Some past exam problems in Control Systems - Part 1

January, 2017 - updates: 12/2017 - 03/01/2020 - 04/01/2020

01) Let a plant be represented by the transfer function between the input u and the output y given by

$$P(s) = \frac{1}{s-2}$$

- Determine, in a feedback control scheme, a controller such that at steady-state the controlled output y is independent from a constant unknown disturbance d acting at the plant's input.
- Show stability of the resulting control system by applying the Nyquist criterion and plotting the root-locus.

01 - Solution summary) The controller should have a pole in $s = 0$ and ensure closed-loop stability, obtained for example through the controller synthesis based upon the root locus. After the introduction of the pole in $s = 0$ we have $n - m = 2$ and a positive center of asymptotes. We have to choose a zero/pole pair to move the center of asymptotes and a sufficiently high positive gain (determined, for example, through Routh criterion). The loop shaping technique cannot be used since the plant has a pole with positive real part.

02) Find a numeric example of a three dimensional unstable system ($n = 3$) of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad C : \text{generic}$$

with $A_{11} : 2 \times 2$, A_{22} scalar and B_1 a two dimensional vector such that

- there exists a two dimensional uncontrollable subsystem
- and the system is stabilizable

02 - Solution summary) A_{22} (scalar) uncontrollable and therefore needs to be stable; the pair (A_{11}, B_1) needs to be not completely controllable with a stable uncontrollable eigenvalue and an unstable controllable one.

03) Let the plant be

$$P(s) = \frac{1}{s+100}$$

Design a control system such that the following specifications are met

- the plant's output asymptotically (i.e. at steady-state) tracks the reference $r(t) = t\delta_{-1}(t)$ with no error

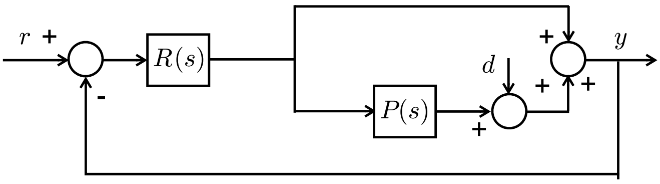

Figure 1: Control scheme

- the crossover frequency is $\omega_c^* = 1$ rad/sec with a phase margin of at least 30°.

03 - Solution summary) Two poles in $s = 0$ are necessary in the controller, plus standard loop shaping technique. No constraint on the absolute value of the controller gain thus a lead function will be sufficient in order to obtain the right phase value at the desired crossover frequency and a gain to obtain $\omega_c^* = 1$ rad/sec (applying also Bode's stability theorem).
Alternatively, noticing that the controller already has two poles (the two poles in $s = 0$), we can use a negative zero to increase the phase. For example a zero in $s = -1$ with a proper choice of the gain (40 dB for the 1/100 constant factor and -3 dB for the magnitude of the binomial term $(1 + j\omega)$ in $\omega = 1$ rad/s) will guarantee both the required crossover frequency and phase margin; the controller would be

$$C(s) = \frac{100}{\sqrt{2}} \frac{(s+1)}{s^2}$$

04) For the control scheme of Figure 1 with

$$R(s) = \frac{1}{s} \qquad P(s) = \frac{3}{s+1}$$

show that a constant disturbance d, at steady-state, does not affect the output y.

04 - Solution summary) Compute the transfer function from d to y and verify that it has a zero in $s = 0$ and that it has all its poles in the open left half plane (so we can apply the final value theorem to $y(s) = W_{dy}(s)d(s)$).

In particular, denoting with $a(s)$ the Laplace transform of the output of the $R(s)$ transfer function, setting $r = 0$, we have

$$y(s) = a(s) + d(s) + P(s)a(s)$$
$$a(s) = -R(s)y(s)$$

and solving we obtain

$$W_{dy}(s) = \frac{y(s)}{d(s)} = \frac{s(s+1)}{s^2 + 2s + 4}$$

Setting $d(t) = d\,\delta_{-1}(t)$ and thus $d(s) = d/s$, we have (final value theorem can be applied)

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sW_{dy}(s)\frac{d}{s} = \lim_{s\to 0} \frac{s(s+1)}{s^2 + 2s + 4} = 0$$

05) (Cruise control) A car of total mass m travels at speed $v(t)$. A simple force balance can represent the system dynamics along the horizontal axis

$$m\frac{dv(t)}{dt} + cv(t) = f(t) - mg\sin\alpha_i$$

with $f(t)$ the driving force (control input) and c a friction coefficient. The overall goal is to drive at a precise given constant cruise (i.e. steady-state) speed v_d independently of the slopes characterised by the unknown angles α_i when the cruise control in on.

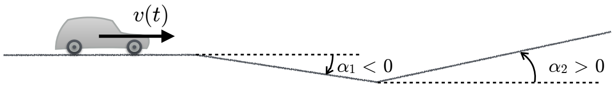

Figure 2: Cruise control

1. Draw a control scheme and design a controller which guarantees the desired behaviour.
2. Explain, possibly using root locus arguments, if for a better closed-loop transient behaviour when switching from a constant desired cruise velocity v_{1d} to v_{2d} you would prefer a low or high friction coefficient value c.

05 - Solution summary) The slope effect is a piecewise constant disturbance acting on the plant's input; we therefore have a type 1 and astatism requirement (including closed loop stability). The plant's transfer function, from the sum of the control force and the disturbance to the car velocity, is characterized by the single pole in $-c/m$. Adding the pole in $s = 0$ will lead to a closed loop system which remains asymptotically stable (verify through root locus or Bode stability theorem). The second point can be addressed by considering the friction as a varying positive parameter and study the root locus w.r.t. this parameter's variations (see "Other use of the RL" in the root locus slides).

