Dottorato di Ricerca in Ingegneria dei Sistemi

Control of Nonholonomic Systems

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LECTURE OUTLINE

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   - kinematic constraints
   - integrability of kinematic constraints
   - a control viewpoint
   - dynamics vs. kinematics
   - more general nonholonomic constraints

2. **Modeling Examples**
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   (by M. Vendittelli)
   - minimum time problems
   - application to WMRs
     * extracting information from PMP
     * type A trajectories
     * type B trajectories

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relevant for Lectures 1–5

A. De Luca, G. Oriolo*
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A. De Luca, G. Oriolo, C. Samson*
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relevant for Lecture 7

. . . and the references therein

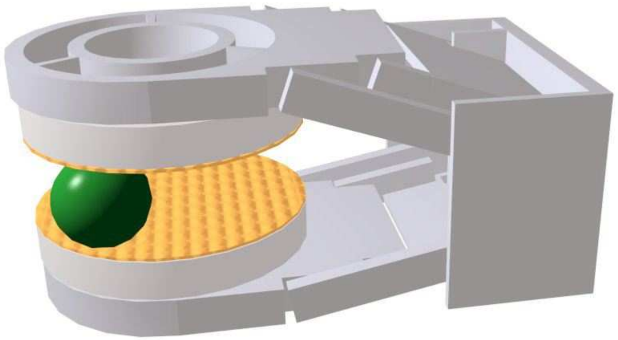
* downloadable from
`http://www.dis.uniroma1.it/~labrob/people/oriolo/oriolo.html`

† downloadable from
`http://http://www.laas.fr/~jpl/book-toc.html`

INTRODUCTION

Nonholonomic systems? Among the others. . .

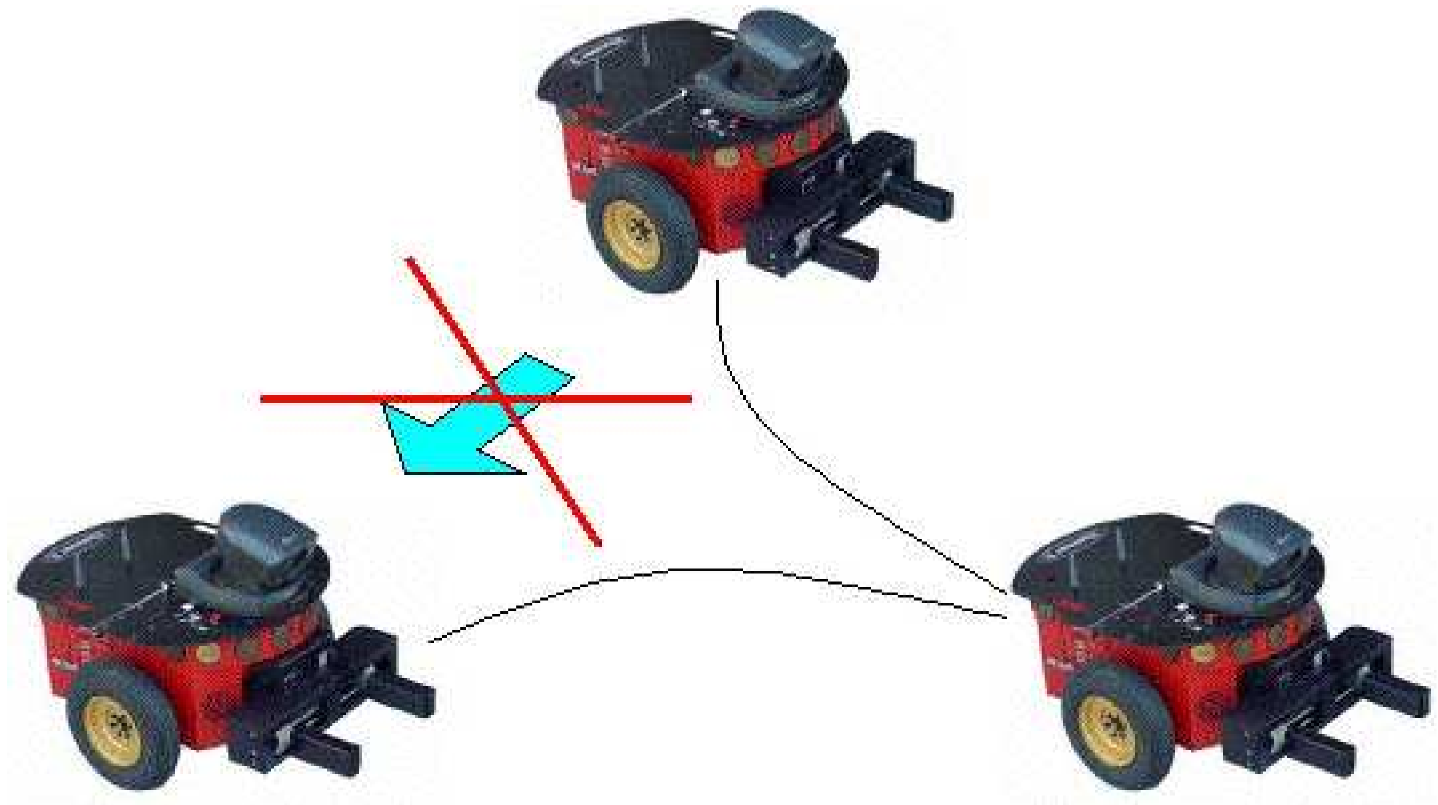
wheeled mobile robots (WMRs)



rolling manipulation

what is nonholonomy?

