Elective in Robotics 2014/2015

Analysis and Control of Multi-Robot Systems

Elements of Passivity Theory

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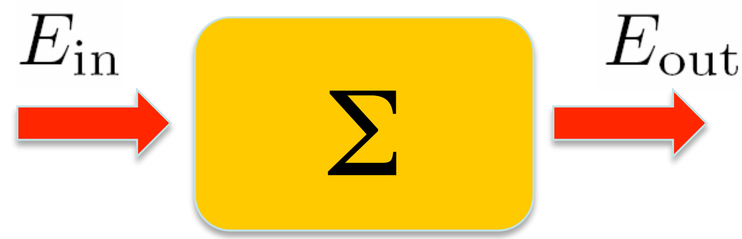
Interconnected Systems

- A way to look at interconnected systems:



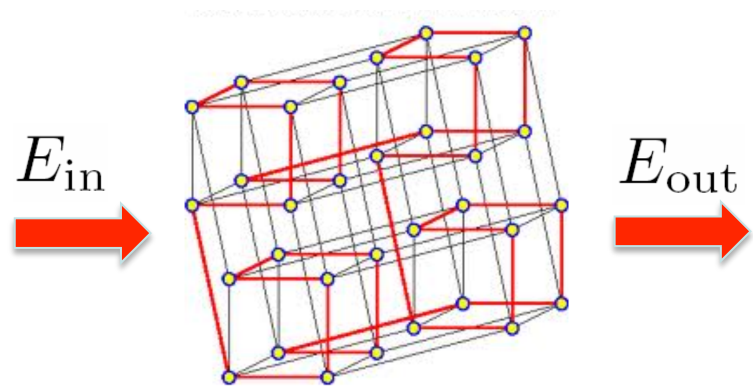
$$E_{\text{out}} \leq E_{\text{in}}$$

$$\Sigma : \dot{x} = f(x(t), u(t), t)$$

- It is often very useful to consider "Input/Output" characterizations of dynamical systems
  - e.g., passivity theory (**not** the only possibility!)

- What if Σ is made of a "network" of simpler systems?

- Can we infer global features out of:
  - The network (graph) topology
  - The individual I/O properties of the single subsystems?

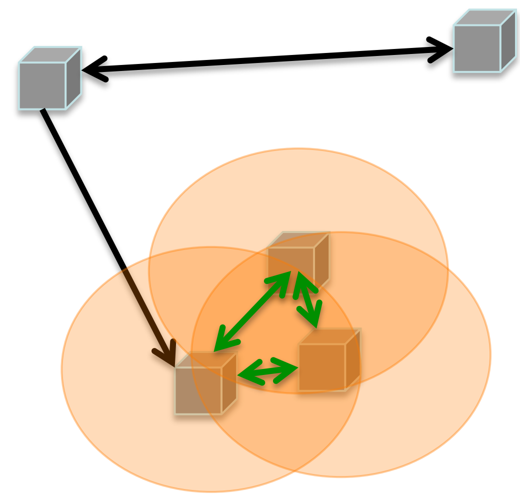


$$E_{\text{out}} \leq E_{\text{in}}$$

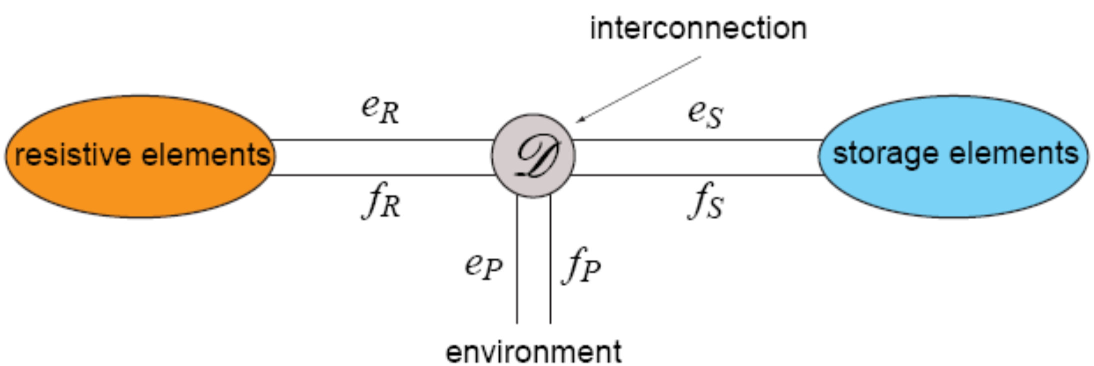
- Is this helpful for modeling and control of multi-robot systems?

