controllable and/or observable.
2. Is the plant stabilizable from the output with a static gain? In case of a positive answer, determine the controller.
3. Is the plant stabilizable with a dynamic controller based on the separation principle? In case of a positive answer, determine the controller.
   - decompose w.r.t. controllability, choose $\tilde{F}$ to assign the desired eigenvalues and go back to the original coordinates;
   - decompose w.r.t. observability, choose $\tilde{K}$ and write the observer in the original coordinates;
   - determine the controller transfer function (from y to u).
4. Is it possible to stabilize the given system with a controller of the form

   $$C_3(s) = \frac{k_c}{s+p}$$

   If possible, determine k_c and p and give an interpretation in terms of root locus.

1 - Solution

• Eigenvalues

$$\det\begin{pmatrix}\lambda-2 & 3\\ -1 & \lambda+2\end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1) \qquad \Rightarrow \qquad \lambda_1 = -1, \quad \lambda_2 = 1$$

PBH test

$$\text{rk}\begin{pmatrix}\lambda-2 & 3 & 3\\ -1 & \lambda+2 & 1\end{pmatrix}_{\lambda_1=-1} = \text{rk}\begin{pmatrix}-3 & 3 & 3\\ -1 & 1 & 1\end{pmatrix} = 1, \qquad \text{rk}\begin{pmatrix}\lambda-2 & 3 & 3\\ -1 & \lambda+2 & 1\end{pmatrix}_{\lambda_2=1} = \text{rk}\begin{pmatrix}-1 & 3 & 3\\ -1 & 3 & 1\end{pmatrix} = 2$$

$$\mathrm{rk}\begin{pmatrix}\lambda-2 & 3\\ -1 & \lambda+2\\ 2 & -2\end{pmatrix}_{\lambda_1=-1} = \mathrm{rk}\begin{pmatrix}-3 & 3\\ -1 & 1\\ 2 & -2\end{pmatrix} = 1, \qquad \mathrm{rk}\begin{pmatrix}\lambda-2 & 3\\ -1 & \lambda+2\\ 2 & -2\end{pmatrix}_{\lambda_2=1} = \mathrm{rk}\begin{pmatrix}-1 & 3\\ -1 & 3\\ 2 & -2\end{pmatrix} = 2$$

therefore $\lambda_1 = -1$, and the corresponding natural mode e^{-t}, is both uncontrollable and unobservable while $\lambda_2 = 1$, and the corresponding natural mode e^t, is both controllable and observable. As an alternative one could answer this question after the decomposition w.r.t. controllability and observability.

• (Alternative) Checking the rank of the controllability matrix (see next) which turns out to be 1, from the PBH test for only one eigenvalue one understands the property of the other. For example, being $\lambda_1 = -1$ uncontrollable, the other eigenvalue will for sure be controllable (the uncontrollable subsystem has dimension equal to the rank of the controllability matrix).

• Since only λ_2 is both controllable and observable the transfer function will contain the only pole $p = \lambda_2$ and no zero[1] so we have a minimum-phase system with $n - m = 1$ and therefore it is clearly stabilizable with high gain. Check

$$\begin{aligned}
P(s) &= \begin{pmatrix}2 & -2\end{pmatrix}\begin{pmatrix}s-2 & 3\\ -1 & s+2\end{pmatrix}^{-1}\begin{pmatrix}3\\ 1\end{pmatrix} = \frac{1}{(s+1)(s-1)}\begin{pmatrix}2 & -2\end{pmatrix}\begin{pmatrix}s+2 & -3\\ 1 & s-2\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}\\
&= \frac{1}{(s+1)(s-1)}\begin{pmatrix}2s+2 & -2s-2\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix} = \frac{4s+4}{(s+1)(s-1)} = \frac{4}{s-1}
\end{aligned}$$

A static controller $C(s) = K$ will lead to the closed loop polynomial equal to

$$s - 1 + 4K$$

which has a negative pole if $K > 1/4$. Choose a value for K (it was required to determine a controller).

• System is both stabilizable (via state feedback) and detectable (from the output), therefore it is stabilizable with a dynamic controller (for example based on the separation principle) using only the output as feedback measurement (output feedback stabilization).

