Self assessment - 00A

February 29, 2024

1 Exercise

Given the matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

1. Find the nullspace of A_1 and A_2.
2. Prove that vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ generate the same subspace than $\mathbf{w}_3$ and $\mathbf{w}_4$ with

$$\mathbf{w}_1 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

3. Prove that both $(\mathbf{w}_1, \mathbf{w}_2)$ and $(\mathbf{w}_3, \mathbf{w}_4)$ generate the nullspace of A_2.

2 Exercise

Given the matrices

$$A_1 = \begin{pmatrix} 3 & 1 & 1 \\ -3 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

1. Find the eigenvalues of A_1 and their geometric multiplicities.
2. Find the eigenvalues of A_2 and their geometric multiplicities.

3 Exercise

Consider the following plant with $\alpha \in \mathbb{R}$ a real parameter.

$$\begin{aligned} \dot{x}_1 &= x_1 + x_3 + u \\ \dot{x}_2 &= u \\ \dot{x}_3 &= -2x_3 \\ y &= \alpha x_1 + x_2 + x_3 \end{aligned}$$

1. Find (A, B, C, D) of the state space representation.

2. Compute the eigenvalues of A and their corresponding eigenvectors. What are the natural modes of the system?
3. Which initial conditions guarantee that the state ZIR will converge to zero asymptotically?
4. Which initial conditions guarantee that the state ZIR will not diverge?
5. Can we avoid, with a proper choice of the output through α, the divergence of the output ZIR for every initial condition?
6. Can we avoid divergence of the impulse response with a proper choice of α?

4 Exercise

Consider the horizontal motion of a point mass under the action of a force $f(t)$ and a friction force proportional, with coefficient $\mu > 0$, to the mass velocity. The following questions need to be solved symbolically, without assigning particular numeric values for the system parameters m and μ.

1. Find the state space representation by considering that we are also interested in the mass position.
2. If possible, find the change of coordinates (similarity transformation) that will diagonalize the dynamic matrix.
3. Write the matrix exponential in the original state (position displacement and velocity).
4. Assuming the mass is pushed from its rest position with a unit impulse force $f(t) = \delta(t)$, where will the mass stop?
5. Find explicitly the position $p(t)$ and velocity $\dot{p}(t)$ time evolution when no input is applied but the system starts from a generic initial condition $(p_0, \dot{p}_0)$, in other words find the state Zero Input Response (ZIR). How is the found ZIR related to the natural modes of the system?
6. For the state ZIR, find the relationship between $\dot{p}(t)$ and $p(t)$, i.e. write the solution $\dot{p}(t)$ in terms of the solution $p(t)$ so that we can plot the ZIR in the $(p, \dot{p})$ phase plane. The obtained relationship will also depend upon the initial condition $(p(0), \dot{p}(0)) = (p_0, \dot{p}_0)$. Comment the typical system trajectories in the phase plane.
7. Find the set of initial conditions $(p_0, \dot{p}_0)$ such that the ZIR tends asymptotically to the origin $(0, 0)$. Plot this set in the phase plane $(p, \dot{p})$.
8. Find explicitly the position and velocity time evolution when the system starts from the rest configuration $(p_0, \dot{p}_0) = (0, 0)$ and a unit constant force $f(t) = 1$ is applied from $t = 0$.
9. Assume that the constant unit force is applied only for a finite time interval of length T, i.e. $f(t) = 1$ for $t \in [0, T]$ and $f(t) = 0$ for $t > T$. Write the state forced response.
10. Write the state evolution when the constant applied force during the interval of duration T has amplitude α, i.e. $f(t) = \alpha$ for $t \in [0, T]$?
11. Assume we start for the initial condition $(p_0, 0)$, we want to find α (if it exists) such that the input $f(t) = \alpha$ for $t \in [0, T]$ will lead to a state evolution that will asymptotically tend to the origin. To do so, note that the given input will transfer the state from its original value to a new value reached at time $t = T$. From that state the system evolves with no input applied. Use the previous results in order to solve the problem.