due to the presence of wheels, a WMR **cannot move sideways**



this is the **rolling without slipping** constraint, a special case of **nonholonomic** behavior

in general: a **nonholonomic** mechanical system **cannot move in arbitrary directions** in its configuration space problems:

- our everyday experience indicates that WMRs are controllable, but can it be proven?

  $\hookrightarrow$ we need a mathematical characterization of nonholonomy

- in any case, if the robot must move between two configurations, a **feasible** path is required (i.e., a motion that complies with the constraint)

  $\hookrightarrow$ we need appropriate path planning techniques

- the feedback control problem is much more complicated, because:

  ◇ a WMR is **underactuated**: less control inputs than generalized coordinates

  ◇ a WMR is **not smoothly stabilizable** at a point

  $\hookrightarrow$ we need appropriate feedback control techniques

Kinematic Constraints

- the configuration of a mechanical system can be uniquely described by an n-dimensional vector of **generalized coordinates**

$$q = (q_1 \quad q_2 \quad \ldots \quad q_n)^T$$

- the configuration space $\mathcal{Q}$ is in general an n-dimensional smooth manifold, locally diffeomorphic to $\mathbb{R}^n$

- the **generalized velocity** at a generic point of a trajectory $q(t) \subset \mathcal{Q}$ is the tangent vector

$$\dot{q} = (\dot{q}_1 \quad \dot{q}_2 \quad \ldots \quad \dot{q}_n)^T$$

- **geometric constraints** may exist or be imposed on the mechanical system

$$h_i(q) = 0 \qquad i = 1, \ldots, k$$

  restricting the possible motions to an $(n - k)$-dimensional submanifold

- a mechanical system may also be subject to a set of **kinematic constraints**, involving generalized coordinates and their derivatives; e.g., first-order kinematic constraints

$$a_i(q, \dot{q}) = 0 \qquad i = 1, \ldots, k$$

- in most cases, the constraints are **Pfaffian**

$$a_i^T(q)\dot{q} = 0 \quad i = 1, \ldots, k \qquad \text{or} \qquad A^T(q)\dot{q} = 0$$

  i.e., they are linear in the velocities

- kinematic constraints may be **integrable**, that is, there may exist k functions h_i such that

$$\frac{\partial h_i(q(t))}{\partial q} = a_i^T(q) \qquad i = 1, \ldots, k$$

  in this case, the kinematic constraints are indeed geometric constraints

a set of Pfaffian constraints is called **holonomic** if it is integrable (a geometric limitation); otherwise, it is called **nonholonomic** (a kinematic limitation) holonomic/nonholonomic constraints affect mobility in a **completely different** way:

for illustration, consider a single Pfaffian constraint

$$a^T(q)\dot{q} = 0$$

- if the constraint is **holonomic**, then it can be integrated as

  $$h(q) = c$$

  with $\dfrac{\partial h}{\partial q} = a^T(q)$ and c an integration constant

  $$\Downarrow$$

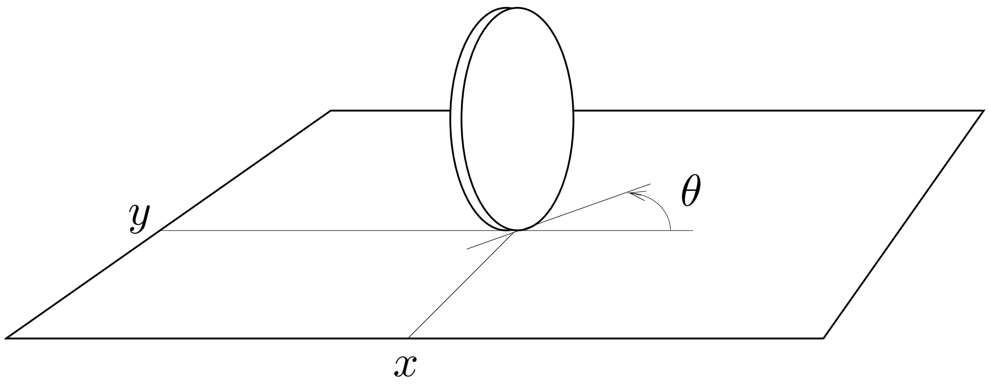
  the motion of the system is confined to lie on a particular level surface (**leaf**) of h, depending on the initial condition through $c = h(q_0)$

