Stability Theory for Nonlinear Systems

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Introduction

consider a **nonlinear** time-invariant dynamic system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x) \end{aligned}$$

with state $x \in I\!R^n$, input $u \in I\!R^p$, output $y \in I\!R^q$

typical problem

compute, given $x_0 = x(0)$ and $u_{[0,t]}$, the state $x(t)$ and/or the output $y(t)$ for any $t > 0$

e.g., in linear systems, where $f(x,u) = Ax + Bu$, one has

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

however:

often, one is not interested in computing the explicit solution, but rather in studying some properties such as **boundedness**, **asymptotic behavior**, ...

$\Rightarrow$ **qualitative** theory of differential equations (Poincaré 1880, Lyapunov 1892, LaSalle and Lefschetz 1947...)

basic idea

study the qualitative behavior of the system under **perturbations** of the initial state and of the input with respect to nominal values

denoting by $x(t)$ the state solution corresponding to x_0 and $u_{[0,t]}$, we wonder:

- what happens if $x_0 \to x_0 + \Delta x_0$?
- what happens if $u_{[0,t]} \to u_{[0,t]} + \Delta u_{[0,t]}$?

in particular:

- how **close** is the perturbed evolution to the nominal evolution?
- under which conditions the two solutions tend to **coincide** for $t \to \infty$?

it seems natural to call

- **stable** a system in which small perturbations give rise to small variations
- **unstable** a system in which small perturbations give rise to large variations

stability theory consists of

definitions

stability properties (different kinds depending on system behavior or application needs)

conditions

that a system must satisfy to possess these various properties

criteria

to check whether these conditions hold or not, without computing explicitly the perturbed solution of the system

e.g., in linear systems

- definition of stability, asymptotic stability, instability
- asymptotic stability condition: $\lim_{t\to\infty} x(t)|_{u\equiv 0} = \lim_{t\to\infty} e^{At}x_0 = 0$
- asymptotic stability criteria:
  - all eigenvalues of A must have negative real part
  - Routh criterion
  - Nyquist criterion for feedback systems

typically one considers the behavior of systems **in free evolution**

$$\dot{x} = f(x)$$

with respect to **perturbations of the initial state** x_0

motivation:

- consider $\dot{x} = f(x, u)$ with a feedback control law $u = h(x)$; the **closed-loop** dynamics becomes

  $$\dot{x} = f(x, h(x)) = f'(x)$$

  that is, a (new) system in free evolution

- consider $\dot{x} = f(x, u)$; if the input perturbation is **non-persistent**

  $$\tilde{u}(t) = \begin{cases} u(t) + \delta(t) & t \in [0, t_1] \\ u(t) & t > t_1 \end{cases}$$

  the problem can be recast as studying the effect of a perturbed initial state (i.e., $x(t_1)$)

Definitions

an important preliminary concept: equilibrium point

a state $x_e \in I\!R^n$ is an **equilibrium point** of system $\dot{x} = f(x)$ if setting $x_0 = x_e$ implies $x(t) = x_e$, $\forall t > 0$ (a **degenerate** trajectory of the system)

hence

x_e is an **equilibrium point** if $f(x_e) = 0$

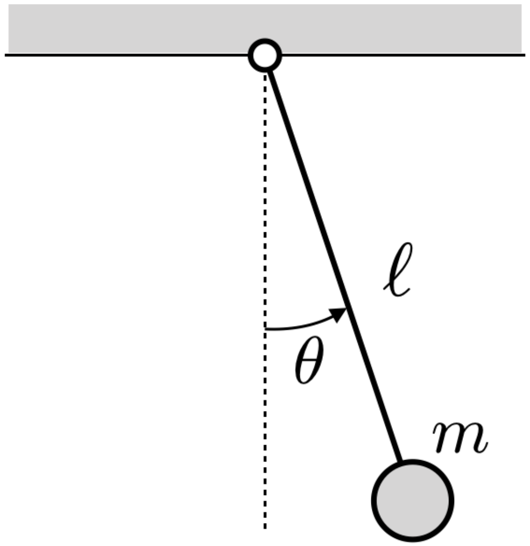
i.e., equilibrium points are the **zeros** of the vector function $f(x)$

e.g., in linear systems $\dot{x} = Ax$, equilibrium points must satisfy

$$Ax_e = 0 \qquad \text{i.e.,} \quad x_e \in \mathcal{N}(A)$$

- if A is nonsingular, the **only** equilibrium point is the origin
- if A is singular, the equilibrium points are **infinite** and **contiguous**: geometrically, they are hyperplanes passing through the origin (lines if $\dim(\mathcal{N}(A)) = 1$, planes if $\dim(\mathcal{N}(A)) = 2$, ...)

e.g., pendulum of length ℓ and mass m in the presence of viscous friction with coefficient d



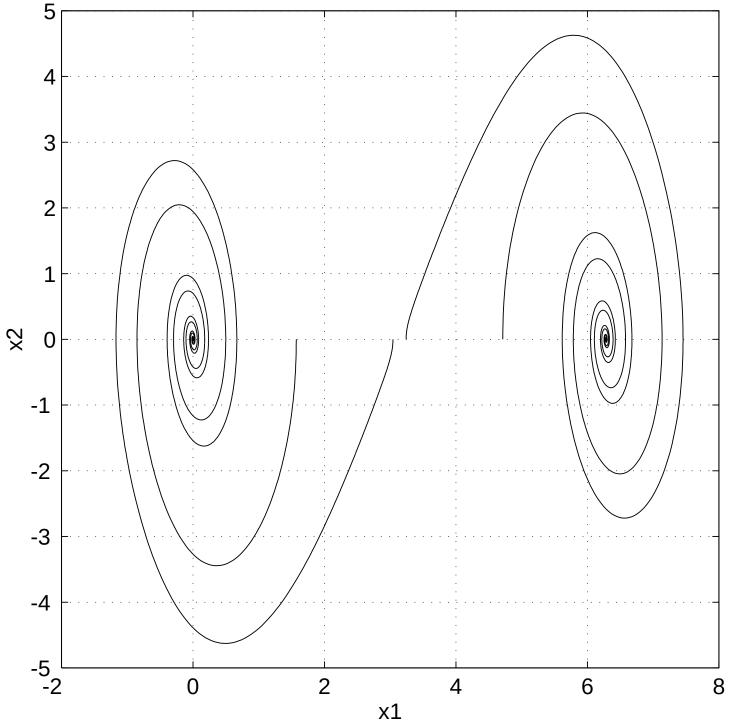
$$m\,\ell^2\,\ddot\theta + d\,\dot\theta + m\,g\,\ell\sin\theta = 0$$

setting $x = (x_1, x_2) = (\theta, \dot\theta)$, the state space equations are

$$\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= -\frac{g}{\ell}\sin x_1 - \frac{d}{m\ell^2}\,x_2 \end{aligned}$$

$\Rightarrow f(x) = (x_2 \quad -\frac{g}{\ell}\sin x_1 - \frac{d}{m\ell^2}\,x_2)^T$; a **nonlinear** system!

equilibrium points are characterized by $x_1 = j\pi$ ($j = 0, 1$) and $x_2 = 0$ (pendulum pointing up/down and at rest) here are the pendulum trajectories in the plane $(x_1, x_2) = (\theta, \dot{\theta})$ (**phase plane**)



e.g., another nonlinear system

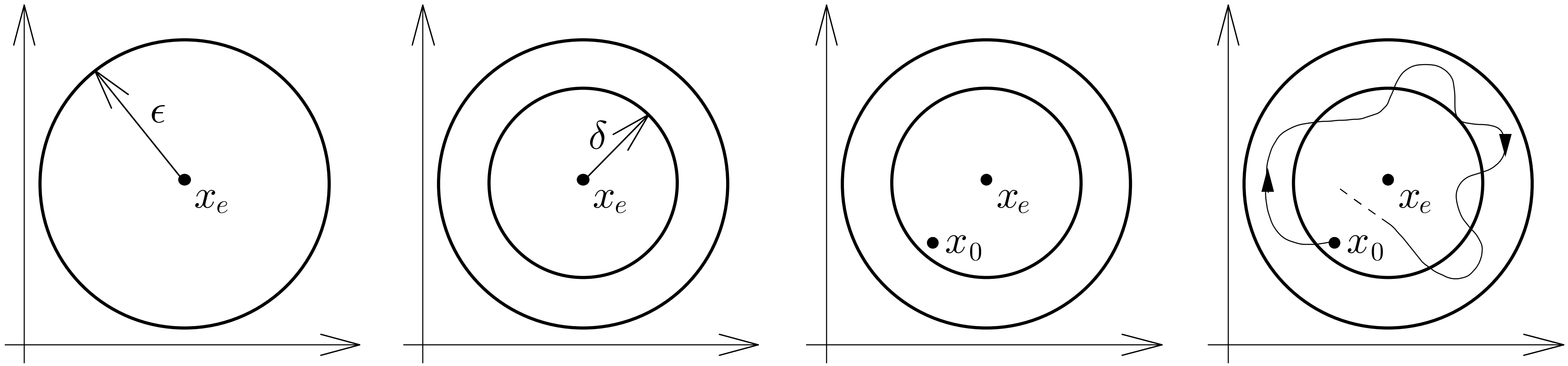
$$
\begin{aligned}
\dot{x}_1 &= 1 - x_1^3 \\
\dot{x}_2 &= x_1 - x_2^2
\end{aligned}
$$