Interconnected Systems

- A very useful tool: port-based network modeling
  - aka: <u>Port-Hamiltonian Modeling, Generalized Bond-Graphs, etc.</u>



- General framework that captures
  - I/O - external - overall behavior
  - Internal interconnection (graph) of simpler subsystems

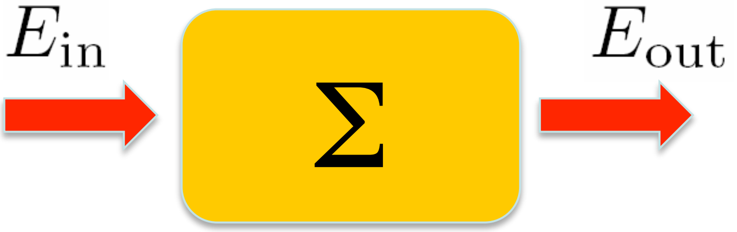


- We will see how to embed within this machinery some of the graph-related topics discussed so far
  - Mainly, graph theory, consensus/agreement protocols, distributed sensing

Introduction to Passivity

- What is passivity?

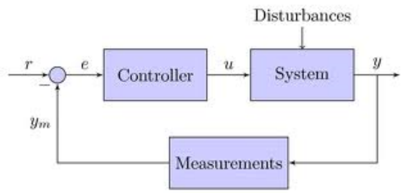
- Intuitively: something that does not produce internal energy

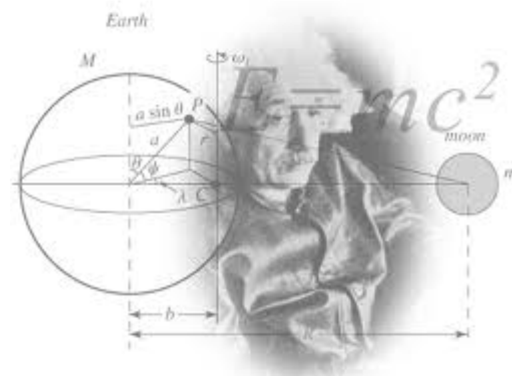


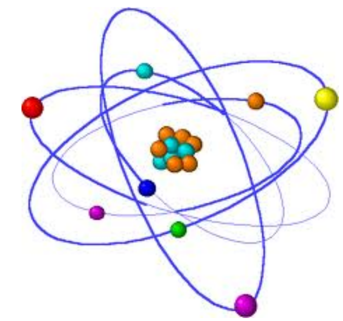
- Stems from circuit theory
- Describes input/output (I/O) behaviors
- Seamlessly applies to linear and nonlinear systems

Introduction to Passivity

- Passivity-like concepts are common to many scientific areas
  - Mathematics
  - Physics
  - Electronics
  - Control



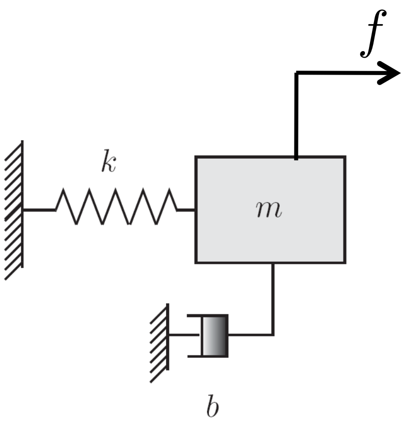




- Basic idea: most physical systems have common I/O characteristics dictated by
  - Energy conservation
  - Energy transportation
  - Energy dissipation

- Energy plays a fundamental role
  - Common unifying language across all physical domains

Introduction to Passivity



- Consider this simple mechanical system with dynamics

$$m\ddot{x} + b\dot{x} + kx = f$$

and energy $E(x,\, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \geq 0,\ \forall x,\, \dot{x}$

- How is the “energy flowing” within the system?

$$\frac{\mathrm{d}}{\mathrm{d}t}E(x,\, \dot{x}) = m\dot{x}\ddot{x} + kx\dot{x} = \boxed{f\dot{x}} - \boxed{b\dot{x}^2}$$