- if the constraint is **nonholonomic**, then it cannot be integrated

  $$\Downarrow$$

  although at each configuration the instantaneous motion (velocity) of the system is restricted to an $(n-1)$-dimensional space (the null space of the constraint matrix $a^T(q)$), **it is still possible to reach any configuration in $\mathcal{Q}$**

a canonical example of nonholonomy: the rolling disk

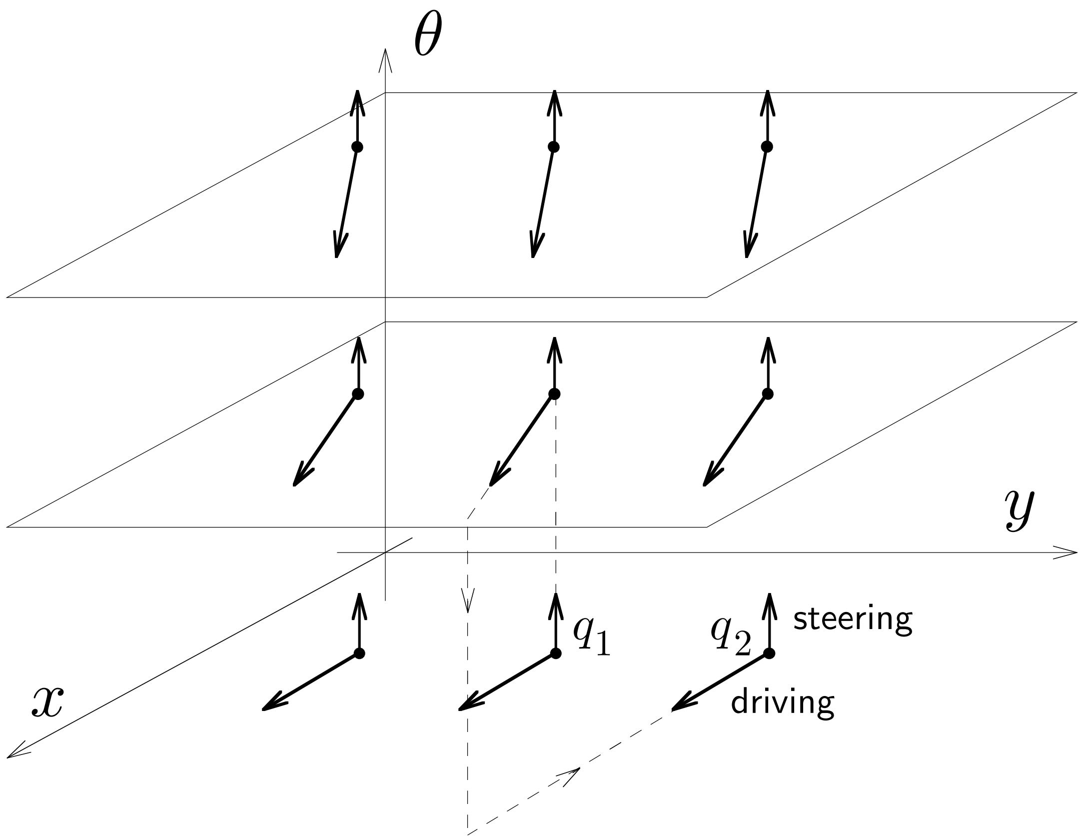


- generalized coordinates $q = (x, y, \theta)$
- **pure rolling** nonholonomic constraint $\quad \dot{x}\sin\theta - \dot{y}\cos\theta = 0 \quad \left(\frac{\dot{y}}{\dot{x}} = \tan\theta\right)$
- feasible velocities are contained in the null space of the constraint matrix

$$a^T(q) = (\sin\theta \quad -\cos\theta \quad 0) \qquad \Longrightarrow \qquad \mathcal{N}(a^T(q)) = \text{span}\left\{\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right\}$$

- **any** configuration $q_f = (x_f, y_f, \theta_f)$ can be reached:
  1. rotate the disk until it aims at (x_f, y_f)
  2. roll the disk until until it reaches (x_f, y_f)
  3. rotate the disk until until its orientation is θ_f

nonholonomy **in the configuration space** of the rolling disk



- at each q, only two instantaneous directions of motion are possible
- to move from q_1 to q_2 (**parallel parking**) an appropriate **maneuver** (sequence of moves) is needed; one possibility is to follow the dashed line

a less canonical example of nonholonomy: the fifteen puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

- generalized coordinates $q = (q_1, \ldots, q_{15})$
- each q_i may assume 16 different values corresponding to the cells in the grid; **legal** configurations are obtained when $q_i \neq q_j$ for $i \neq j$
- depending on the current configuration, a limited number (2 to 4) moves are possible
- **any** configuration with an **even** number of inversions can be reached by an appropriate sequence of moves

Integrability of Kinematic Constraints

- when is a single kinematic Pfaffian constraint

$$a^T(q)\dot{q} = \sum_{j=1}^{n} a_j(q)\dot{q}_j = 0$$

integrable as $h(q) = 0$?

since $\dot{h}(q) = \sum_{j=1}^{n} \frac{\partial h}{\partial q_j}\dot{q}_j = 0$, integrability requires

$$\gamma(q)a_j(q) = \frac{\partial h}{\partial q_j}(q) \qquad j = 1, \ldots, n$$

with $\gamma(q) \neq 0$ **integrating factor**, or equivalently

$$\frac{\partial(\gamma a_k)}{\partial q_j} = \frac{\partial(\gamma a_j)}{\partial q_k}, \qquad j, k = 1, \ldots, n$$

where a system of PDE's must be solved

- for k kinematic Pfaffian constraints, one must check integrability **not only** of each constraint but **also** of independent combinations

$$\sum_{i=1}^{k} \gamma_i(q)a_i^T(q)\dot{q} = 0$$

even if each constraint is not integrable by itself, a subset (or even the whole set) of them may be integrable!