equilibrium points are described by $x_1 = 1$ and $x_2 = \pm 1$ ■

note: the equilibrium points of a nonlinear system can be finite (2 in the previous examples, but any other number is possible, including zero) or infinite, and they can be **isolated** points in state space

stability definitions [Lyapunov]
(in the following, $|\cdot|$ denotes any norm in $\mathbb{R}^n$)

an equilibrium point x_e is **stable** (S) if: $$\forall \epsilon, \exists\, \delta(\epsilon) : |x_0 - x_e| < \delta \Rightarrow |x(t) - x_e| < \epsilon, \forall t > 0$$



$\forall \epsilon$ $\qquad$ $\exists\, \delta(\epsilon)$ $\qquad$ $|x_0 - x_e| < \delta$ $\qquad$ $|x(t) - x_e| < \epsilon, \forall t > 0$

an equilibrium point x_e of a dynamic system is stable if it possible to keep the system evolution **arbitrarily close** to x_e by choosing the initial condition x_0 **sufficiently close** to x_e; that is, if it is possible to arbitrarily bound the solution in the neighborhood of x_e by suitably bounding the perturbation

obviously: an equilibrium point x_e is **unstable** (U) if it is not stable

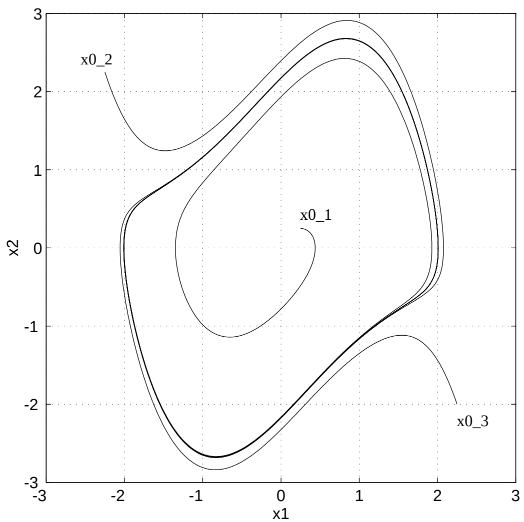
- stability is a **property of equilibrium points**: a system may have both stable and unstable equilibrium points (only happens in nonlinear systems, e.g., the pendulum)

- the definition of stability does **not** require the perturbed evolution to **converge** to x_e

- on the other hand, instability does **not** mean that the perturbed evolution **diverges**

  e.g., Van der Pol oscillator (mass-spring-damper with position-dependent damping)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2)x_2 \end{aligned}$$



regardless of the initial condition, trajectories converge to a **limit cycle**: hence, it is impossible to bound arbitrarily the displacement from 0 (e.g., for $\epsilon = 1$ there esists no δ)

⇒ the origin is an unstable equilibrium point for this system ∎

in practice, often **simple** stability is not enough:

an equilibrium point x_e is **asymptotically stable** (AS) if:

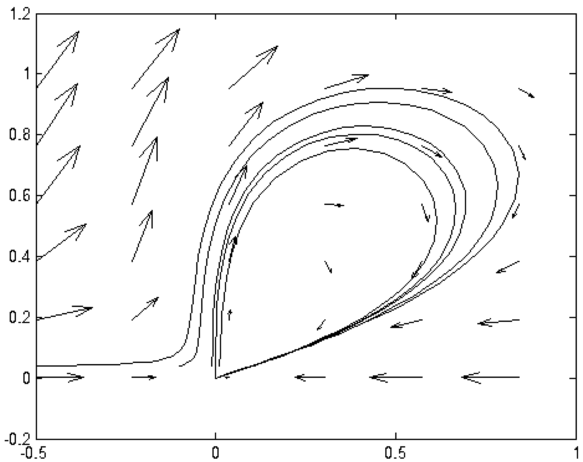
1. it is stable
2. $\exists\, \delta_a : |x_0 - x_e| < \delta_a \Rightarrow \lim_{t\to\infty} |x(t) - x_e| = 0$

- in addition to stability, AS requires that the state converges to x_e for initial conditions sufficiently close to x_e
- asymptotic stability is a **local** concept, i.e., convergence is guaranteed provided that x_0 belongs to the spherical neighborhood of x_e of radius δ_a (**basin of attraction**); if x_0 is outside this neighborhood, $x(t)$ may not converge or **even diverge**!
- 2. **does not imply** 1.; that is, one may have convergence without stability (equilibrium points of this kind are sometimes called **quasi-asymptotically stable**, but they are actually unstable)

e.g., [Vinograd]

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}$$

$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}$$



regardless of how close x_0 is to the origin, if $x_{1,0} < 0$ the trajectory will converge only after going a **finite distance** away from 0: hence, it is not possible to bound at will the solution around the origin

$\Rightarrow$ the origin is an unstable (quasi-asymptotically stable) equilibrium point for this system ∎

in applications, it is useful to have an **estimate** of the time needed by the perturbed evolution to return to a given neighborhood of x_e

an equilibrium point x_e is **exponentially stable** (ES) if there exist positive constants α, λ and c such that:

$$|x(t) - x_e| \leq \alpha |x_0 - x_e| e^{-\lambda t}, \quad \forall t > 0, \quad \forall |x_0 - x_e| < c$$

- ES requires that there exists a neighborhood of x_e from which perturbed solutions converge to x_e with at least exponential speed; with respect to the definition of AS, condition 1 may be **omitted** because it is implied by exponential convergence

- λ is called **exponential convergence rate**; setting $\alpha = e^{\lambda \tau_0}$, an easy computation shows that after $(\tau_0 + 1/\lambda)$ seconds the distance from x_e is reduced to at least $1/e$ (around 35%) of its initial value

- ES **implies** AS but the opposite is **not true**

  e.g., the origin is asymptotically but not exponentially stable for system

  $$\dot{x} = -x^2$$

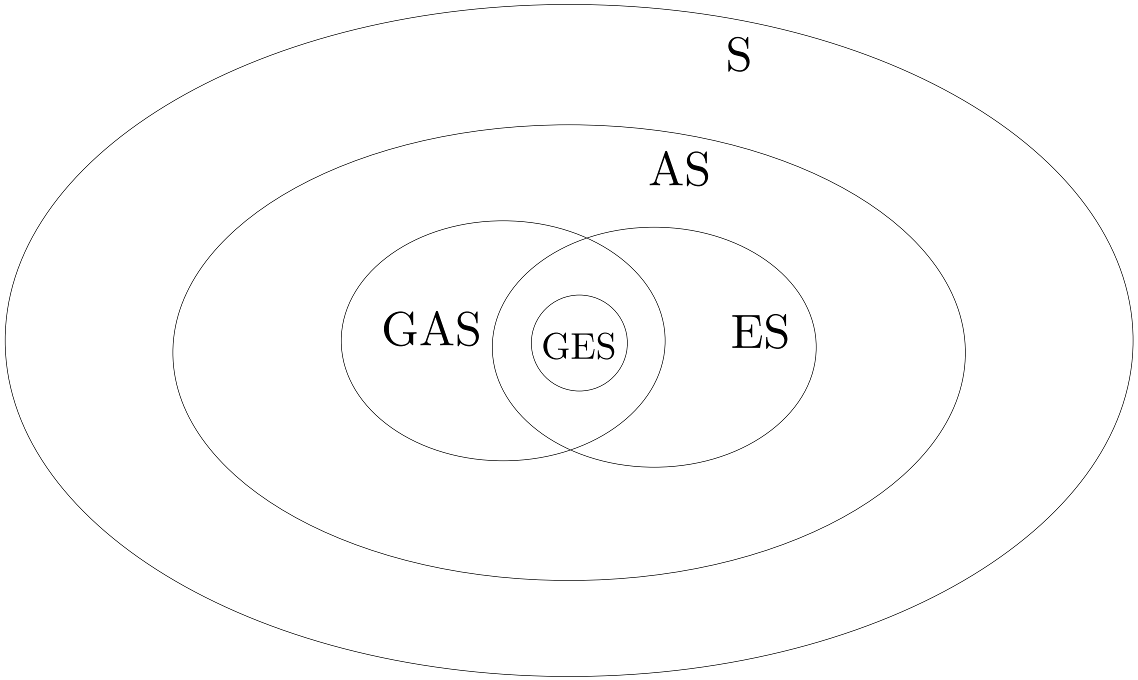
  in fact, its solution is $x(t) = \frac{x_0}{1 + t x_0}$, whose convergence to zero is slower than any exponential function ∎

asymptotic stability and exponential stability, which are intrinsically local properties, become **global** when the domain of attraction coincides with $I\!R^n$:

an equilibrium point x_e is **globally asymptotically stable** (GAS) if it is stable and the state converges to x_e for **any** initial state

an equilibrium point x_e is **globally exponentially stable** (GES) if the state converges exponentially to x_e for **any** initial state

summarizing, we have the following classification of **stable** equilibrium points



note: a necessary condition for an x_e to be GAS is that it is the **only** equilibrium point