06) Let the open-loop system be given by

$$F(s) = \frac{K(s^2 + 25)}{(s+1)(s^2+1)}$$

- Study the closed-loop stability using the Nyquist stability criterion with $K \in \mathbb{R}$. Check the obtained result with the Routh criterion.
- Knowing that there are no singular points, find a compatible root locus (positive and negative).

06 - Solution summary) Do not get confused by the trinomial term with zero damping coefficient at the numerator, it will make the Nyquist plot go through the origin (see Fig. 3). The Routh table for the closed-loop pole polynomial

$$p(s, K) = (s+1)(s^2+1) + K(s^2+25) = s^3 + (1+K)s^2 + s + 1 + 25K$$

(necessary condition requires $K > -1/25$) is

$$
\begin{array}{|cc}
1 & 1 \\
1+K & 1+25K \\
-\dfrac{24K}{1+K} & \\
1+25K &
\end{array}
$$

and therefore the closed-loop system is asymptotically stable iff $-1/25 < K < 0$.

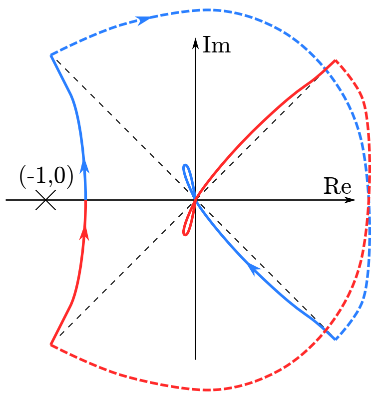

Figure 3: Exercise 6: Nyquist plot for a value of K in $(-1/25, 0)$

There may be different compatible root locus plots compatible with the basic rules seen during the course but only one has no singular points. In particular, from the previous Routh analysis, there are no positive values of K such that all three closed-loop poles lie in the left half plane; more precisely, for $K > 0$ there are two sign changes in the first column of the Routh table and therefore two positive branches lie in the right half plane. When K is negative, if $K > -1/25$ all three closed-loop poles – and therefore the corresponding portion of the root locus branches – lie in the left half plane, while if $K < -1/25$ there is only one change of sign (note that if $-1 < K < -1/25$ the signs of the first column are $+++-$, while for $K < -1$ it's $+---$) and this corresponds to the closed-loop real pole moving from the left to the right half plane. Therefore the probable root locus is shown in Fig. 4.

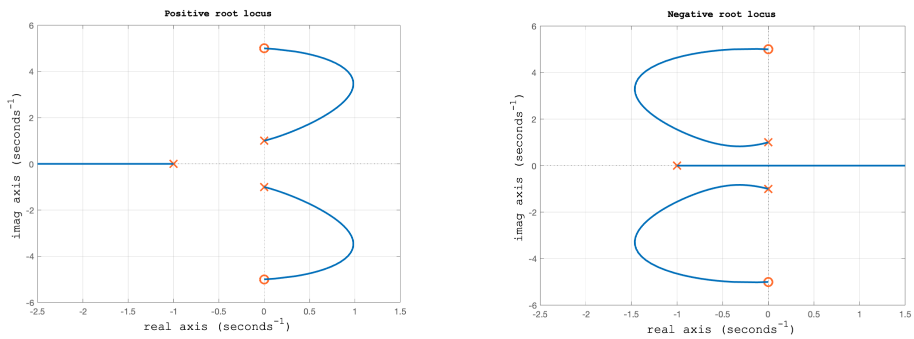

Figure 4: Exercise 6: positive and negative root locus

07) Let the plant be

$$P(s) = \frac{s - 100}{(s+1)(s+100)}$$

Design a control system such that the following specifications are met

- for the reference $r(t) = t\delta_{-1}(t)$ the maximum allowed error in absolute value is 0.1 and
- the crossover frequency is $\omega_c^* = 10$ rad/sec with a phase margin of at least 30°.

07 - Solution summary) Steady state specification: part of the controller should have a pole in $s = 0$ and a gain $K_c = -10$. To fulfill the dynamic specification we need to amplify by 20 dB and increase the phase by at least 30°. This can be achieved by adding a lead function which will introduce the required phase increase of at least 30° and some amplification (for example for the $m_a = 4$ and normalized frequency 1, we obtain also a 2.5 dB amplification). The remaining required amplification can be achieved through a second controller gain (for the chosen example $K_{c2}|_{dB} = 17.5$ dB).

08) Let a system be defined by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

- Is the system stabilizable with state feedback?
- Is the system stabilizable with output feedback?
- Any idea of what poles does the system have?

08 - Solution summary) To answer the first point we first observe that the system has the following three eigenvalues $\{-1, 0, -2\}$. In order for the system to be stabilizable through state feedback, the eigenvalue 0 needs to be controllable. Then either we use the PBH test or we do the Kalman decomposition through the change of coordinates. There exists an uncontrollable (hidden dynamics) subsystem of dimension 2 characterized by the eigenvalues $\{-1, -2\}$. So the system is stabilizable through state feedback.

In order to be stabilizable through output feedback, since we already checked that it is stabilizable through state feedback (necessary condition for being output stabilizable), we only need to check if 0 is observable (for example through the PBH test). The answer is yes. The only eigenvalue that will become a pole is the controllable and observable eigenvalue, that is 0.