• Kalman decomposition w.r.t. controllability

$$P = \begin{pmatrix}3 & 3\\ 1 & 1\end{pmatrix} \quad\Rightarrow\quad \mathcal{R} = \mathrm{gen}\left\{\begin{pmatrix}3\\ 1\end{pmatrix}\right\}, \quad T_R^{-1} = \begin{pmatrix}3 & 0\\ 1 & 1\end{pmatrix}, T_R = \frac{1}{3}\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}$$

leads in the new coordinates to

$$\begin{aligned}
A_R &= \frac{1}{3}\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}\begin{pmatrix}2 & -3\\ 1 & -2\end{pmatrix}T_R^{-1} = \frac{1}{3}\begin{pmatrix}2 & -3\\ 1 & -3\end{pmatrix}\begin{pmatrix}3 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix},\\
B_R &= \frac{1}{3}\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}, \qquad C_R = \begin{pmatrix}2 & -2\end{pmatrix}\begin{pmatrix}3 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}4 & -2\end{pmatrix}
\end{aligned}$$

[1] The system has $D = 0$ so the transfer function will be strictly proper. Moreover we know we have a cancellation, and therefore there has been a numerator/denominator cancellation in the computation of the transfer function. After the cancellation we have only one pole left and no zero. We know the structure will be $P(s) = K/(s-1)$ but we do not know the gain K unless we compute explicitly the transfer function.

A stabilizing feedback in these coordinates would take on the form

$$\tilde{F} = \begin{pmatrix} f_1 & 0 \end{pmatrix} \quad \Rightarrow \quad F = \tilde{F}\, T_R = \begin{pmatrix} \frac{f_1}{3} & 0 \end{pmatrix}$$

where f_1 has been chosen s.t. $1 + f_1 = \lambda_1^*$ (which is equivalent to applying the Ackermann formula to the controllable subsystem).

- Observer design from original system. Being

$$O = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, \quad \mathcal{I} = \text{gen}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \Rightarrow \quad T_I^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_I = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

leads in the new coordinates to

$$A_I = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & -2 \end{pmatrix} T_I^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$B_I = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad C_I = \begin{pmatrix} 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

The chosen $\tilde{K}$ will be of the form

$$\tilde{K} = T_I\, K \quad \Rightarrow \quad K = T_I^{-1}\, \tilde{K} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ 0 \end{pmatrix} = \begin{pmatrix} k_1 \\ 0 \end{pmatrix}$$

where k_1 has been chosen s.t. $1 - 2k_1 = \lambda_1^{\rm d}$ (which is equivalent to applying the Ackermann formula to the observable subsystem). The observer is

$$\dot{\xi} = (A - KC)\xi + Bu + Ky = \ldots$$

Note that there are several checks to see if your computation were right; for example if you find a pole which is not an eigenvalue, there is clearly an error somewhere. Similarly if after the change of coordinates the matrices do not have the expected structure and/or different eigenvalues.

- The dynamic controller, after using $u = F\xi$, has input y and output u

$$\begin{aligned} \dot{\xi} &= (A - KC + BF)\xi + Ky \\ u &= F\xi \end{aligned}$$

and therefore its transfer function is

$$C(s) = F\,[sI - (A + BF - KC)]^{-1}\, K = \ldots$$

- With the proposed controller, the closed loop polynomial becomes

$$L(s) = C_3(s)P(s) = \frac{4k_c}{(s-1)(s+p)} \quad \Rightarrow \quad d_{cl}(s) = s^2 + s(p-1) + 4k_c - p$$

and thus is stable iff $p > 1$ and $k_c > p/4$. In terms of root locus interpretation, since the system is minimum-phase with $n - m = 2$ and the center of asymptotes $s_0 = (1-p)/2 < 0$ if $p > 1$, it is stabilizable with positive high-gain. The crossing through the origin (one root moves from positive to negative) clearly happens for $k_c = p/4$ so the singular point is negative.

2) Let the plant be represented by the scheme in Fig. 1 with

$$P_1(s) = \frac{1}{s-1}, \qquad P_2(s) = \frac{5}{s+10}$$

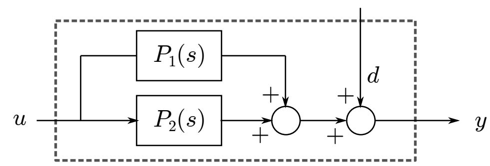

Figure 1: Plant

1. Determine a controller which guarantees, at steady state, no effect on the output of a constant disturbance d acting on the plant.
2. Draw the corresponding Nyquist plot.