- if there exist $p \leq k$ functions h_i such that, $\forall q$

$$\text{span}\left\{\frac{\partial h_1}{\partial q}(q), \ldots, \frac{\partial h_p}{\partial q}(q)\right\} \subset \text{span}\left\{a_1^T(q), \ldots, a_k^T(q)\right\}$$

  then the system motion is restricted to the $(n-p)$-dimensional manifold of level surfaces of the h_i's

$$\{q : h_1(q) = c_1, \ldots, h_p(q) = c_p\}$$

- motion reduction due to kinematic constraints

$$\begin{aligned} p = k \quad &\Longleftrightarrow \quad \text{holonomic} \\ 0 < p < k \quad &\Longleftrightarrow \quad \text{partially holonomic} \\ p = 0 \quad &\Longleftrightarrow \quad \text{(completely) nonholonomic} \end{aligned}$$

- assessing integrability is **not obvious**: complete (N&S conditions) and constructive answers are obtained by differential geometric tools

A Control Viewpoint

- holonomy/nonholonomy of constraints may be conveniently studied through a dual approach: look at the

directions in which motion is **allowed**

rather than

directions in which motion is **prohibited**

- there is a strict relationship between

capability of accessing every configuration

and

nonholonomy of the velocity constraints

- the interesting question is:

given two arbitrary points q_i and q_f,
when does a connecting trajectory $q(t)$ exist
which satisfies the kinematic constraints?

$\Downarrow$

. . . this is indeed a **controllability** problem!

- associate to the set of kinematic constraints a basis for their null space, i.e. a set of vectors g_j such that

$$a_i^T(q)g_j(q) = 0 \quad i = 1, \dots, k \quad j = 1, \dots, n-k$$

  or in matrix form

$$A^T(q)G(q) = 0$$

- feasible trajectories of the mechanical system are the solutions $q(t)$ of

$$\boxed{\dot{q} = \sum_{j=1}^{m} g_j(q)u_j = G(q)u} \qquad (*)$$

  for some input $u(t) \in I\!R^m$, $m = n - k$ (u: also called **pseudovelocities**)

- $(*)$ is a **driftless** (i.e., $u = 0 \Rightarrow \dot{q} = 0$) nonlinear system known as the **kinematic model** of the constrained mechanical system

- **controllability** of its whole configuration space is equivalent to **nonholonomy** of the original kinematic constraints

Dynamics versus Kinematics

- use Lagrange formalism to obtain the dynamics of a mechanical system with n degrees of freedom, subject to k Pfaffian kinematic constraints

$$A^T(q)\dot{q} = 0$$

- Lagrangian = Kinetic Energy − Potential Energy

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - U(q) = \frac{1}{2}\dot{q}^T B(q)\dot{q} - U(q)$$

  with inertia matrix $B(q) > 0$

- **Euler-Lagrange** equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = A(q)\lambda + S(q)\tau$$

  where

  - $S(q)$ is a $n{\times}m$ matrix mapping the m external inputs τ into forces/torques performing work on the generalized coordinates q ($m = n - k$)
  - $\lambda \in I\!R^m$ is the vector of **Lagrange multipliers**

- the **dynamic model** of the mechanism subject to constraints is

$$\begin{aligned} B(q)\ddot{q} + n(q,\dot{q}) &= A(q)\lambda + S(q)\tau \qquad\qquad (\diamond) \\ A^T(q)\dot{q} &= 0 \end{aligned}$$

  with

$$n(q,\dot{q}) = \dot{B}(q)\dot{q} - \frac{1}{2}\left(\frac{\partial}{\partial q}\left(\dot{q}^T B(q)\dot{q}\right)\right)^T + \left(\frac{\partial U(q)}{\partial q}\right)^T$$

- to eliminate the Lagrange multipliers, being

$$A^T(q)G(q) = 0$$

  multiply $(\diamond)$ by $G^T(q)$ to obtain a reduced set of $m = n-k$ differential equations

$$G^T(q)\left(B(q)\ddot{q} + n(q,\dot{q})\right) = G^T(q)S(q)\tau$$

- assume now an hypothesis of 'enough control'

$$\det G^T(q)S(q) \neq 0$$

- merge the kinematic and dynamic models into the **reduced state-space model**

$$\begin{aligned} \dot{q} &= G(q)v \\ M(q)\dot{v} + m(q,v) &= G^T(q)S(q)\tau \end{aligned}$$

where $v \in I\!R^m$ are the pseudovelocities and

$$\begin{aligned} M(q) &= G^T(q)B(q)G(q) > 0 \\ m(q,v) &= G^T(q)B(q)\dot{G}(q)v + G^T(q)n(q,G(q)v) \end{aligned}$$

where

$$\dot{G}(q)v = \sum_{i=1}^{m} \left(v_i \frac{\partial g_i}{\partial q}(q) \right) G(q)v$$

