Autonomous and Mobile Robotics
Solution of Final Class Test, 2011/2012

Note: only solutions to Problems 1 and 3 are provided. Problem 2 is actually a discussion and many different choices are possible.

Solution of Problem 1

The angular momentum conservation constraint is linear in the generalized velocities, and therefore Pfaffian. In particular, it can be rewritten as

$$\boldsymbol{a}^T(\boldsymbol{q})\,\dot{\boldsymbol{q}} = 0 \qquad \text{with} \quad \boldsymbol{a}^T(\boldsymbol{q}) = (a_1(q_2)\ \ a_2(q_2))$$

The corresponding kinematic model describes the admissible velocities as $\dot{\boldsymbol{q}} \in \mathcal{N}(\boldsymbol{a}^T(\boldsymbol{q}))$. The number of generalized coordinates is $n = 2$ while the number of constraints is $k = 1$; therefore, the dimension of the null space of $\boldsymbol{a}^T(\boldsymbol{q})$ is $n - k = 1$. One possible basis for this null space is clearly given by

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_2(q_2) \\ -a_1(q_2) \end{pmatrix}$$

Correspondingly, the kinematic model associated to the conservation of angular momentum is

$$\dot{\boldsymbol{q}} = v\,\boldsymbol{g}(\boldsymbol{q}) = v \begin{pmatrix} a_2(q_2) \\ -a_1(q_2) \end{pmatrix}$$

where $v \in I\!R$ is the velocity input for this model (no direct physical meaning).

The controllability of the above kinematic model is characterized by its accessibility distribution $\Delta_{\mathcal{A}}$, i.e., the involutive closure of $\Delta = \text{span}(\boldsymbol{g})$. Since Δ is generated by a single vector field, it is necessarily involutive, i.e., $\Delta_{\mathcal{A}} = \Delta$. Then we have $\dim \Delta_{\mathcal{A}} = \dim \Delta = 1 < 2$ and the system is *not controllable*. This means that the angular momentum conservation constraint is *holonomic*, i.e., it can be written as a geometric constraint $h(q_1, q_2) = 0$. It is therefore impossible to move the coordinates q_1 and q_2 to arbitrary values; once one of them is given, the other follows from the constraints.

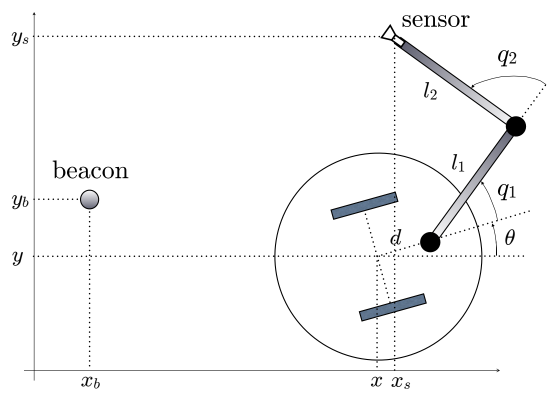
Note that the angular momentum is conserved only if no external forces act on the system. In the considered case, this means that the only available input is the torque at the rotational joint between the two bodies. We may then write a kinematic model whose velocity input is 'closer' to the actual input. To this end, consider the following alternative[1] basis for the null space of $\boldsymbol{a}^T(\boldsymbol{q})$

$$\boldsymbol{g}'(\boldsymbol{q}) = \begin{pmatrix} -\dfrac{a_2(q_2)}{a_1(q_2)} \\ 1 \end{pmatrix}$$

and the corresponding kinematic model

$$\dot{\boldsymbol{q}} = v_2 \begin{pmatrix} -\dfrac{a_2(q_2)}{a_1(q_2)} \\ 1 \end{pmatrix}$$

with the velocity input $v_2 = \dot{q}_2$, i.e., the velocity of the actuated rotational joint. Obviously, this model is also not controllable.



Solution of Problem 3

Refer to the figure for the definition of the relevant variables and quantities. The configuration vector of the mobile manipulator is $\boldsymbol{q} = (x \;\; y \;\; \theta \;\; q_1 \;\; q_2)^T$. The corresponding kinematic model is

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ u_1 \\ u_2 \end{pmatrix}$$

with obvious meaning for v, ω, u_1, u_2. Using Euler integration and including noise, a discrete-time nonlinear model of this system is obtained as

$$\boldsymbol{q}_{k+1} = \begin{pmatrix} x_{k+1} \\ y_{k+1} \\ \theta_{k+1} \\ q_{1,k+1} \\ q_{2,k+1} \end{pmatrix} = \begin{pmatrix} x_k + v_k\,T_s\cos\theta_k \\ y_k + v_k\,T_s\sin\theta_k \\ \theta_k + \omega_k\,T_s \\ q_{1,k} + u_{1,k}\,T_s \\ q_{2,k} + u_{2,k}\,T_s \end{pmatrix} + \begin{pmatrix} w_{p1,k} \\ w_{p2,k} \\ w_{p3,k} \\ w_{p4,k} \\ w_{p5,k} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}_k, \boldsymbol{u}_k) + \boldsymbol{w}_{p,k}$$

where T_s is the sampling interval, $\boldsymbol{u}_k = (v_k \;\; \omega_k \;\; u_{1,k} \;\; u_{2,k})^T$ is the input vector in $[t_k, t_{k+1}]$, and $\boldsymbol{w}_{p,k} = (w_{p1,k} \;\; \ldots \;\; w_{p5,k})^T$ is a white gaussian process noise with zero mean and covariance matrix $\boldsymbol{W}_{p,k}$. As usual, v_k and ω_k will be reconstructed from wheel encoder readings (see, e.g., the formulas in the AMR slides "Odometric Localization"); whereas $u_{1,k}$, $u_{2,k}$ will be reconstructed from joint encoder readings, e.g., by numerical differentiation.