5 Solution Exercise 1

1. The two nullspaces are given by

$$A_1\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \quad \rightarrow \quad \text{Ker}(A_1) = \text{gen}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$A_2\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \quad \rightarrow \quad \text{Ker}(A_2) = \text{gen}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} = \text{gen}\{\mathbf{v}_a, \mathbf{v}_b\}$$

Note that $\mathbf{v}_a$ and $\mathbf{v}_b$ are linearly independent and therefore we could have chosen any other vector $\mathbf{w}$ linear combination of $\mathbf{v}_a$ and $\mathbf{v}_b$ as another equivalent base vector. For example

$$\mathbf{w}_a = \mathbf{v}_a - \mathbf{v}_b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \rightarrow \quad \text{Ker}(A_2) = \text{gen}\{\mathbf{v}_a, \mathbf{v}_b\} = \text{gen}\{\mathbf{v}_a, \mathbf{w}_a\}$$

or even

$$\mathbf{w}_b = 3\mathbf{v}_a - 2\mathbf{v}_b = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} \quad \rightarrow \quad \text{Ker}(A_2) = \text{gen}\{\mathbf{v}_a, \mathbf{v}_b\} = \text{gen}\{\mathbf{v}_a, \mathbf{w}_a\} = \text{gen}\{\mathbf{w}_a, \mathbf{w}_b\}$$

since $\mathbf{w}_b$ is linearly independent from $\mathbf{w}_a$, while we cannot write

$$\text{Ker}(A_2) = \text{gen}\{\mathbf{w}_a, \mathbf{w}_c\} \quad \text{with} \quad \mathbf{w}_c = 2\mathbf{v}_a - 2\mathbf{v}_b = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{w}_a$$

since $\mathbf{w}_a$ and $\mathbf{w}_c$ are not linearly independent.

2. We want to prove that $\text{gen}\{\mathbf{w}_1, \mathbf{w}_2\} = \text{gen}\{\mathbf{w}_3, \mathbf{w}_4\}$. First notice that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent since the two vectors are not parallel (for two vectors this is equivalent to saying that the two vectors are linearly independent). Then it can be readily[1] seen that

$$\mathbf{w}_3 = \mathbf{w}_1 - \mathbf{w}_2 \qquad \text{and} \qquad \mathbf{w}_4 = -\frac{5}{3}\mathbf{w}_1 + \frac{4}{3}\mathbf{w}_2$$

i.e. both vectors can be generated from the base $\{\mathbf{w}_1, \mathbf{w}_2\}$ and thus belong to the same subspace. Moreover since $\mathbf{w}_3$ and $\mathbf{w}_4$ are not parallel they can be chosen as a base (for the same subspace generated by $\{\mathbf{w}_1, \mathbf{w}_2\}$).

3. Similarly, being

$$\mathbf{w}_1 = \mathbf{v}_a + 2\mathbf{v}_b \quad \mathbf{w}_2 = 2\mathbf{v}_a + \mathbf{v}_b \quad \mathbf{w}_3 = -\mathbf{v}_a + \mathbf{v}_b \quad \mathbf{w}_1 = \mathbf{v}_a - 2\mathbf{v}_b$$

[1] The second relation can be found by solving the three equations in the two unknowns a and b such that $\mathbf{w}_4 = a\mathbf{w}_1 + b\mathbf{w}_2$, i.e. $-3a - 3b = 1$, $a + 2b = 1$ and $2a + b = -2$. If no solution (a, b) can be found then $\mathbf{w}_4$ is not obtainable as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$ and therefore does not belong to the subspace generated by $\{\mathbf{w}_1, \mathbf{w}_2\}$. For example the vector $\mathbf{w}_5^T = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^T$ does not belong to $\text{gen}\{\mathbf{w}_1, \mathbf{w}_2\}$.

6 Solution Exercise 2

Due to the particular block triangular structure, the eigenvalues of the two matrices are

$$\text{eig}(A_1) = \text{eig}\begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix} \cup \{0\} = \{2, 0, 0\}, \qquad \text{eig}(A_2) = \text{eig}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cup \{0\} = \{0, 0, 0\}$$

that is A_1 has eigenvalues $\lambda_1 = 2$ with algebraic multiplicity 1 and $\lambda_2 = 0$ with algebraic multiplicity 2, while A_3 has eigenvalues $\lambda_1 = 0$ with algebraic multiplicity 3.