- define external input τ as a **nonlinear feedback law** from the state (q,v)

$$\tau = \left(G^T(q)S(q)\right)^{-1} (M(q)a + m(q,v)) \qquad (\triangle)$$

where $a \in I\!R^m$ is a vector of **pseudoaccelerations**

- in the absence of constraints, $(\triangle)$ reduces to the **computed torque** law $\Rightarrow$ linear & decoupled closed-loop dynamics (double integrators)

- due to the presence of constraints, the resulting system is

$$\begin{aligned} \dot{q} &= G(q)v \quad && \textbf{kinematic model} \\ \dot{v} &= a && \textbf{dynamic extension} \end{aligned}$$

- letting $x = (q, v)$ and $a = u$, the **state-space model** of the closed-loop system is rewritten in compact form as

$$\dot{x} = f(x) + g(x)u = \begin{pmatrix} G(q)v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u$$

  i.e., a nonlinear control system **with drift** also known as the **second-order kinematic model** of the constrained mechanism

$$\Downarrow$$

⋆ an invertible *feedback control law* can eliminate dynamic parameters

⋆ moving from kinematics to dynamics essentially requires some *input smoothness* assumptions (need $a = \dot{v}$)

⋆ most nonholonomic problems can be addressed at a *first-order* kinematic level

More General Nonholonomic Constraints

- one may also find Pfaffian constraints of the form

$$a_i^T(q)\dot{q} = c_i, \; i = 1, \ldots, k \quad \text{or} \quad A^T(q)\dot{q} = c$$

  with constant c_i

- these constraints are **differential** but **not** of a kinematic nature; for example, this form arises from conservation of an initial **non-zero** angular momentum in space robots

- the mechanism subject to constraint is transformed into an equivalent control system by describing the feasible trajectories $q(t)$ as solutions of

$$\dot{q} = f(q) + \sum_{i=1}^{m} g_i(q) u_i$$

  i.e., a nonlinear control system **with drift**, where $g_1(q), \ldots, g_m(q)$ are a basis of the null space of $A^T(q)$ and the drift vector f is computed through pseudoinversion

$$f(q) = A^{\#}(q)c = A(q)\left(A^T(q)A(q)\right)^{-1} c$$

MODELING EXAMPLES

source of nonholonomic constraints on motion:

- bodies in **rolling contact without slipping**
  - wheeled mobile robots (WMRs) or automobiles (wheels rolling on the ground with no skid or slippage)
  - dextrous manipulation with multifingered robot hands (rounded fingertips on grasped objects)

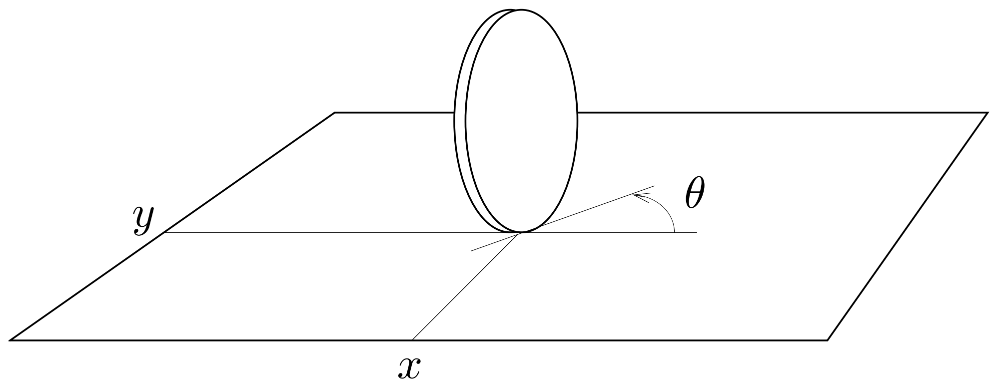
- **angular momentum conservation** in multibody systems
  - robotic manipulators floating in space (with no external actuation)
  - dynamically balancing hopping robots, divers or astronauts (in flying or mid-air phases)
  - satellites with reaction (or momentum) wheels for attitude stabilization

- special **control operation**

$$\dot{q} = G(q)u \qquad q \in \mathbb{R}^n \;\; u \in \mathbb{R}^m \;\; (m < n)$$

  - non-cyclic inversion schemes for redundant robots (m task commands for n joints)
  - floating underwater robotic systems ($m = 4$ velocity inputs for $n = 6$ generalized coords)

unicycle



- generalized coordinates $q = (x, y, \theta)$
- nonholonomic constraint $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$
- a matrix whose columns span the null space of the constraint matrix is

$$G(q) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \end{pmatrix}$$

- hence the kinematic model

$$\dot{q} = G(q)u = g_1(q)u_1 + g_2(q)u_2$$

with $u_1 =$ **driving**, $u_2 =$ **steering** velocity inputs unicycle dynamics

- define

$$\begin{aligned} m &= \text{mass of the unicycle} \\ I &= \text{inertia around vertical axis at contact point} \\ u_1 &= \text{driving force} \\ u_2 &= \text{steering torque} \end{aligned}$$