Input/Output Mechanical power

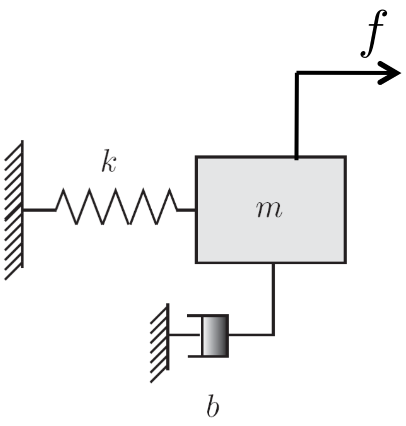
Internal dissipated power

- Integrating back, we get

$$E(x(t),\, \dot{x}(t)) = \boxed{E(x(t_0),\, \dot{x}(t_0))} + \int_{t_0}^{t} f(\tau)\dot{x}(\tau)\mathrm{d}\tau - \int_{t_0}^{t} b\dot{x}^2(\tau)\mathrm{d}\tau$$

Initial stored energy

Introduction to Passivity



- if $f = 0, b = 0 \Rightarrow E = E(x(0), \dot{x}(0)) = const$

  no I/O "energy flow", but still an internal dynamics

$$m\ddot{x} + kx = 0$$

- if $b \geq 0, \;\; f \neq 0$

$$E(x(t), \dot{x}(t)) - E(x(t_0), \dot{x}(t_0)) = \int_{t_0}^{t} f(\tau)\dot{x}(\tau)\mathrm{d}\tau - \int_{t_0}^{t} b\dot{x}^2(\tau)\mathrm{d}\tau \leq \int_{t_0}^{t} f(\tau)\dot{x}(\tau)\mathrm{d}\tau$$

  and since $E(x(t), \dot{x}(t)) \geq 0$ we have

$$\boxed{-E(x(t_0), \dot{x}(t_0)) \leq \int_{t_0}^{t} f(\tau)\dot{x}(\tau)\mathrm{d}\tau}$$

- The total extractable energy is limited by the initial stored energy

Introduction to Passivity

- Passivity: a property of a physical system (but also, more in general, of a linear/nonlinear dynamical system)

- Based on the concept of "energy"

- Describes the energy flow (power) through the system

- It is an I/O characterization

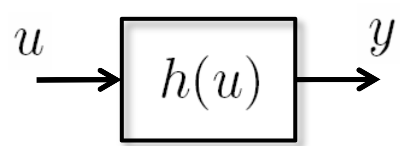
- Usually, passivity is a robust property (e.g., w.r.t. parametric variations)

- It is (of course) related to classical Lyapunov stability concepts

- **Proper compositions of passive systems are passive** -> very useful property (later)

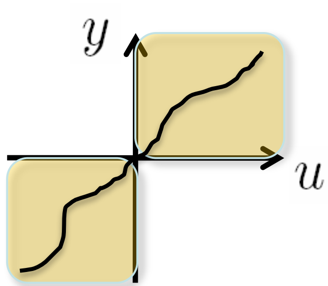
Passivity: formal definitions

- A first definition of passivity can be given for memoriless (static) functions $y = h(u)$



- The function is said to be passive if $u^T y \geq 0, \; \forall u$
  - "Power" flowing into the system is never negative
  - The system does not produce energy (can only absorb and dissipate)
  - Example: the familiar electrical resistance $y = Ru, \;\; R > 0$, the power is $u^T y = Ru^2 \geq 0$

- For the scalar case, passivity imposes a constraint on the graph of $y = h(u)$
  - It must lie in the first and third quadrant



- But we are interested in MIMO dynamical systems

Passivity: formal definitions

- Consider a generic nonlinear system (affine in the input)

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases}$$

with state/input/output $x \in \mathbb{R}^n,\ u \in \boxed{\mathbb{R}^m},\ y \in \boxed{\mathbb{R}^m}$

- The system is **dissipative** if there exists a continuous (differentiable) lower bounded function of the state (storage function)

$$V(x) \in \mathcal{C}^1 : \mathbb{R}^n \to \mathbb{R}^+$$

and a function of the input/output pair (supply rate) $w(u,\, y) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$

such that (~equivalently) $$\begin{cases} V(x(t)) - V(x(t_0)) &\leq \int_{t_0}^{t} w(u(s),\, y(s))\mathrm{d}s \\ \dot{V}(x(t)) &\leq w(u(t),\, y(t)) \end{cases}$$

Passivity: formal definitions

- When the supply rate is

$$w(u,\, y) = y^T u + \delta u^T u + \epsilon y^T y, \qquad \delta \geq 0,\ \epsilon \geq 0$$

the system is said passive (w.r.t. the supply rate w and with storage function V)

