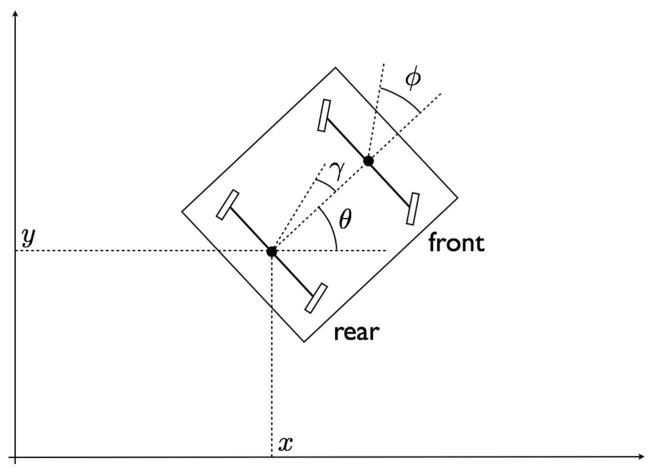
Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2015/2016

Solution of Problem 1

1. A convenient choice of generalized coordinates is $\boldsymbol{q} = (x \;\; y \;\; \theta \;\; \phi \;\; \gamma)^T$ (see figure). The dimension of the configuration space is $n = 5$.



Both two-wheel axles can be assimilated to a single wheel located at the axle midpoint. The equivalent robot has then two wheels: the front wheel, which can be steered, and the rear wheel, which can be steered and driven. The $k = 2$ kinematic constraints acting on the robot are therefore (one pure rolling condition for each wheel):

$$\dot{x}\,\sin(\theta+\gamma) - \dot{y}\,\cos(\theta+\gamma) = 0 \quad (1)$$
$$\dot{x}_f\,\sin(\theta+\phi) - \dot{y}_f\,\cos(\theta+\phi) = 0, \quad (2)$$

where (x_f, y_f) are the Cartesian coordinates of the front axle midpoint. Denoting by ℓ the distance between the two axle midpoints, we have $x_f = x + \ell\,\cos\theta$ and $y_f = y + \ell\,\sin\theta$. By using these, constraint (2) can be rewritten as follows

$$\dot{x}\,\sin(\theta+\phi) - \dot{y}\,\cos(\theta+\phi) - \dot{\theta}\,\ell\,\cos\phi = 0. \quad (3)$$

Constraints (1) and (3) can be put in Pfaffian form:

$$\begin{pmatrix} \sin(\theta+\gamma) & -\cos(\theta+\gamma) & 0 & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\,\cos\phi & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1^T(\boldsymbol{q}) \\ \boldsymbol{a}_2^T(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}.$$

Since $\boldsymbol{A}^T$ is a 2×5 $(k \times n)$ matrix, its null space has dimension $5 - 2 = 3$. By inspection, a basis of $\mathcal{N}(\boldsymbol{A}^T)$ can then be easily written as

$$\boldsymbol{g}_1(\boldsymbol{q}) = \begin{pmatrix} \cos(\theta+\gamma) \\ \sin(\theta+\gamma) \\ \alpha \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{g}_2(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{g}_3(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where α in $\boldsymbol{g}_1(\boldsymbol{q})$ must be such that $\boldsymbol{a}_2^T(\boldsymbol{q})\,\boldsymbol{g}_1(\boldsymbol{q}) = 0$. An easy computation provides

$$\alpha = \frac{\sin(\phi - \gamma)}{\ell \cos\phi}.$$

The kinematic control system of the Cycab is then

$$\dot{\boldsymbol{q}} = \boldsymbol{g}_1(\boldsymbol{q})\,v + \boldsymbol{g}_2(\boldsymbol{q})\,\omega_f + \boldsymbol{g}_3(\boldsymbol{q})\,\omega_r,$$

where v is the driving[1] velocity and ω_f, ω_r are steering velocities of the front and rear wheels, respectively. The model can be written in a more explicit format as

$$\dot{x} = v\cos(\theta + \gamma) \quad (4)$$
$$\dot{y} = v\sin(\theta + \gamma) \quad (5)$$
$$\dot{\theta} = v\,\frac{\sin(\phi - \gamma)}{\ell\cos\phi} \quad (6)$$
$$\dot{\phi} = \omega_f \quad (7)$$
$$\dot{\gamma} = \omega_r. \quad (8)$$