Stability of Linear Systems

theorem

if a linear system admits multiple equilibrium points, stability (instability) of one of them implies stability (instability) of all the others

proof it is enough to show that, if the generic equilibrium point x_e is stable, then the origin is stable, and vice versa

by hypothesis, we have: $\forall \epsilon, \exists\, \delta(\epsilon) : |x_0 - x_e| < \delta \Rightarrow |x(t) - x_e| < \epsilon, \forall t > 0$

$x(t) - x_e$ is the difference between the solutions starting from x_0 and x_e, respectively $\Rightarrow$ due to linearity, $x(t) - x_e$ is also the solution starting from $x_0 - x_e = z_0$, denoted by $x_{z_0}(t)$

we have then: $\forall \epsilon, \exists\, \delta(\epsilon) : |z_0| < \delta \Rightarrow |x_{z_0}(t)| < \epsilon, \forall t > 0$; that is, the origin is stable

the vice versa is shown similarly ∎

theorem

in a linear system:

1. only the origin can be AS, and only when there exist no other equilibrium points
2. if the origin is AS, it is also GAS

proof

1: trivial (see slide 5)

2: trivial for finite-dimensional time-invariant systems: local convergence of the free evolution $x(t) = e^{At}x_0$ requires the eigenvalues of A to have negative real part; but then, convergence is global ∎

theorem

in a linear system, the origin is ES if and only if it is AS

proof

necessity: trivial

sufficiency: trivial for finite-dimensional time-invariant systems, because if the origin is AS then the free evolution is a combination of converging exponentials ■

summarizing, in linear systems:

- if the origin is the only equilibrium point, it can be S, AS (actually, ES), or U
- if there are multiple equilibrium points, they are infinite, contiguous, and they are either all S or all U
- in any case, one may directly say that **the system is S, AS (actually, ES), or U**

the following stability criterion is immediate

theorem

a finite-dimensional time-invariant linear system is S if and only if

1. eigenvalues of A with geometric multiplicity equal to algebraic multiplicity have non-positive real part
2. eigenvalues of A with geometric multiplicity lower than algebraic multiplicity have negative real part

the system is ES if and only if all the eigenvalues of A have negative real part

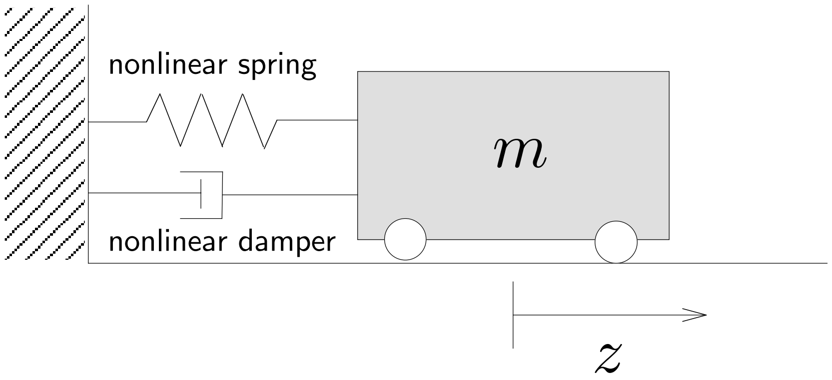
to avoid computing eigenvalues, apply Routh criterion to characteristic polynomial of A

Direct Lyapunov Method

basic idea

if the total energy of a system (mechanical, electrical, . . .) is continuously **dissipated**, the system (linear or nonlinear) tends to an equilibrium ⇒ one may be able to prove stability/instability by looking at the variation of a **single scalar function**

e.g., nonlinear mass-spring-damper system, state $x = (z, \dot{z})$



$$m\ddot{z} + b\dot{z}|\dot{z}| + (k_0 z + k_1 z^3) = 0$$

one cannot study stability of the origin using the definition, because it is impossible to solve the above differential equation in closed-form: let us look then at mechanical energy

$$V(x) = V_{\text{kin}}(\dot{z}) + V_{\text{pot}}(z) = \frac{1}{2}m\dot{z}^2 + \int_0^z (k_0\zeta + k_1\zeta^3)d\zeta = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}k_0 z^2 + \frac{1}{4}k_1 z^4$$

energy/stability relationships

- energy is zero only at the equilibrium point $z = 0$, $\dot{z} = 0$, i.e., the origin
- if energy (always) converges to zero, then the origin is (globally) asymptotically stable
- if energy diverges, then the origin is unstable

how does energy change when the system moves? it is sufficient to differentiate V with respect to t (of which V is a composite function) and replace $\ddot{z}$ with its expression derived from the dynamic model

$$\dot{V}(x) = m\dot{z}\ddot{z} + (k_0 z + k_1 z^3)\dot{z} = -b|\dot{z}|^3 \leq 0$$

$\Rightarrow$ intuitively, energy is **continuously dissipated** until the system converges to a state with zero velocity ($\dot{z} = 0$); moreover, since in any position different from $z = 0$ the mass would be subject to a nonzero elastic force $-k_0 z - k_1 z^3$, we may conclude that **state trajectories actually converge to the origin** ($z = 0$, $\dot{z} = 0$) ■

the direct Lyapunov method is based on a generalization (and rigorous formalization) of the above reasoning: one looks for a suitable *energy-like* scalar function for the nonlinear system under consideration, and studies its evolution in time as the system moves in the following, we refer to the nonlinear time-invariant systems in free evolution

$$\dot{x} = f(x) \qquad x \in I\!R^n$$

and denote by x_e the equilibrium point under study; thus, $f(x_e) = 0$

preliminary definitions: consider a scalar function $V(x)$, continuously differentiable with respect to x (i.e., $V \in C^1$), and a spherical neighborhood $S(x_e, r)$ of x_e with radius r

- $V(x)$ is **positive definite** (PD) in $S(x_e, r)$ if
  - a) $V(x_e) = 0$
  - b) $V(x) > 0, \forall x \in S(x_e, r), x \neq x_e$

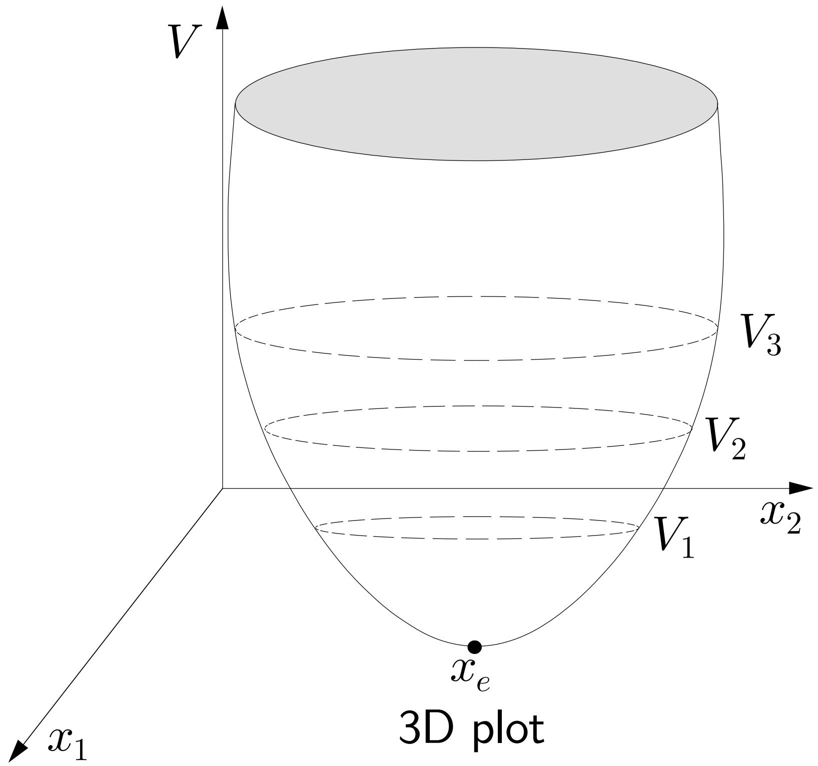
- $V(x)$is **positive semidefinite** (PSD) in $S(x_e, r)$ if
  - a) $V(x_e) = 0$
  - b) $V(x) \geq 0, \forall x \in S(x_e, r), x \neq x_e$