- In particular,
  - lossless if $\delta = 0,\ \epsilon = 0$ and $\boxed{\dot{V} = y^T u}$
  - input strictly passive (ISP) if $\delta > 0$
  - output strictly passive (OSP) $\epsilon > 0$
  - very strictly passive (VSP) $\delta > 0,\ \epsilon > 0$

- If there exists a positive definite function $S(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ such that

$$\dot{V}(x) \leq y^T u - S(x)$$

then the system is said strictly passive, and $S(x)$ is called dissipation rate

Passivity: interpretation

- Some (physical) interpretation:

- The storage function $V(x)$ represents the internal stored energy

- The supply rate $y^T u$ is the power (energy flow) exchanged with the external world

- The basic passivity condition can be interpreted as

Current energy is at most equal to the initial energy + supplied energy from outside

$$\boxed{V(x(t))} \leq \boxed{V(x(t_0))} + \boxed{\int_{t_0}^{t} y^T(s)u(s)\mathrm{d}s}$$

equivalent to **<u>"no internal generation of energy"</u>**

Passivity: interpretation

- Exctractable energy is bounded from below

$$\int_{t_0}^{t} y^T(s)u(s)\mathrm{d}s \geq V(x(t)) - V(x(t_0)) \geq -V(x(t_0)) \geq -c^2, \quad c \in \mathbb{R}$$

  - One cannot extract an infinite amount of energy from a passive system
  - The maximum amount of extractable energy (net of the energy supplied from outside) is the initial stored energy (recall the example before)

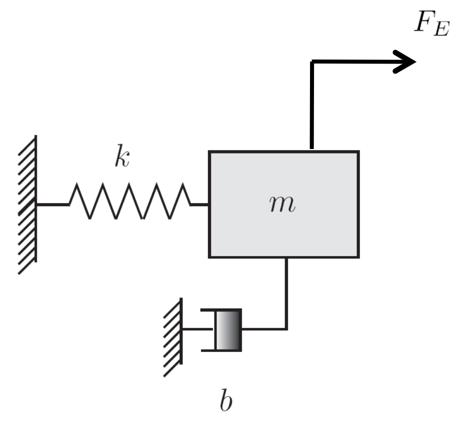
- This yields an (additional) equivalent passivity condition: a system is passive if

$$\boxed{\int_{t_0}^{t} y^T(\tau)u(\tau)\mathrm{d}\tau \geq -c^2,\ c \in \mathbb{R}, \quad \forall u,\ \forall t \geq t_0}$$

- This alternative definition is sometimes useful in proofs and general considerations on the system at hand
  - No formal need of a storage function

Passivity: review of the example

- Consider again the initial example: $m\ddot{x} + b\dot{x} + kx = f$



- Take $u = f$ as the input and $y = \dot{x}$ as the output

- Take the total energy $\boxed{V = E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2}$ as storage function

- Is the system passive w.r.t. the input/output pair $(u,\ y)$?

- By differentiating V, we get $\dot{V} = f\dot{x} - b\dot{x}^2 = yu - by^2 \leq yu$

- Therefore, the system is passive, in particular output strictly passive

Passivity: another example

- The integrator $\begin{cases} \dot{x} &= u \\ y &= x \end{cases}$

  is passive (lossless) w.r.t. the storage function $V(x) = \frac{1}{2}x^2$ since

$$\dot{V} = xu = yu$$

- Similarly, the integrator with nonlinear output $\begin{cases} \dot{x} &= u \\ y &= h(x) \end{cases}$ (■)

  with $\int_0^x h(\sigma)\mathrm{d}\sigma \geq 0, \quad \forall x$

  is passive (lossless) w.r.t. the storage function $V(x) = \int_0^x h(\sigma)\mathrm{d}\sigma$

- This fact will be heavily exploited later on as (■) will constitute the fundamental energy storage element with associated energy function $V(x)$

Passivity: what is it good for?

- Passivity, so far:
  - I/O characterization
  - Nice energetic interpretation
  - Used to describe how the "energy flows" within a system
  - Several equivalent definitions

- But what is it good for? How can we use it?

