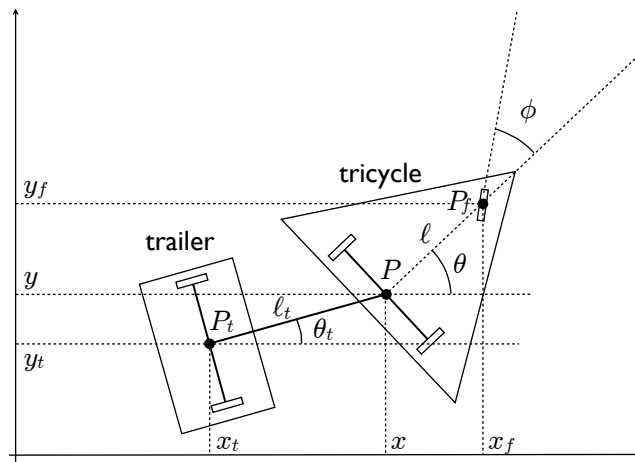


## Autonomous and Mobile Robotics Solution of Class Test no. 1

### Solution of Problem 1

A convenient choice of generalized coordinates is  $\mathbf{q} = (x \ y \ \theta \ \phi \ \theta_t)^T$  (see figure), i.e., a set of generalized coordinates for the tricycle plus an additional coordinate for the trailer. In the following, both two-wheel axles are assimilated to a single wheel located at the axle midpoint. The robot has then three wheels: the tricycle front wheel, the tricycle rear wheel, and the trailer wheel.



The kinematic constraints acting on the robot are therefore three (one “pure rolling” condition for each wheel):

$$\begin{aligned} \dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) &= 0 \\ \dot{x} \sin \theta - \dot{y} \cos \theta &= 0 \\ \dot{x}_t \sin \theta_t - \dot{y}_t \cos \theta_t &= 0, \end{aligned}$$

where  $(x_f, y_f)$  and  $(x_t, y_t)$  are the Cartesian coordinates of  $P_f$  (the centre of the tricycle front wheel) and  $P_t$  (the trailer axle midpoint), respectively. Being

$$\begin{aligned} x_f &= x + l \cos \theta \\ y_f &= y + l \sin \theta \end{aligned}$$

and

$$\begin{aligned} x_t &= x - l_t \cos \theta_t \\ y_t &= y - l_t \sin \theta_t \end{aligned}$$

it is easy to obtain the following expression for the kinematic constraints

$$\begin{aligned} \dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \dot{\theta} l \cos \phi &= 0 \\ \dot{x} \sin \theta - \dot{y} \cos \theta &= 0 \\ \dot{x} \sin \theta_t - \dot{y} \cos \theta_t + l_t \dot{\theta}_t &= 0, \end{aligned}$$

or, in Pfaffian form

$$\begin{pmatrix} \sin \theta & -\cos \theta & 0 & 0 & 0 \\ \sin(\theta + \phi) & -\cos(\theta + \phi) & -\ell \cos \phi & 0 & 0 \\ \sin \theta_t & -\cos \theta_t & 0 & 0 & \ell_t \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\theta}_t \end{pmatrix} = \mathbf{A}^T(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}.$$

Note that the submatrix consisting of the first two rows and the first four columns of  $\mathbf{A}^T$  coincides with the constraint matrix for the bicycle. A basis for the null space of  $\mathbf{A}^T$  can then be easily written by “completing” the basis found for the bicycle (in particular, the rear-wheel drive version, since we have a real-wheel drive tricycle here) with a suitable fifth element. One easily obtains

$$\mathbf{G}(\mathbf{q}) = \begin{pmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ \tan \phi / \ell & 0 \\ 0 & 1 \\ -\sin(\theta_t - \theta) / \ell_t & 0 \end{pmatrix} = (\mathbf{g}_1(\mathbf{q}) \quad \mathbf{g}_2(\mathbf{q})).$$

The kinematic control system is then

$$\dot{\mathbf{q}} = \mathbf{g}_1(\mathbf{q})v + \mathbf{g}_2(\mathbf{q})\omega,$$

where  $v$  and  $\omega$  are respectively the driving and the steering velocity of the tricycle.