2. Controllability of the Cycab can be established by computing Lie Brackets of the input vector fields and using the accessibility rank condition. Alternatively, one may prove controllability of the Cycab *constructively* relying on controllability of the car-like robot. In fact, the following maneuver can be used to steer the Cycab between two arbitrary configurations $\boldsymbol{q}_1 = (x_1 \;\; y_1 \;\; \theta_1 \;\; \phi_1 \;\; \gamma_1)^T$ and $\boldsymbol{q}_2 = (x_2 \;\; y_2 \;\; \theta_2 \;\; \phi_2 \;\; \gamma_2)^T$:

1. rotate the rear wheels from γ_1 to zero using the ω_r command;
2. drive the Cycab from $(x_1 \;\; y_1 \;\; \theta_1 \;\; \phi_1 \;\; 0)^T$ to $(x_2 \;\; y_2 \;\; \theta_2 \;\; \phi_2 \;\; 0)^T$ keeping $\gamma \equiv 0$ by setting ω_r to zero (a trajectory for doing this certainly exists, because the Cycab with $\gamma \equiv 0$ becomes a car-like robot, which is controllable);
3. rotate the rear wheels from zero to γ_2 using the ω_r command.

Solution of Problem 2

Since we are only interested in controlling θ, we may take it as output and try to perform input-output linearization. The time derivative of θ is directly given by (6)

$$\dot{\theta} = v\,\frac{\sin(\phi - \gamma)}{\ell\cos\phi}.$$

Only one input appears (the driving velocity v) but this is sufficient because the output is scalar. In fact, by using the input transformation

$$v = \frac{\ell\cos\phi}{\sin(\phi - \gamma)}u, \quad (9)$$

we obtain a linear map between the derivative of the output and the new input u:

$$\dot{\theta} = u.$$

This simple integrator dynamics can be made globally asymptotically stable around the desired output trajectory $\theta_d(t)$ by letting

$$u = \dot{\theta}_d + k(\theta_d - \theta).$$

The control law in terms of the original input v is readily obtained by using the last expression of u in (9).

Note that the input transformation (9) becomes singular when $\phi = \gamma$. This singularity can be easily avoided by properly choosing the other two control inputs ω_f and ω_r, which are not used by the input-output linearization controller. In particular, if at the initial instant t_0 we have $\phi_0 \neq \gamma_0$, it is sufficient to set $\omega_f = \omega_r = 0$ to guarantee that the singularity is never met. If instead $\phi_0 = \gamma_0$, one may perform a brief burst of duration ϵ with one of two steering velocities so as to achieve $\phi_\epsilon \neq \gamma_\epsilon$; and then set $\omega_f = \omega_r = 0$.

[1]In fact, one may verify that $\dot{x}^2 + \dot{y}^2 = v^2$. Other kinematic models can be written choosing a different $\boldsymbol{g}_1$, but this is the only choice appropriate for rear-wheel drive.

Solution of Problem 3

A straightforward approach for designing the required localization system is to use an Extended Kalman Filter.

The discrete-time nonlinear model of the Cycab (process dynamics) is easily obtained from (4–8) using Euler integration

$$
\begin{aligned}
x_{k+1} &= x_k + v_k\, T_s \cos(\theta_k + \gamma_k) \\
y_{k+1} &= y_k + v_k\, T_s \sin(\theta_k + \gamma_k) \\
\theta_{k+1} &= \theta_k + v_k\, T_s \frac{\sin(\phi_k - \gamma_k)}{\ell \cos \phi_k} \\
\phi_{k+1} &= \phi_k + \omega_{f,k} T_s \\
\gamma_{k+1} &= \gamma_k + \omega_{r,k} T_s.
\end{aligned}
$$

where T_s is the sampling interval. As usual, the discrete-time velocity inputs v_k and ω_k can be reconstructed from wheel encoder readings (see, e.g., the formulas in the AMR slides 'Odometric Localization').

Denote by (x_c, y_c) the Cartesian coordinates of the charging station, which represents the landmark used by our localization system. Since at time t_k the sensor is located at (x_k, y_k), the output equation (measurement model) is

$$
h_k = \sqrt{(x_k - x_c)^2 + (y_k - y_c)^2}.
$$

The rest of the solution is trivial: linearize the process dynamics and measurement model and then derive the EKF equations.