- Key features:
  - Strong link to Lyapunov stability
  - Proper (and useful) interconnections of passive systems are passive (modularity)
  - A system can be made passive
    - By a choice of the "right output"
    - By a feedback action
  - A passive system is "easily stabilizable" from the output
  - And... many real-world systems are passive

Passivity vs. Lyapunov

- Short summary about Lyapunov stability

- Given a system $\dot{x} = f(x) \quad f(0) = 0$ (■)

  the equilibrium $x = 0$ is
  - Stable if $\forall\epsilon > 0\, \exists\delta(\epsilon) > 0 \mid \|x(t_0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \quad \forall t \geq t_0$
  - Unstable if it is not stable
  - Asymptotically stable if stable and $\|x(t_0)\| \leq \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$

- The Lyapunov Theorems allow to establish (asympt.) stability of (■) without explicitly computing the solution of (■)

- Pivotal is the concept of Lyapunov function, i.e., a positive definite function $V(x)$

$$V(x) \in \mathcal{C}^1 : D \to \mathbb{R}^+, \quad D \subset \mathbb{R}^n$$

$$V(0) = 0,\ V(x) > 0 \text{ in } D - \{0\}$$

Passivity vs. Lyapunov

- If there exists a $V(x)$ such that
  - $\dot{V}(x) \leq 0 \text{ in } D$ then the system is stable
  - $\dot{V}(x) < 0 \text{ in } D - \{0\}$ then the system is (locally) asympt. stable (LAS)
  - If $V(x)$ is radially unbounded, i.e., $D = \mathbb{R}^n$ and $\|x\| \to \infty \Rightarrow V(x) \to \infty$, and it still holds $\dot{V}(x) < 0 \text{ in } D - \{0\}$, then the system is globally asympt. Stable (GAS)

- Also in the case 1), let $S = \{x \in D |\ \dot{V}(x) = 0\}$

- **LaSalle Th.**:The system will converge towards M, the largest invariant set in S

- If $M = \{0\}$, i.e., only $x(t) \equiv 0$ can stay identically in S, then the system is LAS (GAS)

Passivity vs. Lyapunov

- Let us go back to the passivity conditions
- System dynamics $\begin{cases} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{cases}$ and there exists a storage function

$V(x)$ such that $\dot{V}(x) \le y^T u$

- Assume that $V(0) = 0$, then $V(x)$ is a Lyapunov candidate around 0 and
  - If $u \equiv 0$ then $\dot{V} \le 0$, i.e., the system is stable
  - If $y \equiv 0$ then $\dot{V} \le 0$, i.e., the zero-dynamics of the system is stable

- The system can be easily stabilized by a static output feedback

$$u = -\phi(y), \quad y^T \phi(y) > 0 \, \forall y \neq 0$$

for instance $\boxed{u = -ky, \quad k > 0}$

Passivity: output feedback

- By setting $u = -\phi(y)$ we obtain
  - Non increasing storage function $\dot{V} \leq -y^T\phi(y) \leq 0$ $\Longrightarrow$ bounded state trajectories
  - Convergence to a manifold $\dot{V} \equiv 0 \Leftrightarrow y = h(x) \equiv 0$ (the set S of before)

- Remember LaSalle: if the system is zero-state observable

$$\boxed{h(x(t)) \equiv 0 \Rightarrow x(t) \equiv 0}$$

then $u = -\phi(y)$ provides asympt. stability (LAS)

- Global results (GAS) if the storage function $V(x)$ is radially unbounded

Passivity: output feedback

- Passivity of a system is w.r.t. an input/output pair

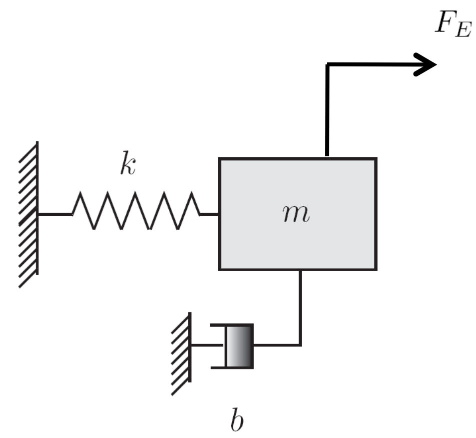
- One can also look for a good output w.r.t. which the system is passive

- Consider the state evolution $\dot{x} = f(x) + g(x)u$ and assume we can find a $V(x)$ such that $\frac{\partial V}{\partial x} f(x) \leq 0$, i.e., a stable free evolution ($u \equiv 0$)

- Then, the system is passive w.r.t. the output $y = \left[\frac{\partial V}{\partial x} g(x)\right]^T$

- The feedback $u = -ky = -k \left[\frac{\partial V}{\partial x} g(x)\right]^T$ makes the system LAS (GAS)