1. For A_1, the geometric multiplicity of λ_1 is 1 (being always $0 <$ geom. mult. $\leq$ alg. mult.) while we need to determine the dimension of $\text{Ker}(A_1 - \lambda_2 I)$ to find the geometric multiplicity of λ_2. Since

$$\text{Ker}(A_1 - \lambda_2 I) = \text{Ker}(A_1) = \text{Ker}\begin{pmatrix} 3 & 1 & 1 \\ -3 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{gen}\left\{ \begin{pmatrix} -1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

so clearly the dimension of $\text{Ker}(A_1 - \lambda_2 I)$ is 2 and therefore the geometric multiplicity of $\lambda_2 = 0$ is 2. Note that there is no need to find a basis of $\text{Ker}(A_1 - \lambda_2 I)$ since we are only interested in its dimension. We could therefore instead use the *rank-nullity theorem* (applied to a generic square $n \times n$ matrix M) which states that

$$\dim(\text{Ker}(M)) + \text{rank}(M) = n$$

Since we are not able to find a non-zero minor of dimension 2 in the matrix $A_1 - \lambda_2 I = A_1$, then the rank is 1 (some elements, which are minors of dimension 1, are different from 0) and therefore we have

$$\dim(\text{Ker}(A_1 - \lambda_2 I)) = 3 - 1 = 2$$

which implies that the geometric multiplicity of $\lambda_2 = 0$ is 2.

2. For A_2, to find the geometric multiplicity of the unique eigenvalue $\lambda_1 = 0$, again we need to find the dimension of $\text{Ker}(A_2 - \lambda_1 I) = \text{Ker}(A_2)$. Since the rank of A_2 is clearly 1, the dimension of the nullspace is

$$\dim(\text{Ker}(A_2)) = 3 - 1 = 2$$

which is confirmed by

$$\text{Ker}(A_2 - \lambda_1 I) = \text{Ker}(A_2) = \text{Ker}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{gen}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

7 Solution Exercise 3

1. From direct inspection we have

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha & 1 & 1 \end{pmatrix}$$

2. Being the matrix A upper triangular, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = -2$. To compute the eigenvectors we solve

$$(A - \lambda_1 I)u_1 = 0 \quad \rightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} u_1 = 0 \quad \rightarrow \quad u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda_2 I)u_2 = 0 \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} u_2 = 0 \quad \rightarrow \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A - \lambda_3 I)u_3 = 0 \quad \rightarrow \quad \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} u_3 = 0 \quad \rightarrow \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

The natural modes are

$$e^{\lambda_1 t} = e^t, \quad e^{\lambda_2 t} = 1, \quad e^{\lambda_3 t} = e^{-2t}$$

3. Since the modes are diverging (e^t), bounded (1) and converging (e^{-2t}), the only initial conditions in the state ZIR that will guarantee a converging state evolution are those parallel to u_3, that is $x(0) = a\, u_3$ with a non-zero (i.e. $x(0)$ belonging to the eigenspace associated to $\lambda_3 = -2$). In this way there are no components along the other two eigenspaces

$$e^{At}x(0) = \sum_{i=1}^{3} e^{\lambda_i t} u_i v_i^T a\, u_3 = a\, e^{-2t} u_3 \quad \text{since} \quad v_1^T u_3 = v_2^T u_3 = 0$$

4. We need to choose the initial condition with no component in the eigenspace relative to λ_1 or, equivalently, we can choose any initial condition belonging to the subspace generated by $\{u_2, u_3\}$ i.e.

$$x(0) = au_2 + bu_3 = \begin{pmatrix} b \\ a \\ -3b \end{pmatrix}, \qquad a, b \in \mathbf{R}$$

5. The output ZIR is given by

$$Ce^{At}x(0) = \sum_{i=1}^{3} e^{\lambda_i t} C u_i v_i^T x(0)$$

The only way to cancel out the contribution of the unstable mode $e^{\lambda_1 t}$ in the output ZIR (independently from the value of the initial condition) is by choosing C such that $Cu_1 = 0$. This can be achieved with $\alpha = 0$. In this case, the output ZIR will never diverge.

6. Similarly, being the impulse response

$$Ce^{At}B = \sum_{i=1}^{3} e^{\lambda_i t} C u_i v_i^T B$$

since $v_1^T B \neq 0$ the only possibility is to choose again $\alpha = 0$.