- the general dynamic model

$$B(q)\ddot{q} + n(q,\dot{q}) = a(q)\lambda + S(q)\tau$$

being $B(q) = B$, $n = 0$ particularizes in this case to

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \lambda + \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

subject to $a^T(q)\dot{q} = 0$

from the reduction procedure, being

$$\begin{aligned} G(q) &= S(q) \\ G^T(q)S(q) &= I_{2\times 2} \\ G^T(q)B\dot{G}(q) &= 0 \end{aligned}$$

we obtain the reduced state-space model

$$\begin{aligned} \dot{q} &= G(q)v \\ G^T(q)BG(q)\dot{v} &= \tau \end{aligned}$$

or the five dynamic equations

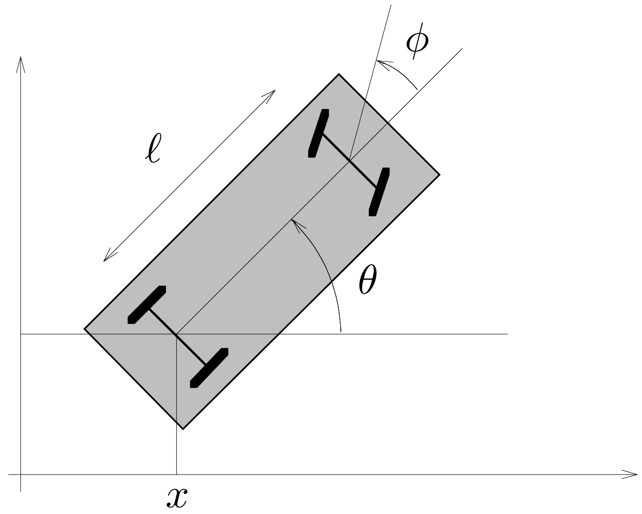
$$\begin{aligned} \dot{x} &= \cos\theta\, v_1 \\ \dot{y} &= \sin\theta\, v_1 \\ \dot{\theta} &= v_2 \\ m\,\dot{v}_1 &= \tau_1 \\ I\,\dot{v}_2 &= \tau_2 \end{aligned}$$

that can be put in the form

$$\dot{X} = f(X) + g_1(X)\tau_1 + g_2(X)\tau_2$$

with $X = (x, y, \theta, v_1, v_2)$

car-like robot



- 'bicycle' model: front and rear wheels collapse into two wheels at the axle midpoints
- generalized coordinates $q = (x, y, \theta, \phi)$ ϕ: steering angle
- nonholonomic constraints

$$\begin{aligned} \dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) &= 0 \quad \textbf{(front wheel)} \\ \dot{x} \sin\theta - \dot{y} \cos\theta &= 0 \quad \textbf{(rear wheel)} \end{aligned}$$

- being the front wheel position

$$x_f = x + \ell \cos\theta \qquad y_f = y + \ell \sin\theta$$

the first constraint becomes

$$\dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \dot{\theta}\, \ell \cos\phi = 0$$

the constraint matrix is

$$A^T(q) = \begin{pmatrix} \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 \\ \sin\theta & -\cos\theta & 0 & 0 \end{pmatrix}$$

there are two physical alternatives for the controls:

(RD) choosing

$$G(q) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{\ell}\tan\phi & 0 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad \dot{q} = g_1(q)u_1 + g_2(q)u_2$$

where $u_1 =$ **rear driving**, $u_2 =$ **steering** inputs

- ◇ a 'control singularity' at $\phi = \pm\pi/2$, where vector field g_1 diverges

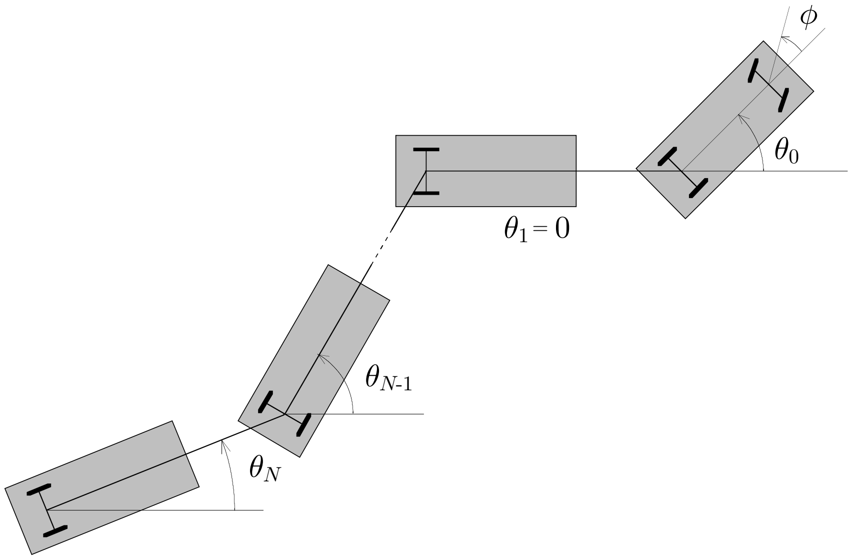
(FD) choosing

$$G(q) = \begin{pmatrix} \cos\theta\cos\phi & 0 \\ \sin\theta\cos\phi & 0 \\ \frac{1}{\ell}\sin\phi & 0 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad \dot{q} = g_1(q)u_1 + g_2(q)u_2$$