The Cartesian coordinates $\boldsymbol{p}_s = (x_s, y_s)$ of the sensor are expressed as

$$\begin{aligned} x_s &= x + d\cos\theta + l_1\cos(\theta + q_1) + l_2\cos(\theta + q_1 + q_2) \\ y_s &= y + d\sin\theta + l_1\sin(\theta + q_1) + l_2\sin(\theta + q_1 + q_2) \end{aligned}$$

The output equation is therefore

$$z_k = \sqrt{(x_{s,k} - x_b)^2 + (y_{s,k} - y_b)^2} + w_{m,k} = h(\boldsymbol{q}_k) + w_{m,k}$$

where $w_{m,k}$ is a white gaussian measurement noise with zero mean and (co)variance $W_{m,k}$. Note that z_k, $h(\boldsymbol{q}_k)$, $w_{m,k}$ and $W_{m,k}$ are all scalars.

The linearization of the process and output equations, respectively evaluated at the previous estimate $\hat{\boldsymbol{q}}_k$ and at the prediction $\hat{\boldsymbol{q}}_{k+1|k}$, gives

$$\boldsymbol{F}_k = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}_k = \hat{\boldsymbol{q}}_k} = \begin{pmatrix} 1 & 0 & -v_k T_s \sin\hat{\theta}_k & 0 & 0 \\ 0 & 1 & v_k T_s \cos\hat{\theta}_k & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[1] We are assuming that $a_1(q_2) \neq 0$, $\forall q_2$.

and

$$\boldsymbol{H}_{k+1} = \left.\frac{\partial h}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}_k=\hat{\boldsymbol{q}}_{k+1|k}} = \left.\frac{\partial h}{\partial \boldsymbol{p}_{s,k}}\frac{\partial \boldsymbol{p}_{s,k}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}=\hat{\boldsymbol{q}}_{k+1|k}}$$

$$= \frac{1}{\sqrt{(\hat{x}_{s,k+1|k}-x_b)^2+(\hat{y}_{s,k+1|k}-y_b)^2}}\begin{pmatrix}\hat{x}_{s,k+1|k}-x_b & \hat{y}_{s,k+1|k}-y_b\end{pmatrix}\left.\frac{\partial \boldsymbol{p}_{s,k}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}=\hat{\boldsymbol{q}}_{k+1|k}}$$

Here, $\hat{x}_{s,k+1|k}, \hat{y}_{s,k+1|k}$ are the sensor coordinates at the predicted configuration $\hat{\boldsymbol{q}}_{k+1|k}$. The 2×5 Jacobian matrix $\partial \boldsymbol{p}_{s,k}/\partial \boldsymbol{q}_k\big|_{\boldsymbol{q}=\hat{\boldsymbol{q}}_{k+1|k}}$ is easily computed from the above expressions for x_s and y_s.

The EKF equations are finally obtained as follows.

1. State and covariance prediction:

$$\begin{aligned}\hat{\boldsymbol{q}}_{k+1|k} &= \boldsymbol{f}(\hat{\boldsymbol{q}}_k, \boldsymbol{u}_k)\\ \boldsymbol{P}_{k+1|k} &= \boldsymbol{F}_k\boldsymbol{P}_k\boldsymbol{F}_k^T + \boldsymbol{W}_{p,k}\end{aligned}$$

2. Correction:

$$\begin{aligned}\hat{\boldsymbol{q}}_{k+1} &= \hat{\boldsymbol{q}}_{k+1|k} + \boldsymbol{R}_{k+1}\nu_{k+1}\\ \boldsymbol{P}_{k+1} &= \boldsymbol{P}_{k+1|k} - \boldsymbol{R}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\end{aligned}$$

where the innovation

$$\nu_{k+1} = z_{k+1} - \sqrt{(x_{s,k+1|k}-x_b)^2+(y_{s,k+1|k}-y_b)^2}$$

is a scalar quantity and the Kalman gain matrix

$$\boldsymbol{R}_{k+1} = \boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + W_{m,k+1})^{-1}$$

is a 5×1 matrix (note that $\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + W_{m,k+1}$ is actually a scalar, so no matrix inverse computation is required).

In these equations, $\boldsymbol{P}_k$ obviously denotes the covariance of the estimate, which will be initialized at a certain value reflecting the uncertainty on the initial estimate $\hat{q}_0$.