8 Solution Exercise 4

[*This solution clearly exceeds what is expected from a student work but should rather be seen as a detailed analysis of an easy example useful to clarify the theory seen in class.*]

The differential equation relating position $p(t)$, velocity $\dot{p}(t)$ and acceleration $\ddot{p}(t)$ is

$$m\ddot{p} + \mu\dot{p} = f(t)$$

1. Choosing as state $x^T = \begin{pmatrix} p & \dot{p} \end{pmatrix}^T$, $f(t)$ as input u and the position p as output, we have the following state space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\mu/m \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u = Ax + Bu$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x = Cx$$

2. The system has two distinct eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -\mu/m$. The associated eigenvectors, needed to diagonalize the dynamic matrix, are

$$\lambda_1 = 0 \quad \rightarrow \quad (A - \lambda_1 I)u_1 = Au_1 = 0 \quad \rightarrow \quad u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = 0 \quad \rightarrow \quad (A - \lambda_2 I)u_2 = \begin{pmatrix} \mu/m & 1 \\ 0 & 0 \end{pmatrix} u_2 = 0 \quad \rightarrow \quad u_2 = \begin{pmatrix} 1 \\ -\mu/m \end{pmatrix}$$

and therefore using the similarity matrix

$$T^{-1} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -\mu/m \end{pmatrix} \quad \rightarrow \quad T = \begin{pmatrix} 1 & m/\mu \\ 0 & -m/\mu \end{pmatrix}$$

we get

$$\bar{A} = TAT^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -\mu/m \end{pmatrix}, \quad \bar{B} = TB = \begin{pmatrix} 1/\mu \\ -1/\mu \end{pmatrix}, \quad \bar{C} = CT^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The new diagonalizing coordinates are

$$z = Tx = \begin{pmatrix} 1 & m/\mu \\ 0 & -m/\mu \end{pmatrix} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p + \frac{m}{\mu}\dot{p} \\ -\frac{m}{\mu}\dot{p} \end{pmatrix}$$

3. The matrix exponential is found through the diagonalized matrix $\bar{A}$ as

$$e^{At} = e^{T^{-1}\bar{A}Tt} = T^{-1}e^{\bar{A}t}T = \begin{pmatrix} 1 & 1 \\ 0 & -\mu/m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\mu t/m} \end{pmatrix} \begin{pmatrix} 1 & m/\mu \\ 0 & -m/\mu \end{pmatrix}$$
$$= \begin{pmatrix} 1 & m(1 - e^{-\mu t/m})/\mu \\ 0 & e^{-\mu t/m} \end{pmatrix}$$

4. We need to find the impulse response, which is independent from the choice of the state,

$$w(t) = \bar{C}e^{\bar{A}t}\bar{B} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\mu t/m} \end{pmatrix} \begin{pmatrix} 1/\mu \\ -1/\mu \end{pmatrix} = \frac{1}{\mu}\left(1 - e^{-\mu t/m}\right)$$

Therefore the output (position p) will tend, after an impulsive force has been applied, to the constant value

$$\bar{p} = \lim_{t \to \infty} w(t) = \frac{1}{\mu}$$

To find the impulsive response we could have also computed the transfer function (by doing the Laplace transform of the differential equation starting from zero initial position and velocity)

$$(ms^2 + \mu s)p(s) = f(s) \quad \to \quad W(s) = \frac{\text{Output}(s)}{\text{Input}(s)} = \frac{p(s)}{f(s)} = \frac{1}{s(ms + \mu)}$$

which can be expanded in partial fractions

$$W(s) = \frac{1/m}{s(s + \mu/m)} = \frac{R_1}{s} + \frac{R_2}{s + \mu/m}$$

with the residues being

$$R_1 = \{sW(s)\}_{s=0} = \frac{1}{\mu}$$
$$R_2 = \{(s + \mu/m)W(s)\}_{s=-\mu/m} = -\frac{1}{\mu}$$

and take its inverse Laplace transform

$$w(t) = \mathcal{L}^{-1}(W(s)) = \mathcal{L}^{-1}\left(\frac{1}{\mu}\left(\frac{1}{s} - \frac{1}{s + \mu/m}\right)\right) = \frac{1}{\mu}\left(1 - e^{-\mu t/m}\right)$$

5. Since we already computed the matrix exponential, we can directly write the state ZIR

$$x_{ZIR}(t) = e^{At}x(0) = \begin{pmatrix} 1 & m(1 - e^{-\mu t/m})/\mu \\ 0 & e^{-\mu t/m} \end{pmatrix} \begin{pmatrix} p(0) \\ \dot{p}(0) \end{pmatrix} = \begin{pmatrix} p(0) + m(1 - e^{-\mu t/m})\dot{p}(0)/\mu \\ e^{-\mu t/m}\dot{p}(0) \end{pmatrix}$$

Obviously, the ZIR state response is a linear combination of the system natural modes.