where $u_1 =$ **front driving**, $u_2 =$ **steering** inputs

- ◇ no singularities in this case!

N-trailer system



- an FD car-like robot with N trailers, each hinged to the axle midpoint of the previous
- generalized coordinates $q = (x, y, \phi, \theta_0, \theta_1, \ldots, \theta_N) \in I\!R^{N+4}$

$$
\begin{aligned}
x, y &= \text{position of the car rear axle midpoint} \\
\phi &= \text{steering angle of the car (w.r.t. car body)} \\
\theta_0 &= \text{orientation angle of the car (w.r.t. } x\text{-axis)} \\
\theta_i &= \text{orientation angle of } i\text{-th trailer (w.r.t. } x\text{)}
\end{aligned}
$$

- the car is considered as the 0-th trailer

$$
\begin{aligned}
d_0 &= \ell = \text{car length} \\
d_i &= i\text{-th trailer length (hinge to hinge)}
\end{aligned}
$$

nonholonomic constraints:

steering wheel

$$\dot{x}_f \sin(\theta_0 + \phi) - \dot{y}_f \cos(\theta_0 + \phi) = 0$$

with

$$x_f = x + \ell \cos\theta_0 \qquad y_f = y + \ell \sin\theta_0$$

all other wheels

$$\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0 \qquad i = 0, 1, \ldots, N$$

being

$$x_i = x - \sum_{j=1}^{i} d_j \cos\theta_j \qquad y_i = y - \sum_{j=1}^{i} d_j \sin\theta_j$$

the constraints become

$$\begin{aligned} \dot{x} \sin(\theta_0 + \phi) - \dot{y} \cos(\theta_0 + \phi) - \dot{\theta}_0 \, \ell \cos\phi &= 0 \\ \dot{x} \sin\theta_i - \dot{y} \cos\theta_i + \sum_{j=1}^{i} \dot{\theta}_j \, d_j \cos(\theta_i - \theta_j) &= 0 \qquad i = 0, 1, \ldots, N \end{aligned}$$

• the null space of the $N+2$ constraints is spanned by the two columns g_1, g_2 of

$$G(q) = \begin{pmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \\ \frac{1}{\ell}\tan\phi & 0 \\ -\frac{1}{d_1}\sin(\theta_1-\theta_0) & 0 \\ -\frac{1}{d_2}\cos(\theta_1-\theta_0)\sin(\theta_2-\theta_1) & 0 \\ \vdots & \vdots \\ -\frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_j-\theta_{j-1})\right)\sin(\theta_i-\theta_{i-1}) & 0 \\ \vdots & \vdots \\ -\frac{1}{d_N}\left(\prod_{j=1}^{N-1}\cos(\theta_j-\theta_{j-1})\right)\sin(\theta_N-\theta_{N-1}) & 0 \end{pmatrix}$$

• the kinematic model is $\dot{q} = g_1(q)u_1 + g_2(q)u_2$
with $u_1 =$ **(rear) driving**, $u_2 =$ **steering** inputs for the front car

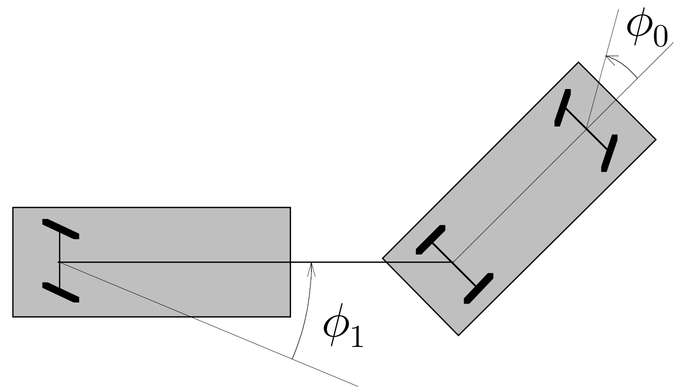
• an alternative way to derive kinematic equations

$$\begin{aligned} \dot{\theta}_i &= -\frac{1}{d_i}\sin(\theta_i-\theta_{i-1})\nu_{i-1} \\ & \qquad\qquad\qquad\qquad i = 1,\ldots,N \\ \nu_i &= \nu_{i-1}\cos(\theta_i-\theta_{i-1}) \end{aligned}$$

with $\nu_i =$ linear (forward) velocity of the i-th trailer ($\nu_0 = u_1$)