- $V(x)$ is **negative definite** (ND) in $S(x_e, r)$ if $-V(x)$ is positive definite, **negative semidefinite** (NSD) in $S(x_e, r)$ if $-V(x)$ is positive semidefinite

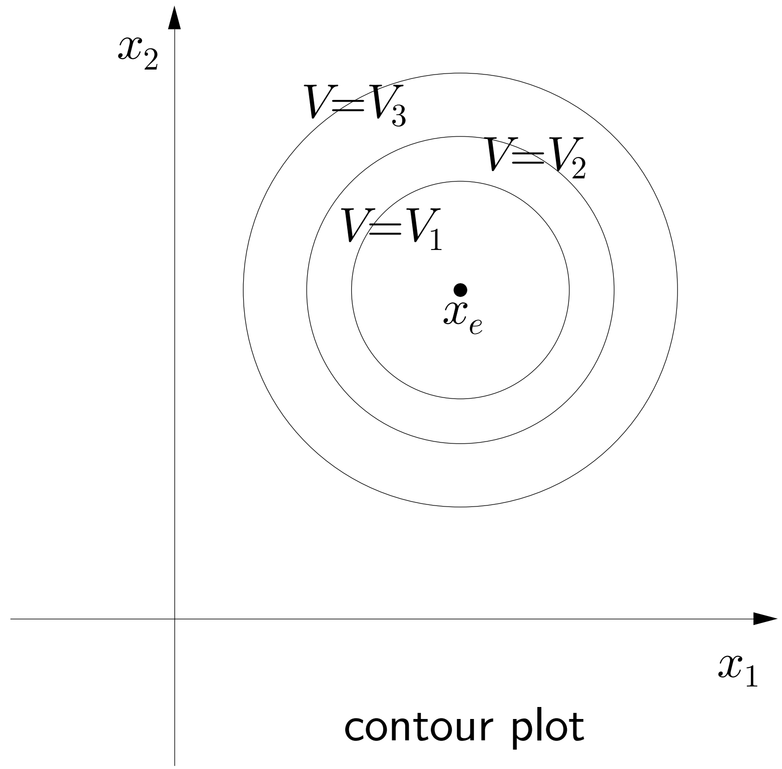
- $V(x)$ is **indefinite** (I) in $S(x_e, r)$ if it is not DP, SDP, DN or SDN

note: $V(x)$ PD (ND) in $S(x_e, r) \Rightarrow V(x)$ PSD (NSD) in $S(x_e, r)$

case $n = 2$: **local** shape of a PD function V around x_e



3D plot



contour plot

e.g., in $\mathbb{R}^2$, function $V(x) = x^T x = x_1^2 + x_2^2$ is PD in any neighborhood of the origin (all level curves are closed)

e.g., in $\mathbb{R}^2$, function $V(x) = x_1^2$ is PSD in any neighborhood of the origin (it is zero on all points of the x_2 axis; no level curves are closed)

e.g., in $\mathbb{R}^2$, function $V(x) = x_1 x_2$ is I in any neighborhood of the origin (there are always neighborhood points where it is positive and neighborhood points where it is negative)

e.g., for the nonlinear mass-spring-damper, mechanical energy is PD in any neighborhood of the origin assume a function $V(x)$ is given, and consider a solution $x(t)$ of $\dot{x} = f(x)$: one may consider $V(x(t))$ as a **composite** function of t, continuously differentiable for any t; we have

$$\dot{V}(t) = \frac{dV(x(t))}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial t} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x(t)) = \dot{V}(x)$$

where $f_i(x(t))$ is the i-th component of vector function $f(x)$

$\dot{V}(x)$, regarded as a function of x, is the **derivative of V along the system trajectories**

$\dot{V}(x)$ can then be positive (negative) definite, positive (negative) semidefinite, or indefinite

e.g., consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_2^2 \\ \dot{x}_2 &= -x_1^2 - x_1 x_2 - x_2 \end{aligned}$$

whose only equilibrium point is the origin, and let $V = (x_1^2 + x_2^2)/2$, which is PD in any neighborhood of the origin; we have

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2 x_2 + x_1 x_2^2 - x_1^2 x_2 - x_1 x_2^2 - x_2^2 = -x_2^2$$

which is NSD in any neighborhood of the origin ■

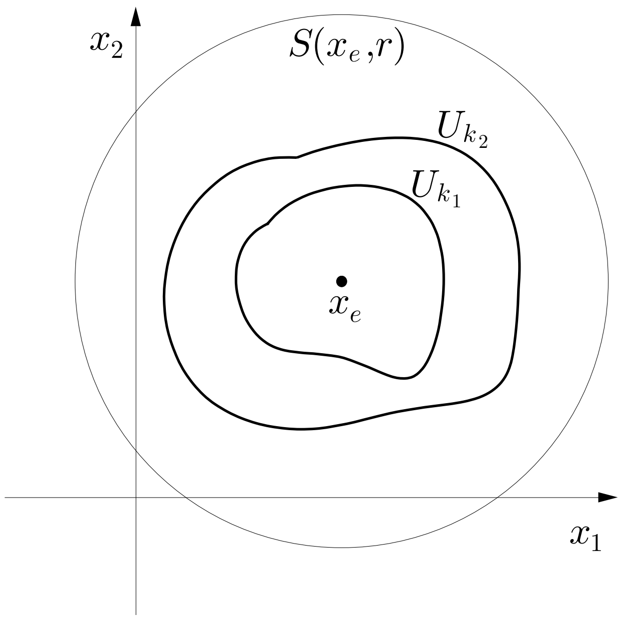
theorem

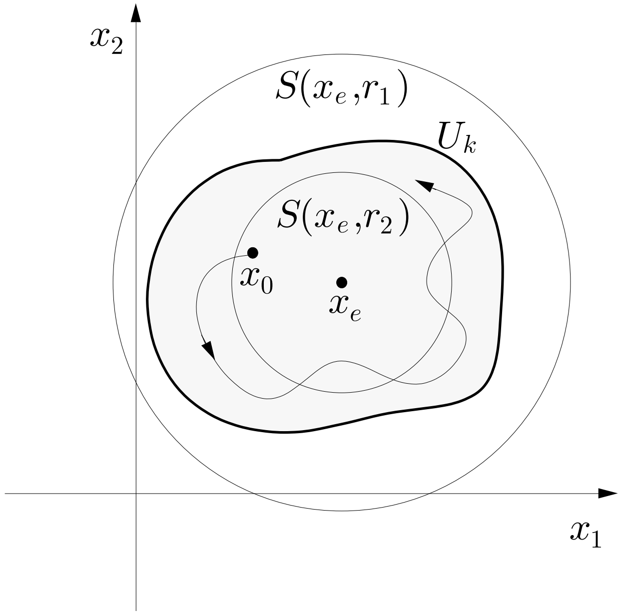
an equilibrium point x_e of $\dot{x} = f(x)$ is **stable** if there exists a function $V(x) \in C^1$ such that

1. $V(x)$ is PD in a neighborhood $S(x_e, r)$
2. $\dot{V}(x)$ is NSD in the same neighborhood

proof based on geometric arguments, case $n = 2$ (but valid in general)

note first that, since $V(x)$ is PD in $S(x_e, r)$, the level curves $U_k = \{x \in I\!R^2 : V(x) = k\}$ are **closed** for sufficiently small k; moreover, if $k_1 < k_2$, U_{k_1} is **inside** U_{k_2}





choose r_1 such that $0 < r_1 \leq r$: then, there exists certainly a k such that U_k is inside $S(x_e, r_1)$ (just take the minimum of V along the boundary of $S(x_e, r_1)$, and a k smaller than such minimum); hence, U_k is closed

moreover, since U_k is a closed curve that contains x_e, it is always possible to find an r_2 such that $S(x_e, r_2)$ is inside U_k

now consider a trajectory starting from $x_0 \in S(x_e, r_2)$; we have $V(x_0) < k$ and, since $\dot{V}$ is negative or zero along the system trajectories contained in $S(x_e, r)$, $V(t)$ is **non-increasing** along the trajectory; hence, we have $V(t) < k$, $\forall t > 0$