6. From the previous expression of $x_{ZIR}(t)$, being the two components respectively $(p(t), \dot{p}(t))$ with

$$\dot{p}(t) = e^{-\mu t/m}\dot{p}(0)$$

one can rewrite the position time evolution $p(t)$ as

$$p(t) = p(0) + \frac{m}{\mu}(1 - e^{-\mu t/m})\dot{p}(0) = p(0) + \frac{m}{\mu}\dot{p}(0) - \frac{m}{\mu}e^{-\mu t/m}\dot{p}(0) = p(0) + \frac{m}{\mu}\dot{p}(0) - \frac{m}{\mu}\dot{p}(t)$$

and therefore the state ZIR components are related as

$$\dot{p}(t) = -\frac{\mu}{m}p(t) + \dot{p}(0) + \frac{\mu}{m}p(0)$$

which is just a straight line of slope $-\mu/m$ in the $(p, \dot{p})$ plane. When the inital conditions change we obtain a set of parallel straight lines. We need however to interpret carefully this result. Let us first look at the equilibrium points of the system. These are the solution of

$$Ax_e = 0 \quad \to \quad x_e = \begin{pmatrix} p_e \\ 0 \end{pmatrix}$$

and therefore this set, in the $(p, \dot{p})$ plane is represented by the horizontal axis. There are a number of interesting observations which confirm our physical intuition.

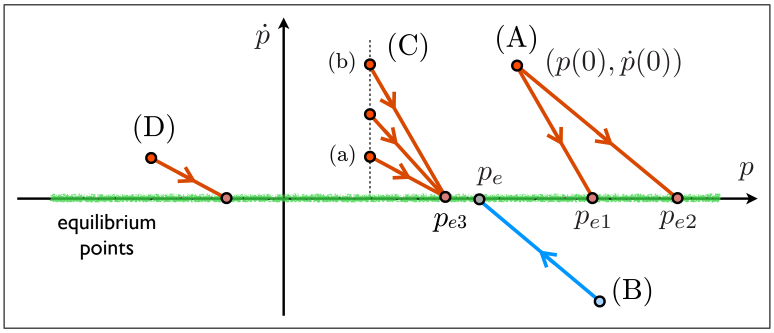

Figure 1: Exercise 4 - phase plane and some ZIR trajectories from different initial conditions

- Starting from the same initial condition $(p(0), \dot{p}(0))$ with positive initial velocity, case (A), we can compare the different resulting trajectories when the friction decreases: with a friction coefficient μ_1 the mass stops (asymptotically) in p_{e1} (one of the infinite equilibrium points) with zero velocity. With a lower friction coefficient $\mu_2 < \mu_1$ the state evolution (a line in the phase plane for this system) will end up in a farther point p_{e2} since the line slope $-\mu_2/m$ has a smaller absolute value w.r.t. $-\mu_1/m$.
- Starting with a negative velocity ($\dot{p}(0) < 0$), case (B), will make the mass move backwards. The mass ends in the equilibrium point p_e.
- Starting from the same initial position $p(0)$, case (C), we need different initial velocities $\dot{p}(0)$ to end in the same final position if we have different masses. The smaller the mass m the larger the initial velocity must be since if $m_1 < m_2$, the slope $-\mu/m_2$ has a smaller absolute value w.r.t. $-\mu/m_1$. In Fig. 1, case (C), the line (a) corresponds to m_2 while line (b) to m_1 with $m_1 < m_2$. We also understand that theoretically, one way to remain in the same position, i.e. to have $p_{e3} = p(0)$, with a non-zero initial velocity (this motion would result in a vertical segment) we need a zero mass (to have a $\pm\infty$ slope). This is more evident from the derivation of x^{∞}_{ZIR} in the next question.
- We know that starting from $x(0) = \begin{pmatrix} p(0) & \dot{p}(0) \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$, the effect of an impulse is equivalent to starting in a ZIR from an initial condition which coincides with the input vector B. For the given system, the state impulse response is equal to the state ZIR from the initial condition