Passivity: output feedback



- Review of the example $m\ddot{x} + b\dot{x} + kx = f$

- Rewrite in canonical state-space form $(x_1,\, x_2) = (x,\, \boxed{m\dot{x}})$

$$\begin{cases} \dot{x}_1 = \dfrac{x_2}{m} \\ \dot{x}_2 = -\dfrac{b}{m}x_2 - kx_1 + u \\ \boxed{y = \dfrac{x_2}{m}} \end{cases}$$

with $u = f$

- Take the storage function (radially unbounded) $V = \dfrac{1}{2}kx_1^2 + \dfrac{1}{2}\dfrac{x_2^2}{m}$

- Passivity condition

$$\dot{V} = \cancel{kx_1\frac{x_2}{m}} - b\frac{x_2^2}{m^2} \cancel{- kx_1\frac{x_2}{m}} + \frac{x_2}{m}u = -b\frac{x_2^2}{m^2} + \frac{x_2}{m}u = -by^2 + yu$$

Passivity: output feedback

- by setting $\boxed{u \equiv 0}$ we obtain $\dot{V} = -by^2 \leq 0$, i.e.,
  - the state trajectories $(x_1(t),\, x_2(t))$ are bounded
  - the output (the velocity) will converge to 0: $y(t) \to 0$

- Let us check the zero-state observability (i.e., LaSalle)

- Zeroing the output means that $y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0$

- The dynamics restricted to this set become $0 = -kx_1 \Rightarrow x_1(t) \equiv 0$

- Therefore, the only possible solution is $(x_1(t),\, x_2(t)) \equiv (0,\, 0)$

- Or, in other words, zeroing the output implies zeroing the complete state

- One can still feedback the output $\boxed{u = -ky}$. This results in faster convergence

$$\dot{V} \leq -(b+k)y^2 < -by^2 \leq 0$$

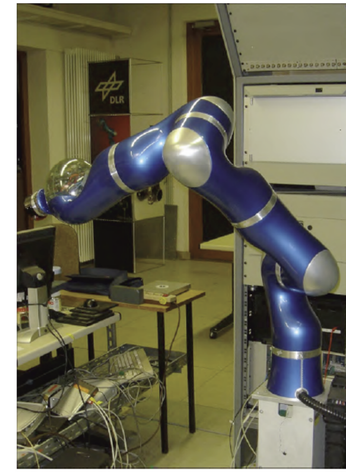
Passivity: output feedback

- Another example: consider the system without output

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u \end{cases}$$

- We want to look for an output $y = h(x)$ that makes the system passive w.r.t. the pair $(u,\ y)$
- Let us consider the (radially unbounded) Storage function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$
- With $u \equiv 0$ it is $\dot{V} = 0$ (stable system). Remember slide 21…
- We can take $y = \frac{\partial V}{\partial x} g(x) = x_2$ and stabilize the system by $u = -kx_2$
- Is the system zero-state observable? (exercise)

Robot Manipulators



- Consider a manipulator arm with joint configuration $q \in \mathbb{R}^n$
- Its dynamical model (Euler-Lagrange form) takes the form

$$M(q)\ddot{q} + C(q,\,\dot{q})\dot{q} + g(q) = \tau$$

Inertia matrix — Centrifugal/Coriolis terms — Gravity vector — Actuation torques (inputs)

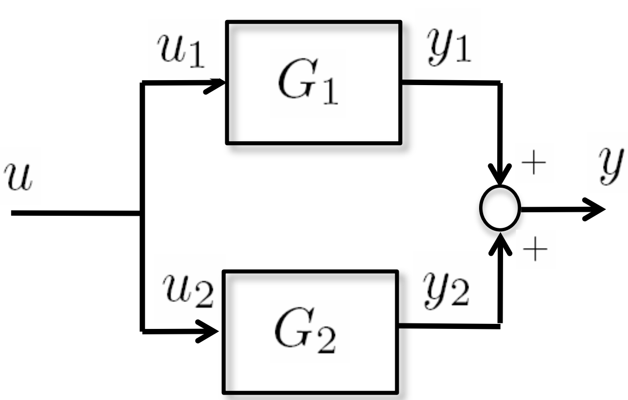
- With these fundamental properties
  - Positive definite Inertia matrix $M(q) > 0$
  - Skew-symmetry of $\dot{M}(q) - 2C(q,\,\dot{q})$
- Apply a feedback pre-compensation (gravity compensation) $\tau = g(q) + \tau'$
- Prove passivity of the new system with storage function $V(q,\,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$ w.r.t. the input/output pair $(\tau',\,\dot{q})$
- Prove zero-state observability