$\Rightarrow$ the trajectory $x(t)$ will **always** remain in $S(x_e, r_1)$

wrapping up, for arbitrarily small r_1 we can always find a sufficiently small r_2 such that

$$|x_0 - x_e| < r_2 \quad \Rightarrow \quad |x(t) - x_e| < r_1, \quad \forall t > 0$$

■

- a $V(x)$ with the properties required by the previous theorem (i.e., such that V, $\dot{V}$ are respectively PD, NSD in a neighborhood of x_e) is called a **Lyapunov function**

- the theorem states that the **existence** of a Lyapunov function is a **sufficient** condition for stability; actually, for finite-dimensional time-invariant system one may show that this condition is also **necessary**

- applying the theorem requires 2 phases, possibly to be repeated:

  1. choose a $V(x)$ that is PD in a neighborhood of x_e (called **Lyapunov candidate**)
  2. compute $\dot{V}$ along the system trajectories and check whether it is NSD in the same neighborhood

  note: if the chosen $V(x)$ does not result to be a Lyapunov function, no conclusion can be drawn; another $V'(x)$ may exist that is a Lyapunov function

- if a system admits a Lyapunov function $V(x)$, then it admits an **infinity** of them; e.g., all the following

$$V'(x) = \beta V^{\gamma}(x) \qquad\qquad \beta > 0, \gamma > 1$$

- coming up with 'good' Lyapunov candidates is obviously essential: total energy is usually a reasonable choice in mechanical or electrical systems, but in general better alternatives may exist without a clear physical meaning

e.g., pendulum (for simplicity, $m = 1$, $d = 1$, $\ell = 1$)

the state vector is $x = (x_1, x_2) = (\theta, \dot{\theta})$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g \sin x_1 - x_2 \end{aligned}$$

for the equilibrium point $x_e^{\text{down}} = (0, 0)$, let us choose mechanical energy as a Lyapunov candidate

$$V(x) = V_{\text{kin}}(x) + V_{\text{pot}}(x) = \frac{1}{2} x_2^2 + g(1 - \cos x_1) \qquad \text{PD in } S(0, 2\pi^-)$$

we find

$$\dot{V}(x) = x_2 \dot{x}_2 + g \sin x_1 \dot{x}_1 = -x_2^2 \quad \text{NSD in the same neighborhood (actually, in all } I\!R^2\text{)}$$

hence x_e^{down} is a **stable** equilibrium point (and $\int_0^t \dot{V} d\tau$ is the dissipated energy) ■

physical intuition suggests that, in the presence of friction, the origin is an **asymptotically** stable equilibrium point for the pendulum ⇒ we need a stronger result to prove it

theorem

an equilibrium point x_e of $\dot{x} = f(x)$ is **asymptotically stable** if there exists a function $V(x) \in C^1$ such that

1. $V(x)$ is PD in a neighborhood $S(x_e, r)$
2. $\dot{V}(x)$ is ND in the same neighborhood

proof first, x_e is certainly stable; in particular, if $x_0 \in S(x_e, r_2)$ (see previous proof) the trajectory will always remain in $S(x_e, r_1) \Rightarrow V(t)$ along the trajectory tends to a limit value $\bar{V} \geq 0$ (because $\dot{V} < 0$ and V is bounded below)

now suppose that $\bar{V} > 0$; since V is continuous and only zero at x_e, then there would exist a neighborhood $S(x_e, \sigma)$ in which the trajectory never enters; since $\dot{V}$ too is continuous and only zero at x_e, there would also exist an $\alpha > 0$ such that $\dot{V} \leq -\alpha$ indefinitely

but then we could write

$$V(t) = V(0) + \int_0^t \dot{V}(\tau) d\tau \leq V(0) - \alpha t$$

and thus V would become negative in finite time, contradicting the hypothesis $\bar{V} > 0$

hence, if $x_0 \in S(x_e, r_2)$ we have $\lim_{t\to\infty} V(t) = \bar{V} = 0$; therefore, being $V(x)$ zero only at $x = x_e$, we conclude that $\lim_{t\to\infty} x(t) = x_e$ ■

note: extrapolating the properties of $S(x_e, r_2)$ in the above proof, one may infer that **any neighborhood of** x_e **contained in** U_{V^*} (where V^* is the minimum of V along the boundary of $S(x_e, r)$) is a **lower estimate** of the **basin of attraction** of x_e

e.g., consider the system

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 + x_2^2 - 1) - x_2 \\ \dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

for which the origin is an equilibrium point

choosing

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \qquad \text{PD in any neighborhood of the origin}$$

we find

$$\dot{V}(x) = (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \qquad \text{ND for } x : x_1^2 + x_2^2 < 1, \text{ i.e., in } S(0, 1^-)$$

the origin is then asymptotically stable for the system

to estimate the basin of attraction:

let $U_{V^*} = \{x \in I\!R^2 : V(x) \leq 1/2\} = S(0, 1^-)$; for any choice of $\rho \in (0, 1)$, the neighborhood $S(0, \rho)$ is contained in U_{V^*}, and hence it represents a lower estimate of the **basin of attraction** of the origin

e.g., pendulum; consider the following Lyapunov candidate (no physical meaning)

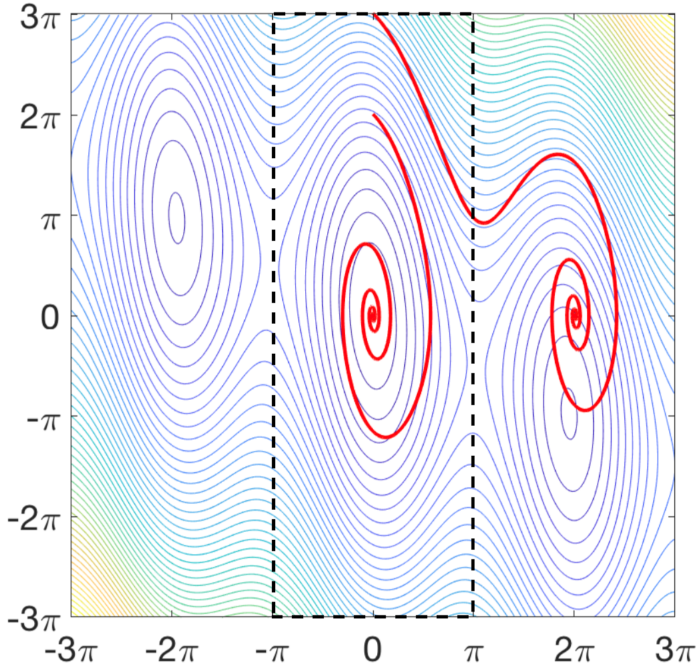
$$V(x) = \frac{1}{2}x_2^2 + 2g(1 - \cos x_1) + \frac{1}{2}(x_1 + x_2)^2$$ PD in any neighborhood of the origin

we get

$$\dot{V}(x) = -x_2^2 - g\, x_1 \sin x_1$$ ND in $S(0, \pi^-)$

hence x_e^{down} is an **asymptotically stable** equilibrium point for the pendulum

domain of attraction: convergence to the origin is guaranteed from initial states **inside level curves that are completely contained in the region where $\dot{V}$ is DN**; if the initial state is inside a level curve that **leaves** such region, divergence **may** occur



■

what happens if we try to apply the previous theorems to the pendulum equilibrium point $x_e^{\text{up}} = (\pi, 0)$? physical intuition tells us that x_e^{up} is **unstable**, but the necessary (and sufficient) condition for stability is the **existence** of a Lyapunov function, which we cannot exclude a priori $\Rightarrow$ an **instability criterion** may be useful

theorem [Cetaev]

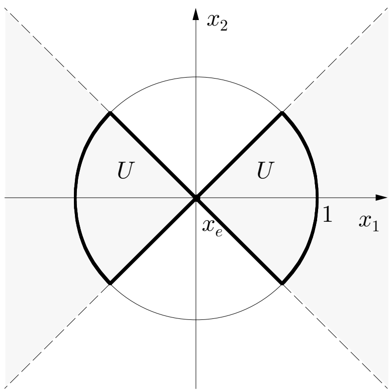
an equilibrium point x_e of $\dot{x} = f(x)$ is **unstable** if there exists a $V(x) \in C^1$ such that