2-sol) Parallel is given by

$$P_1(s) + P_2(s) = \frac{1}{s-1} + \frac{5}{s+10} = \frac{6s+5}{(s-1)(s+10)} = \frac{6(s+5/6)}{(s-1)(s+10)}$$

plus astatism, i.e. add the necessary pole $1/s$. Since we have $n - m = 3 - 1 = 2$ with

$$s_0 = \frac{-10+1+0+5/6}{2} < 0$$

there exists a positive gain, to be determined with the Routh criterion (critical value to be determined), which stabilizes the system. The final controller is

$$C(s) = \frac{K_c}{s}$$

with $K_c > K_c^{crit}$ (computation of K_c^{crit} is left as an exercise). A root locus plot would clarify this stability issue. It should be clear that the considered plant is controlled by a classic unit feedback control scheme with the designed $C(s)$. Note that you need to guarantee asymptotic stability of the closed-loop in order for the system to be astatic (a solution which just adds the pole il $s = 0$ and does not consider stability is wrong/incomplete)

As an alternative one could also add a zero $s = -z$ (with $z > 0$) so to obtain $n - m = 1$ and then choose the gain. In this case the controller is

$$C(s) = \frac{K_c(s+z)}{s}$$

• Draw carefully the Bode plot so that the corresponding Nyquist plot does the counter-clockwise encirclement which confirms closed-loop stability.

3) Let the open loop system be characterized by

$$L(s) = \frac{-K(s+5)}{s(s+3)^2}$$

Illustrate qualitatively how the closed-loop system behaves, as K varies in $\mathbf{R}$ (e.g., steady-state, transient, ...) in response to a constant reference and a sinusoidal output disturbance. You can use approximate Bode plots or other useful graphical tools.

3 - Solution We can say the following (short list):

• Being $n-m=2$ and $s_0 = (-3-3-0+5)/2 = -1/2 < 0$, the closed-loop system will be asymptotically stable for sufficiently negative values of K (i.e. for $-K$ sufficiently large and positive).

• When the closed-loop system is asymptotically stable, the steady-state error w.r.t. the constant reference is null (Type 1 system) independently from the particular value of K (as long as asymptotic stability is guaranteed).

• As $-K$ increases, from the root-locus plot, 2 poles will become complex conjugate with lower and lower damping and higher and higher natural frequency. Therefore the transient will become faster but with unacceptable transient. The third pole will be confined between -3 and -5 so transient can not become any faster than that.

• As $-K$ increases, from the approximate plot of the loop function and the corresponding sensitivity approximate magnitude, the effect at steady-state of the sinusoidal output disturbance will be more attenuated (but with worsing transient). Show plots of $|S(j\omega)|$ for different values of K.

• N.B. Using the approximate (piecewise linear) plot for the Bode phase, we have that the phase goes below $-\pi$ around 30 rad/sec and therefore the phase margin becomes negative for some values of the gain. However, from the root locus, we see that this never happens (closed-loop system always stable for negative K). This is due to the approximation, looking at the real phase plot this does not happen.

4) Let the plant be

$$P(s) = \frac{0.05}{s + 0.01}$$

Design a control scheme and the corresponding controller such that a reference $r(t) = t\delta_{-1}(t)$ is followed at steady state with a maximum error smaller or equal (in absolute value) than 1%. Simultaneously guarantee that the phase margin is at least 30° with a crossover frequency of 0.1 rad/s.

4 - Solution Normal loop shaping design problem.

• Steady-state specifications require the introduction of a pole in $s = 0$ and a sufficiently high (in absolute value) gain $K_c \geq 20$.

• From the modified plant

$$\hat{P}(s) = \frac{100}{s(100s + 1)}$$

Bode plots, we need an increase of phase of at least 30° and an attenuation of 40dB. This requires the use of, in this order, one (or two) lead functions first and then two lag functions with the usual technique (see case IV in the Loop shaping slides).

• Closed-loop stability is guaranteed by Bode's stability theorem (recall the hypothesis).