  $$x_0^i = \begin{pmatrix} p^i(0) \\ \dot{p}^i(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}$$

  i.e. the impulse generates an instantaneous initial velocity. As a check, we use the already computed matrix exponential to find the state impulse response. We have

  $$e^{At}B = \begin{pmatrix} 1 & m(1 - e^{-\mu t/m})/\mu \\ 0 & e^{-\mu t/m} \end{pmatrix} \begin{pmatrix} 0 \\ 1/m \end{pmatrix} = \begin{pmatrix} (1 - e^{-\mu t/m})/\mu \\ e^{-\mu t/m}/m \end{pmatrix}$$

  which coincides with the state ZIR (see previous questions) with $x_0 = B$.

  If we start from a generic initial condition $x(0) = \begin{pmatrix} p(0) & \dot{p}(0) \end{pmatrix}^T$, the state response is the sum of the state impulse response and the state free evolution

  $$x(t) = e^{At}x_0 + e^{At}B = e^{At}(x_0 + B) = e^{At}\bar{x}(0) \quad \text{with} \quad \bar{x}(0) = x_0 + B$$

  that is the state evolution coincides with a state free response from the new initial condition $\bar{x}(0)$. Again the effect of the impulsive force (unit impulse as input) is to instantaneously change the velocity.

7. From the previously computed general state ZIR

$$x_{ZIR}(t) = \begin{pmatrix} p(0) + m(1 - e^{-\mu t/m})\dot{p}(0)/\mu \\ e^{-\mu t/m}\dot{p}(0) \end{pmatrix}$$

we notice that, as $t \to \infty$, the state tends to

$$x_{ZIR}^{\infty} = \begin{pmatrix} p(0) + \frac{m}{\mu}\dot{p}(0) \\ 0 \end{pmatrix}$$

Therefore in order for this final point to be the origin, the initial conditions must satisfy the relation

$$p(0) + \frac{m}{\mu}\dot{p}(0) = 0$$

This set of initial conditions can be expressed either in terms of $p(0)$ or $\dot{p}(0)$ as

$$\begin{pmatrix} p(0) \\ -\dfrac{\mu}{m}\,p(0) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\dfrac{m}{\mu}\dot{p}(0) \\ \dot{p}(0) \end{pmatrix}$$

In the phase plane, this set is the line with slope $-\mu/m$ passing through the origin.

8. When a unit step force is applied and the system starts in $(p(0), \dot{p}(0)) = (0, 0)$ the state evolution is the state forced response – or state Zero State Response (ZSR) – to the input $u = \delta_{-1}(t)$. We can either work in the time or in the Laplace domain.

   - We need to compute explicitly

$$x_{ZSR}(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t \begin{pmatrix} (1 - e^{-\mu(t-\tau)/m})/\mu \\ e^{-\mu(t-\tau)/m}/m \end{pmatrix} d\tau = \begin{pmatrix} p_{ZSR}(t) \\ \dot{p}_{ZSR}(t) \end{pmatrix}$$

   We can compute the first component, the position

$$\begin{aligned} p_{ZSR}(t) &= \frac{1}{\mu}\left[\int_0^t d\tau - e^{-\frac{\mu}{m}t}\int_0^t e^{\frac{\mu}{m}\tau}d\tau\right] \\ &= \frac{1}{\mu}\left[t - \frac{m}{\mu}\left(1 - e^{-\frac{\mu}{m}t}\right)\right] \end{aligned}$$

   We can also do the same for the second component, but since we know it's the velocity we can directly take the time derivative of $p_{ZSR}(t)$ (check as an exercise) and therefore

$$\dot{p}_{ZSR}(t) = \frac{1}{\mu}\left[1 - e^{-\frac{\mu}{m}t}\right]$$

9. The input, now a unit force applied for a time interval T, can be written as

$$u_T(t) = \delta_{-1}(t) - \delta_{-1}(t - T)$$

   Using the Laplace transform translation result, we can compute x_T (the state response to the input u_T) in terms of the state response to the unit step input previously found $x_{ZSR}(t)$ as