1. x_e is an accumulation point for set $P = \{x : V(x) > 0\}$
2. $\dot{V}(x)$ is PD in $U = P \cap S(x_e, r)$, for some $r > 0$

e.g., the equilibrium point $x_e = (0,0)$ is **unstable** for the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

in fact, consider $V(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$, positive in $P = \{x : |x_1| > |x_2|\}$, of which x_e is an accumulation point



we have

$$\dot{V}(x) = x_1^2 + x_1 x_2^2 + x_2^2 = x_1^2 + x_2^2(1 + x_1)$$

that is clearly PD in $U = P \cap S(x_e, 1)$

■

there also exists a criterion for **global** asymptotic stability

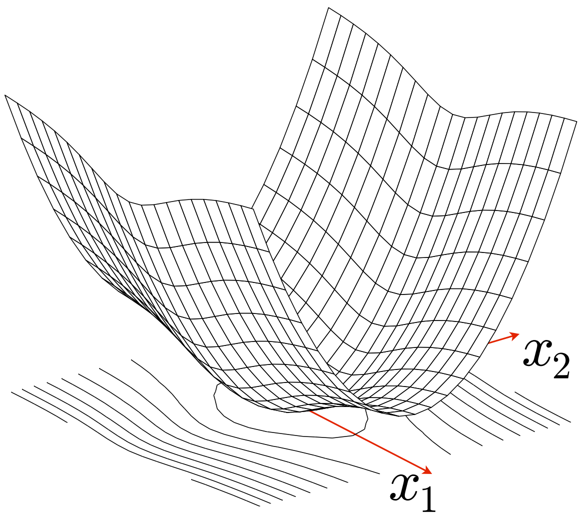
theorem

an equilibrium point x_e of $\dot{x} = f(x)$ is **globally asymptotically stable** if there exists a function $V(x) \in C^1$ such that

1. $V(x)$ is PD in any neighborhood of x_e
2. $\dot{V}(x)$ is ND in any neighborhood of x_e
3. $V(x)$ is **radially unbounded**, i.e.., $\lim_{|x-x_e|\to\infty} V(x) = \infty$

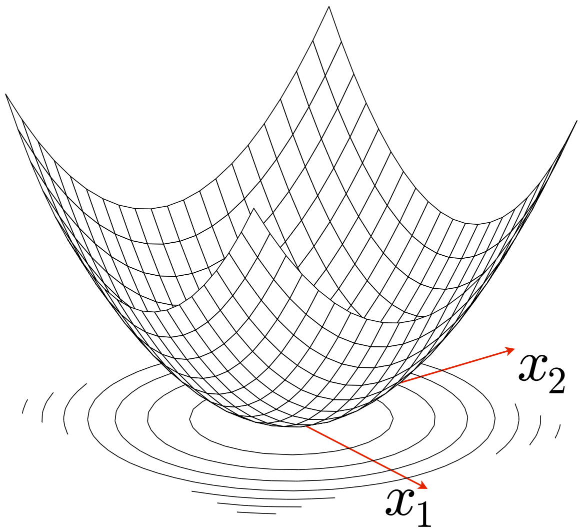
e.g.

$$V = \frac{x_1^2}{1+x_1^2} + x_2^2$$



not radially unbounded

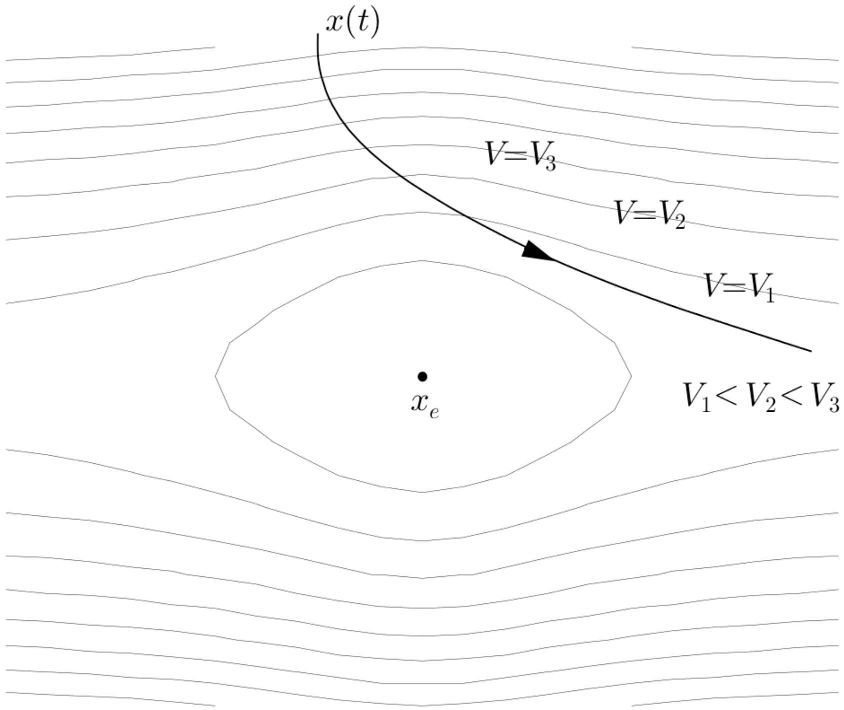
$$V = x_1^2 + x_2^2$$



radially unbounded

proof as in the local case, having observed that the radial unboundedness of V, together with the fact that $\dot{V}$ is ND in all $\mathbb{R}^n$, implies that for any initial condition x_0 the trajectory will remain within the **limited** region defined by $V(x) \leq V(x_0)$ ■

note: if V is **not** radially unbounded, level curves far from x_e are not closed; hence, the state trajectory may diverge from x_e and still remain within the region defined by $V(x) \leq V(x_0)$, while actually crossing level curves that correspond to decreasing values of V



$\Rightarrow$ when x_0 is sufficiently far from x_e, $x(t)$ may **not** converge to x_e

e.g., consider the family of nonlinear systems described by

$$\dot{x} = -c(x), \qquad \text{where} \quad xc(x) > 0, \forall x \neq 0, \text{ and } c(0) = 0$$

and the Lyapunov candidate

$$V(x) = \frac{1}{2}x^2$$

that is PD in any neighborhood of $x_e = 0$ and radially unbounded

we find

$$\dot{V}(x) = x\dot{x} = -xc(x)$$

i.e., $\dot{V}(x)$ is ND in any neighborhood of $x_e = 0$

$\Rightarrow$ x_e is globally asymptotically stable ■

summarizing, the **direct Lyapunov criterion** provides the following results:

	x_e is S	x_e is AS	x_e is GAS	x_e is unstable
$V(x)$	PD in $S(x_e, r)$	PD in $S(x_e, r)$	PD in any $S(x_e, r)$ and radially unbounded	x_e accumulation pt. of $P = \{x : V(x) > 0\}$
$\dot{V}(x)$	NSD in $S(x_e, r)$	ND in $S(x_e, r)$	ND in any $S(x_e, r)$	PD in $P \cap S(x_e, r)$

Building Lyapunov Functions

the main difficulty in applying the direct Lyapunov method for studying an equilibrium point x_e of a nonlinear system $\dot{x} = f(x)$ consists in **choosing** the candidate function $V(x)$; sometimes the physics of the problem provides an inspiration, but in general it may be useful to proceed in a systematic way

a strategy which is often effective is to define $V(x)$ as a **quadratic form** of the kind

$$V(x) = \frac{1}{2}(x - x_e)^T Q (x - x_e)$$

where the $n \times n$ matrix Q is **symmetric** and **positive definite** (i.e., such that $w^T Q w > 0$, $\forall w \neq 0$)

to guarantee that Q is positive definite one may use the following necessary and sufficient **Sylvester condition**

$$Q_{11} > 0, \qquad \begin{vmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{vmatrix} > 0, \qquad \ldots \qquad \det(Q) > 0$$

since Q is symmetric, $\dot{V}(x)$ is computed as

$$\dot{V}(x) = \frac{1}{2}\dot{x}^T Q (x - x_e) + \frac{1}{2}(x - x_e)^T \dot{Q} (x - x_e) + \frac{1}{2}(x - x_e)^T Q \dot{x} = (x - x_e)^T Q \dot{x} + \frac{1}{2}(x - x_e)^T \dot{Q} (x - x_e)$$

e.g., consider the system

$$\begin{aligned}\dot{x}_1 &= k_2 x_1\\ \dot{x}_2 &= -x_2^3 + k_1 x_3\\ \dot{x}_3 &= -2x_2 - x_3^3\end{aligned}$$

with $k_2 < 0$ and $k_1 > 0$; the origin is the only equilibrium point

- choose

  $$V(x) = \frac{1}{2}(x - x_e)^T I_{3\times 3}(x - x_e) = \frac{1}{2}x^T x = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

  which is PD in any neighborhood of the origin and radially unbounded

  we get

  $$\dot{V}(x) = x^T \dot{x} = x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 = k_2 x_1^2 - x_2^4 + (k_1 - 2)x_2 x_3 - x_3^4$$

  in the special case $k_1 = 2$, $\dot{V}(x)$ is ND in any neighborhood of the origin, which is then GAS