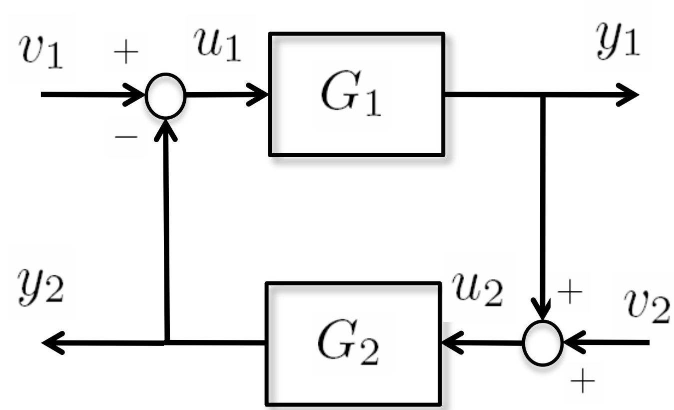
Passivity: interconnection

- Fundamental property: proper interconnections of passive systems are again passive

- This property opens the door to modularity (network modeling):
  - Identify subcomponents
  - Make them passive
  - Interconnect them in a "proper way"
  - The result will be a passive system (stable, etc.)

- We will address two possible interconnections: parallel and feedback

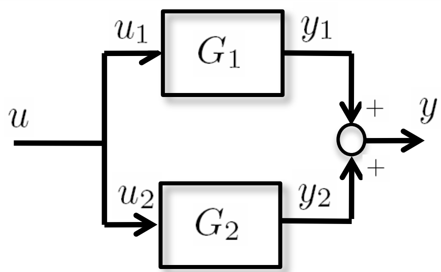




Passivity: interconnection

- Given two passive systems with proper I/O dimensions and storage functions $V_1(x_1)$ and $V_2(x_2)$

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1 \\ y_1 = h_1(x_1) \end{cases} \qquad \begin{cases} \dot{x}_2 = f_2(x_2) + g_2(x_2)u_2 \\ y_2 = h_2(x_2) \end{cases}$$



- For the parallel interconnection, set $u_1 = u_2 = u$ and $y = y_1 + y_2$

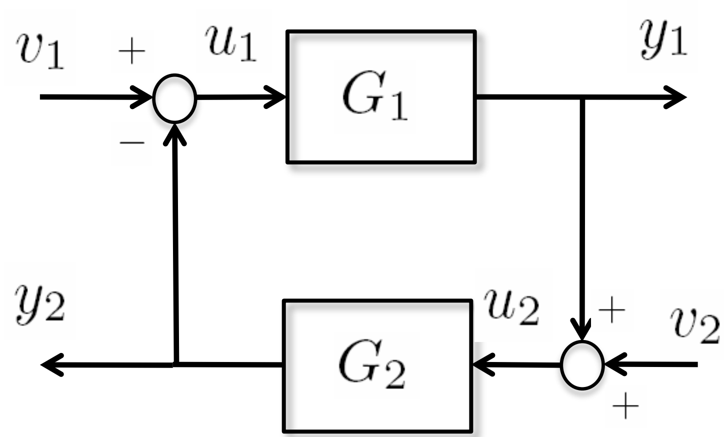
- Let $x = (x_1,\, x_2)$ and $V(x) = V_1(x_1) + V_2(x_2)$ be the storage function

- Then $\dot{V} = \dot{V}_1 + \dot{V}_2 \leq y_1^T u_1 + y_2^T u_2 = (y_1 + y_2)^T u = y^T u$

- The new system is passive w.r.t. the pair $(y_1 + y_2,\, u) = (y,\, u)$

Passivity: interconnection

- Take $\begin{cases} u_1 = \pm y_2 + v_1 \\ u_2 = \mp y_1 + v_2 \end{cases}$ → New (optional) inputs

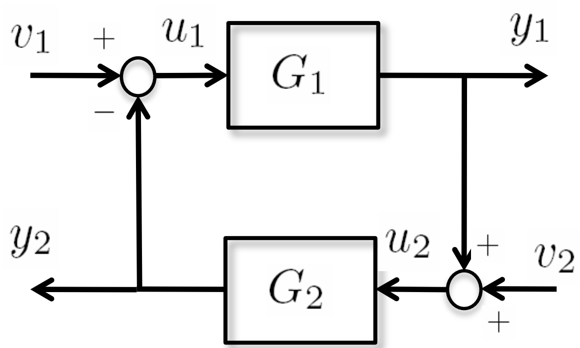


- Prove that the interconnected system is passive with storage function $V(x) = V_1(x_1) + V_2(x_2)$ w.r.t. the (composed) input/output pair

$$\left(\left[y_1^T \; y_2^T \right]^T, \left[v_1^T \; v_2^T \right]^T \right)$$

Passivity: interconnection

- Note the particular structure of the feedback interconnection



$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- The coupling matrix is **skew-symmetric**