$$x_T(t) = x_{ZSR}(t)\delta_{-1}(t) - x_{ZSR}(t - T)\delta_{-1}(t - T)$$

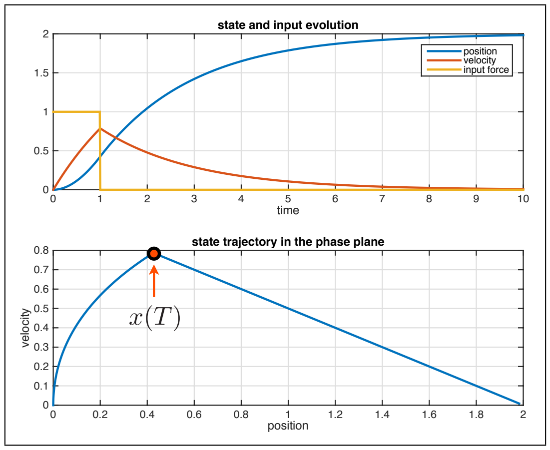

Figure 2: Exercise 4 - State trajectory in t (top) and in the phase plane (bottom) when an input $u_T(t)$ is applied starting from zero initial conditions. From the phase plane plot, the input transfers the state from the origin to $x(T)$. For $t \geq T$ the system evolves with no input applied that is as a ZIR from the initial state $x(T)$.

since the response of a translated input $\delta_{-1}(t-T)$ is equal to the translation (by the same time interval T) of the response $x_{ZSR}(t)$ to the non-translated input $\delta_{-1}(t)$.

The presence of the Heaviside functions in the expression of $x_T(t)$ is needed to remember that $x_{ZSR}(t)$ is null for negative time and thus its translation $x_{ZSR}(t-T)$ is null before $t = T$.

Note that the effect of this input of finite duration is to transfer the state from $x(0)$ (here $x(0) = 0$) to a new state $x(T)$ (the value of the state in $t = T$). After T, the system has no inputs and evolves as a state ZIR from the initial state $x(T)$. A simulation is shown in Fig. 2. In particular the input u_T with $T = 1$ s transfers the state in $x(1) = (0.43, 0.79)$ i.e. in position $p(1) = 0.42$ m with velocity $\dot{p}(1) = 0.79$ m/s. After 1 second the system starts evolving in free evolution. In the phase plane, when the input is applied the state trajectory evolves as $(p_{ZSR}(t), \dot{p}_{ZSR}(t))$. Noting that

$$\dot{p}_{ZSR}(t) = -\frac{\mu}{m} p_{ZSR}(t) + \frac{1}{m} t$$

we obtain the trajectory in the phase plane which is not a straight line due to the presence of the term t/m. During the second phase (ZIR), starting in $t = T$, the state is in free evolution and the phase plane trajectory is a straight line. Asymptotically the mass will stop in $\bar{p} = 2$ m.

We see that the state evolution when $u_T(t)$ is applied can be also computed as follows.

- Compute the value $x_T(T)$ of the state forced evolution (ZSR), i.e. starting from the null state, in $t = T$ when the input $u_T(t)$ is applied.
- Compute the state free evolution (ZIR) from $x_T(T)$, i.e. when no input is applied.

Note, however, that the state $x_T(T)$ is the same reached in $t = T$ by applying only $\delta_{-1}(t)$ (the second step has no effect yet). So the previous remark can be changed into " ... compute $x_{ZSR}(T)$ when the input $\delta_{-1}(t)$ is applied ... ".

10. Being the system linear, the forced response (ZSR) to $\alpha\, u_T(t)$ is just α times the forced response to $u_T(t)$ which has been already computed.

11. We need to put together some of the obtained results.

   - Since the system is linear the state response starting from a non-zero initial condition and subject to an input is given by the sum of the state ZIR and ZSR (free plus forced evolution). Therefore the state response from $(p(0), \dot{p}(0)) = (p_0, 0)$ will be

   $$\begin{pmatrix} p(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha(p_{ZSR}(t)\delta_{-1}(t) - p_{ZSR}(t-T)\delta_{-1}(t-T)) \\ \alpha(\dot{p}_{ZSR}(t)\delta_{-1}(t) - \dot{p}_{ZSR}(t-T)\delta_{-1}(t-T)) \end{pmatrix}$$