- a much more general result is obtained by choosing $Q = \text{diag}(1, \frac{2}{k_1}, 1)$

  $$V(x) = \frac{1}{2}x^T Q x = \frac{1}{2}(x_1^2 + \frac{2}{k_1}x_2^2 + x_3^2) \quad\Rightarrow\quad \dot{V}(x) = k_2 x_1^2 - \frac{2}{k_1}x_2^4 - x_3^4$$

  that, for $k_2 < 0$ and $k_1 > 0$, is always ND $\Rightarrow$ the origin is **always** GAS ■

Invariant Set Theorem

in the application of the direct method, one often finds that the time derivative $\dot{V}(x)$ of the chosen Lyapunov function is **only NSD** (rather than ND); in these conditions, one may infer stability but not asymptotic stability of x_e (e.g., see the first Lyapunov function for the pendulum)

in this situation, the **invariant set theorem** may allow to analyze stability in more detail **without changing** $V(x)$

a subset $G \subset I\!R^n$ of the state space is an **invariant set** for $\dot{x} = f(x)$ if any trajectory $x(t)$ starting from a point $x_0 \in G$ always stays in G

it is a generalization of the concept of equilibrium point; examples of invariant sets are

- equilibrium points
- the basin of attraction of an asymptotically stable equilibrium point
- any trajectory of the system (as long as the system is time-invariant)
- $I\!R^n$ itself

basic idea

if $V(x)$ is PD (i.e., $V(x) > 0$) and $\dot{V}(x)$ is NSD (i.e., $\dot{V}(x) \leq 0$) in a neighborhood of x_e, $V(x)$ must tend to a limit value $\Rightarrow \dot{V}(x)$ should tend to zero, at least under certain conditions

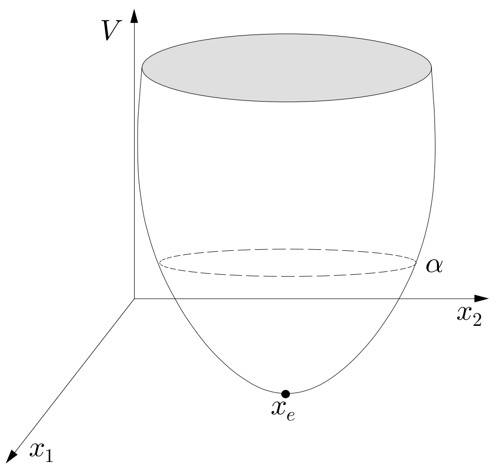
local invariant set theorem [LaSalle]

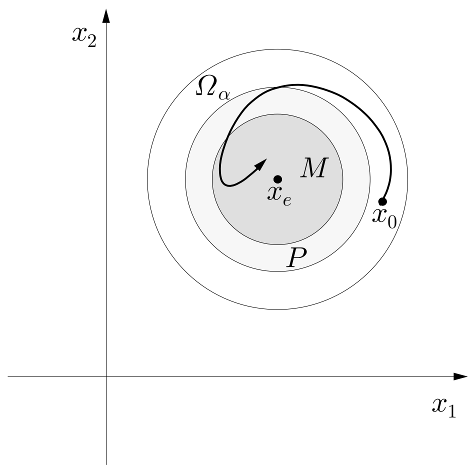
for a system $\dot{x} = f(x)$, assume that there exists a function $V(x) \in C^1$ such that:

1. region $\Omega_\alpha = \{x \in I\!R^n : V(x) \leq \alpha\}$ is bounded, for some $\alpha > 0$
2. $\dot{V}(x) \leq 0$ in Ω_α

and define P, the set of points of Ω_α where $\dot{V} = 0$; then, any trajectory of the system that starts in Ω_α tends asymptotically to M, the **largest invariant set** contained in P

here: largest invariant set contained in P = union of all invariant subsets of P





an immediate consequence is the following

corollary

an equilibrium point x_e of $\dot{x} = f(x)$ is **asymptotically stable** if there exists a function $V(x) \in C^1$ such that

1. $V(x)$ is PD in a set D that contains x_e
2. $\dot{V}(x)$ is NSD in the same set
3. the largest invariant set M in P (the subset of D where $\dot{V} = 0$) consists of x_e only

moreover, denoting by Ω the largest region defined by $V(x) \leq \alpha$, $\alpha > 0$ and contained in D, we have that Ω is an estimate of the **basin of attraction** of x_e

- condition 1 of the corollary implies condition 1 of the local invariant set theorem
- compared with the direct Lyapunov criterion for AS, this corollary 'relaxes' condition 2 (ND $\rightarrow$ NSD) but adds condition 3
- the direct Lyapunov criterion for AS is a special case of this result ($P = x_e$)
- in itself, D is not an estimate of the basin of attraction (in fact, some of the level curves crossing D may be open, and in that case D would not be invariant)

e.g., let us consider again the pendulum with the first Lyapunov function

$$V(x) = \frac{1}{2}x_2^2 + g(1 - \cos x_1) \quad \text{PD in } S(0, 2\pi^-)$$

we have

$$\dot{V}(x) = x_2\dot{x}_2 + g \sin x_1 \dot{x}_1 = -x_2^2 \quad \text{NSD in the same neighborhood (actually in all } I\!R^2\text{)}$$

therefore, $x_e^{\text{down}} = (0, 0)$ is a stable equilibrium point for the pendulum; but the invariant set theorem tells us **more**

set P consists of points for which $\dot{V} = 0$, i.e., states such that $x_2 = 0$; which is the largest invariant set M contained in P?

the dynamics of the system in P is

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -g \sin x_1 \end{aligned}$$

if $x_1 \neq 0$, we would get $\dot{x}_2 \neq 0$ and thus x_2 would change, driving $x(t)$ outside set P

$\Rightarrow$ set M consists only of the origin

hence, x_e^{down} is an **asymptotically stable** equilibrium point for the pendulum

note: the basin of attraction of the origin can be estimated as follows: compute V^*, the minimum of V along the boundary of $S(0, 2\pi^-)$, and identify U_{V^*}, the level curve of V corresponding to V^*; a lower estimate of the basin of attraction is the region of $I\!R^2$ contained in U_{V^*} ■

there exists a global version of the invariant set theorem

global invariant set theorem [LaSalle]

for a system $\dot{x} = f(x)$, assume that there exists a function $V(x) \in C^1$ such that:

1. $V(x)$ is radially unbounded
2. $\dot{V}(x) \leq 0$ in $I\!R^n$

then, any trajectory of the system tends asymptotically to the set M, the **largest invariant set** contained in P, the set of points of Ω_α where $\dot{V} = 0$

note: radial unboundedness of $V(x)$ guarantees that any region $\Omega_\alpha = \{x \in I\!R^n : V(x) < \alpha\}$, $\alpha > 0$, is bounded

and the corresponding

corollary

an equilibrium point x_e of $\dot{x} = f(x)$ is **globally asymptotically stable** if there exists a function $V(x) \in C^1$ such that

1. $V(x)$ is PD in any neighborhood of x_e and radially unbounded
2. $\dot{V}(x)$ is NSD in any neighborhood of x_e
3. the largest invariant set M in P (the subset of D where $\dot{V} = 0$) consists of x_e only

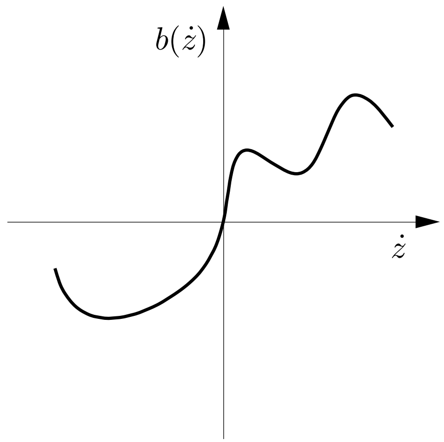
e.g., consider the family of second-order nonlinear systems described by