- This is a fundamental property that allows to retain passivity of the composed system
- We will see later that this is an example of a power-preserving interconnection

Passivity: pre-post multiplication

- Assume G_1 is a passive system with storage function $V(x)$ w.r.t. the pair $(u,\ y)$
- Let $M(x)$ be a (possibly state-dependent) matrix, and let $u = M(x)\tilde{u}$ and $\tilde{y} = M^T(x)y$

$$\tilde{u} \longrightarrow \boxed{M(x)} \xrightarrow{u} \boxed{G_1} \xrightarrow{y} \boxed{M^T(x)} \longrightarrow \tilde{y}$$

- Prove that passivity is preserved by a pre-multiplication of the input by $M(x)$ and a post-multiplication of the output by $M^T(x)$

Summary

- Passivity is a I/O property of a dynamical system, intuitively equivalent to
  - No internal production of “energy”
  - Bounded extractable “energy”

- Seamlessly applies to linear and nonlinear systems

- Linked to Lyapunov stability
  - Stability of the origin in free evolution (asympt. with some observability properties)
  - Stable zero-dynamics
  - Easily stabilizable by static output feedback

- Can be enforced by
  - Finding the “correct” output
  - A proper feedback (passifying) action

- It is a modular property: proper interconnections of passive systems are passive

Review of consensus protocol

- Let us revisit the consensus protocol under the “passivity light”
- Take a passive (lossless) system: single integrator

$$\Sigma : \begin{cases} \dot{x} &= u_1 \\ y_1 &= x \end{cases}$$

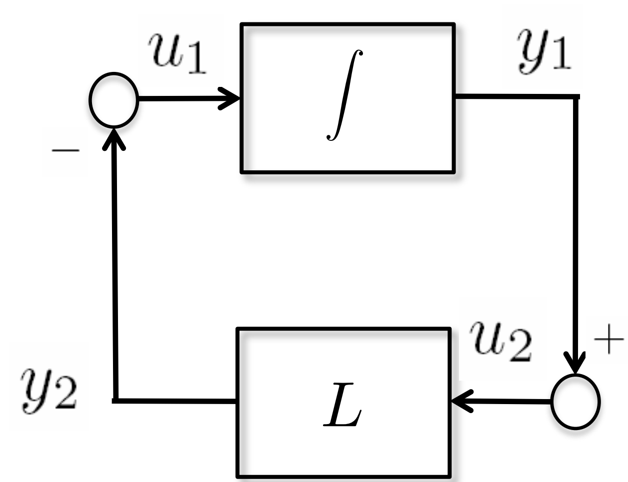
- Consider a static function $y_2(u_2) = Lu_2$
- This is a passive static function $u_2^T y_2 = u_2^T L u_2 \geq 0$
- Interconnect these two passive systems by means of a “feedback interconnection”

$$\begin{cases} u_2 &= y_1 \\ u_1 &= -y_2 \end{cases}$$

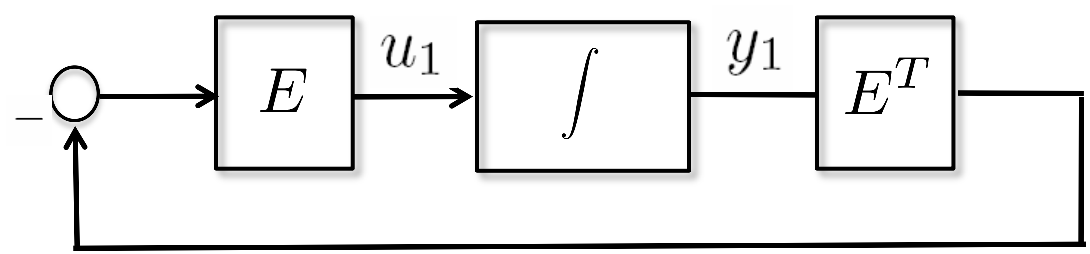
- The resulting system will necessarily be passive….
    - And is nothing but the consensus closed-loop dynamics $\dot{x} = -Lx$

Review of consensus protocol

- As a block scheme



- Another point of view: recall that $L = EE^T$. Then, the consensus protocol is just



- Since the single integrator is passive and a pre-/post-multiplication preserves passivity, we are just closing the loop of a passive with a negative unitary output feedback