Clearly, the above kinematic model is associated to the particular choice of  $\mathbf{q}$  made at the beginning. A different choice would have led to a different model - although equivalent via a change of coordinates. For example, choosing  $\mathbf{q}' = (x \ y \ \theta \ \phi \ \delta)^T$ , with  $\delta = \theta - \theta_t$ , leads to

$$\dot{\mathbf{q}}' = \mathbf{g}'_1(\mathbf{q}')v + \mathbf{g}'_2(\mathbf{q}')\omega,$$

with

$$\mathbf{G}'(\mathbf{q}') = \begin{pmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ \tan \phi / \ell & 0 \\ 0 & 1 \\ \tan \phi / \ell - \sin \delta / \ell_t & 0 \end{pmatrix} = (\mathbf{g}'_1(\mathbf{q}') \quad \mathbf{g}'_2(\mathbf{q}')).$$

## Solution of Problem 2

Consider the following outputs:

$$\begin{aligned} y_1 &= x + \ell \cos \theta + b \cos(\theta + \phi) \\ y_2 &= y + \ell \sin \theta + b \sin(\theta + \phi) \end{aligned}$$

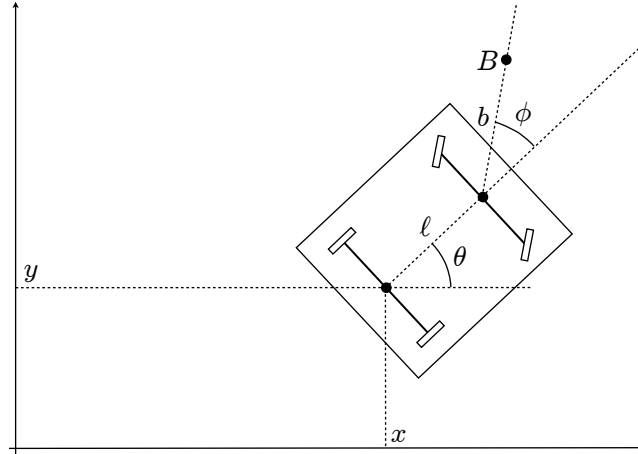
with  $b \neq 0$ . They represent the Cartesian coordinates of a point  $B$  lying on the line passing through the front axle midpoint and oriented as the front wheels; in particular,  $B$  is located at a distance  $b$  from the midpoint (see figure).

The time derivatives of  $y_1$  and  $y_2$  are

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \cos \theta - \tan \phi (\sin \theta + b \sin(\theta + \phi) / \ell) & -b \sin(\theta + \phi) \\ \sin \theta + \tan \phi (\cos \theta + b \cos(\theta + \phi) / \ell) & b \cos(\theta + \phi) \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix} = \mathbf{T}(\theta, \phi) \begin{pmatrix} v \\ \omega \end{pmatrix}.$$

Matrix  $\mathbf{T}(\theta, \phi)$  has determinant  $b / \cos \phi$ , and is therefore always invertible under the assumption that  $b \neq 0$  and  $|\phi(t)| \leq \pi/2$ . It is then sufficient to use the following input transformation

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \mathbf{T}^{-1}(\theta, \phi) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



to put the equations of the bicycle in the input-output linearized form:

$$\begin{aligned}
 \dot{y}_1 &= u_1 \\
 \dot{y}_2 &= u_2 \\
 \dot{\theta} &= \sin \phi (\cos(\theta + \phi) u_1 + \sin(\theta + \phi) u_2) / \ell \\
 \dot{\phi} &= -(\cos(\theta + \phi) \sin \phi / \ell + \sin(\theta + \phi) / b) u_1 \\
 &\quad -(\sin(\theta + \phi) \sin \phi / \ell - \cos(\theta + \phi) / b) u_2.
 \end{aligned}$$

At this point, a simple linear controller such as

$$\begin{aligned}
 u_1 &= \dot{y}_{1d} + k_1(y_{1d} - y_1) \\
 u_2 &= \dot{y}_{2d} + k_2(y_{2d} - y_2),
 \end{aligned}$$

with  $k_1 > 0$ ,  $k_2 > 0$ , guarantees exponential convergence to zero of the Cartesian tracking error, with decoupled dynamics on its two components. Note that the orientation and the steering angle, whose evolutions are governed by the third and fourth equation, are not controlled.