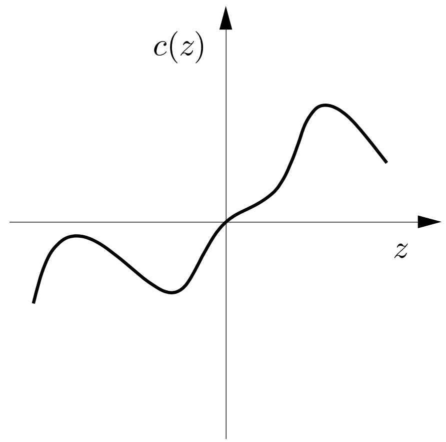
$$\ddot{z} + b(\dot{z}) + c(z) = 0$$

where functions $b(\cdot)$ and $c(\cdot)$ are continuous and such that

$$\dot{z}\, b(\dot{z}) > 0, \forall \dot{z} \neq 0 \qquad z\, c(z) > 0, \forall z \neq 0$$

note that these conditions, combined with continuity, imply that $b(0) = 0$, $c(0) = 0$





this family includes mass-spring-damper mechanical systems (with nonlinear elastic force $c(z)$ and viscous friction $b(\dot{z})$) and RLC electrical systems (with nonlinear inductance $c(z)$ and resistance $b(\dot{z})$)

consider the only equilibrium point $x_e = (z_e, \dot{z}_e) = (0,0)$ and, as a Lyapunov candidate, the total energy of the system (e.g., kinetic + potential)

$$V(x) = \frac{1}{2}\dot{z}^2 + \int_0^z c(y)dy$$

which is PD in any neighborhood of x_e (but not necessarily radially unbounded); then

$$\dot{V}(z) = \dot{z}\ddot{z} + c(z)\dot{z} = -\dot{z}b(\dot{z}) - \dot{z}c(z) + c(z)\dot{z} = -\dot{z}b(\dot{z})$$

is NSD in any neighborhood of $x_e = 0$ in view of the assumptions

set P consists of points for which $\dot{V} = 0$, i.e., states such that $\dot{z} = 0$; which is the largest invariant set M contained in P? the dynamics of the system in P is

$$\ddot{z} = -c(z)$$

if $z \neq 0$, we would get $\ddot{z} \neq 0$ and thus $\dot{z}$ would change, driving $x(t)$ outside set $P \Rightarrow$ set M consists only of the origin

hence, the origin is an **AS** equilibrium point for any system of this family; if, in addition,

$$\lim_{z\to\infty} \int_0^z c(y)dy = \infty$$

then V is radially unbounded and the origin is **GAS**

Barbalat Lemma

invariant set theorems only apply to **time-invariant** systems!

the following result may sometimes prove useful for **time-varying** systems

lemma [Barbalat]

for a system $\dot{x} = f(x, t)$, consider a function $V(x, t) \in C^1$ such that:

1. $V(x)$ is lower bounded
2. $\dot{V}(x, t) \leq 0$
3. $\dot{V}(x, t)$ is **uniformly continuous**

then $\dot{V}(x, t)$ **converges to zero** along the trajectories of the system

- conditions 1 and 2 guarantee that $V(x, t)$ tends to a finite limit
- condition 3 is usually replaced by the (stronger) condition

  3. $\ddot{V}(x, t)$ is bounded

- this lemma can also be useful for time-invariant systems, because it **relaxes** some conditions (e.g., V is not required to be PD)

Indirect Lyapunov Method

basic idea

by analyzing the stability of the **linear approximation** of a nonlinear system around an equilibrium point x_e, it may be possibile to infer conclusions about the stability of x_e **for the original system**

consider a nonlinear system $\dot{x} = f(x)$, with equilibrium point x_e, that is, $f(x_e) = 0$

if $f \in C^\infty$, Taylor expansion around x_e provides

$$f(x) = f(x_e) + \left.\frac{df}{dx}\right|_{x_e} (x - x_e) + h(x - x_e) = J(x_e)(x - x_e) + h(x - x_e)$$

where $h(x - x_e)$ collects the (infinite) terms of degree higher than 1 and $J(x_e)$ is the Jacobian matrix of f with respect to x, computed at x_e

in the new coordinates $\xi = x - x_e$, the dynamics is described by

$$\dot{\xi} = \dot{x} = f(x) = J(x_e)\xi + h(\xi)$$

in the vicinity of the equilibrium point x_e, higher-order terms may be neglected $\Rightarrow$ we obtain the following **linear approximation**

$$\boxed{\dot{\xi} = J(x_e)\xi}$$

whose accuracy in describing the original system is higher as the state is closer to x_e

the analysys of the linear approximation $\dot{\xi} = J(x_e)\xi$ leads to interesting conclusions for the original nonlinear system $\dot{x} = f(x)$

theorem

if matrix $J(x_e)$ is **nonsingular**, x_e is an **isolated** equilibrium point of the nonlinear system

proof by contradiction: if the thesis were false, in any neighborhood of x_e there would be at least one x_e' such that $f(x_e') = f(x_e) = 0$; we would have then

$$f(x_e') = f(x_e) + J(x_e)(x_e' - x_e) + h(x_e' - x_e) = 0 \;\Rightarrow\; J(x_e)(x_e' - x_e) + h(x_e' - x_e) = 0$$

since x_e' changes, this condition requires both terms to be zero; in particular, it must be $J(x_e)(x_e' - x_e) = 0$, which however contradicts the fact that $J(x_e)$ is non singular ■

the opposite is not true; it may happen that x_e is an isolated equilibrium and $J(x_e)$ is singular

e.g., consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1^2 \\ \dot{x}_2 &= x_2 \end{aligned}$$

whose only equilibrium point is the origin; however

$$J(x_e) = \begin{pmatrix} 2x_1 & 0 \\ 0 & 1 \end{pmatrix}\Bigg|_{x_e} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is singular ■

the stronger result is the following

theorem (**indirect Lyapunov criterion**)

consider the linear approximation $\dot{\xi} = J(x_e)\xi$ of a nonlinear system $\dot{x} = f(x)$ around an equilibrium point x_e

1. if all the eigenvalues of $J(x_e)$ have negative real part (i.e., if the linear approximation is AS), x_e is an **asymptotically stable** equilibrium point for the nonlinear system
2. if at least one of the eigenvalues of $J(x_e)$ has positive real part (i.e., if the linear approximation is U), x_e is an **unstable** equilibrium point for the nonlinear system

proof based on the application of direct Lyapunov method: in particular, one shows that there exists a Lyapunov function for the linear approximation which is also a Lyapunov function for the nonlinear system ■

- asymptotic stability of the origin for the linear approximation (which is always global) only implies **local** asymptotic stability of x_e for the nonlinear system

- if no eigenvalue of $J(x_e)$ has positive real part, but some of them have zero real part (i.e., the linear approximation is S — not AS — or U, depending on the relationship between algebraic and geometric multiplicity for these eigenvalues) we are in the **critical case**: **no** conclusion can be drawn on the stability of x_e for the nonlinear system (higher-order terms are decisive)

e.g., consider the pendulum again

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -g\sin x_1 - x_2\end{aligned}$$

the Jacobian matrix is

$$J(x) = \frac{df}{dx} = \begin{pmatrix} 0 & 1 \\ -g\cos x_1 & -1 \end{pmatrix}$$

- around the equilibrium point $x_e^{\text{down}} = (0,0)$ it is

  $$J_e^{\text{down}}(x) = \left.\frac{df}{dx}\right|_{x_e^{\text{down}}} = \begin{pmatrix} 0 & 1 \\ -g & -1 \end{pmatrix}$$

  whose characteristic polynomial is $\lambda^2+\lambda+g$; therefore, the linear approximation of the pendulum around x_e^{down} is AS

  $\Rightarrow$ x_e^{down} is an asymptotically stable equilibrium point for the pendulum

- around the equilibrium point $x_e^{\text{up}} = (\pi,0)$ it is

  $$J_e^{\text{up}}(x) = \left.\frac{df}{dx}\right|_{x_e^{\text{up}}} = \begin{pmatrix} 0 & 1 \\ g & -1 \end{pmatrix}$$

  whose characteristic polynomial is $\lambda^2+\lambda-g$; therefore, the linear approximation of the pendulum around x_e^{up} is U

  $\Rightarrow$ x_e^{up} is an unstable equilibrium point for the pendulum ■

the application of the indirect criterion is **inconclusive** in the critical case; in this situation, one needs to resort to the direct criterion, which is more powerful (and may also allow to determine the basin of attraction, which cannot be analyzed with the indirect method)

e.g., consider the nonlinear system

$$\dot{x} = -x^3$$

whose only equilibrium point is $x_e = 0$; in this case, the Jacobian reduces to a scalar

$$J(x_e) = \left.\frac{df}{dx}\right|_{x_e} = \left.-3x^2\right|_{x_e} = 0$$

and the linear approximation of the system around x_e is $\dot{\xi} = 0 \Rightarrow$ **critical case**

consider the following Lyapunov candidate

$$V(x) = \frac{1}{2}x^2 \qquad \text{PD in any neighborhood of the origin, and radially unbounded}$$

we have

$$\dot{V}(x) = -x^4 \qquad \text{ND in any neighborhood of the origin}$$

$\Rightarrow$ x_e is then GAS for the nonlinear system ■