   - Since we know the set of initial conditions for which the free evolution converges to the origin and we noticed that from $t = T$ the system is in free evolution from $x_{ZRS}(T)$

   $$\begin{pmatrix} p(T) \\ \dot{p}(T) \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha p_{ZSR}(T) \\ \alpha \dot{p}_{ZSR}(T) \end{pmatrix}$$

   we just need to find α (if it exists) such that the input $\alpha\delta_{-1}(t)$ moves the state from $(p_0, 0)$ to a state $(p(T), \dot{p}(T))$ which belongs to the set

   $$S_0 = \left\{ (p(T), \dot{p}(T)) \quad \text{such that} \quad p(T) + \frac{m}{\mu}\dot{p}(T) = 0 \right\}$$

   We need to solve in α the equation

   $$p(T) + \frac{m}{\mu}\dot{p}(T) = p_0 + \alpha p_{ZSR}(T) + \alpha\frac{m}{\mu}\dot{p}_{ZSR}(T) = 0$$

   i.e.

   $$\alpha = -\frac{\mu}{m}\frac{p_0}{\left(\dot{p}_{ZSR}(T) + \frac{\mu}{m}p_{ZSR}(T)\right)}$$

   Finally, recalling that during the forced phase $\dot{p}_{ZSR}(t) = \mu\, p_{ZSR}(t)/m + t/m$, the previous expression simplifies in

   $$\alpha = -\frac{\mu}{T}\, p_0$$

   As an example, see the resulting motion of Fig. 3 where $\mu = 0.7$.

12. Let's use the impulsive response to compute the output ZSR to a sinusoidal input force $f(t) = \sin\bar{\omega}t$ when the output is the mass velocity. With this choice we have

   $$y(t) = \dot{p}(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} p(t) \\ \dot{p}(t) \end{pmatrix} = Cx$$

   and therefore the (output) impulsive response is

   $$w(t) = Ce^{At}B = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} (1 - e^{-\mu t/m})/\mu \\ e^{-\mu t/m}/m \end{pmatrix} = \frac{1}{\mu}e^{-\mu t/m}$$

   The ZSR is then the convolution of the impulsive response with the input, i.e.

   $$y(t) = \dot{p}(t) = \int_0^t w(t-\tau)f(\tau)d\tau = \frac{1}{m}e^{-\mu t/m}\int_0^t e^{-\mu\tau/m}\sin\bar{\omega}\tau\, d\tau$$

   Using

   $$\int e^{cx}\sin bx\, dx = \frac{e^{cx}}{c^2 + b^2}(c\sin bx - b\cos bx)$$

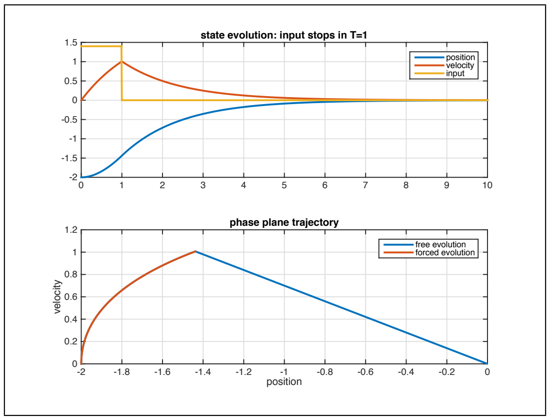

Figure 3: Exercise 4 - State trajectory in t (top) and in the phase plane (bottom) when an input $\alpha u_T(t)$ is applied starting initial conditions $(-2, 0)$. From the phase plane plot, the input transfers the state from the initial state an $x(T)$ belonging to S_0. For $t \geq T$ the system evolves with no input from the initial state $x(T)$.

we obtain

$$\begin{aligned}\dot{p}(t) &= \frac{1}{m} e^{-\mu\, t/m} \left[\frac{e^{\mu\, \tau/m}}{(\frac{\mu}{m})^2 + \bar{\omega}^2} \left(\frac{\mu}{m} \sin \bar{\omega}\tau - \bar{\omega} \cos \bar{\omega}\tau \right) \right]_0^t \\ &= \frac{1}{m} \frac{1}{(\frac{\mu}{m})^2 + \bar{\omega}^2} \left(\frac{\mu}{m} \sin \bar{\omega}\, t - \bar{\omega} \cos \bar{\omega}\, t + \bar{\omega} e^{-\mu/m\, t} \right)\end{aligned}$$