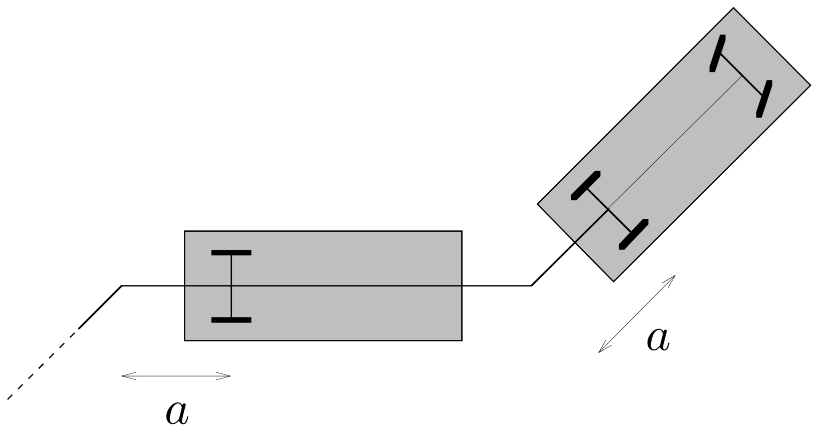
other wheeled mobile robots

- **firetruck**



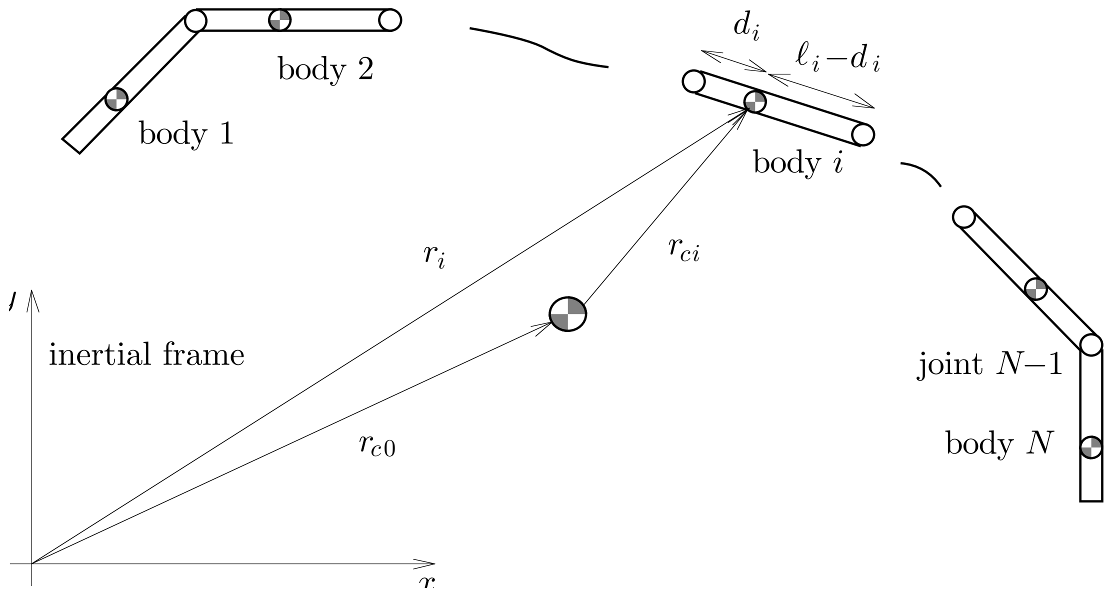
6 configuration variables, 3 differential constraints, 3 control inputs (car driving and steering, trailer steering)

- N-trailer system with **nonzero hooking**



when $a \neq 0$ and $N \geq 2$, this system **cannot** be converted in chained form (later)

Space Robots with Planar Structure



- N planar bodies actuated at the $N-1$ joints (**internal** forces only)

- for the i-th body, let:

$$\begin{aligned} m_i, I_i &= \text{mass and inertia matrix} \\ r_i, v_i &= \text{position and velocity of the center of mass} \\ \omega_i &= \text{angular velocity} \end{aligned}$$

- assume the center of mass of each body is located on the body axis

- no external forces (gravity), no dissipation

$$\Downarrow$$

1. conservation of linear momentum (assumed to be initially zero)

$$\sum_{i=1}^{N} m_i v_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{N} m_i r_i = m_t r_{c0}$$

   i.e., two scalar **holonomic** constraints in the planar case

2. conservation of angular momentum (= zero)

$$\sum_{i=1}^{N} (I_i \omega_i + m_i (r_i \times v_i)) = 0$$

   i.e., a scalar **nonholonomic** constraint in the planar case

- it is convenient to place the inertial frame in the center of mass of the whole system ($r_{c0} = 0$, $r_i = r_{ci}$)

- for a N-body system with kinetic energy

$$T = \frac{1}{2}\dot{q}^T B(q)\dot{q}$$

and $U =$ constant, the vector of **generalized momenta** is

$$p = B(q)\dot{q} \in I\!\!R^N$$

- for a planar system, each component

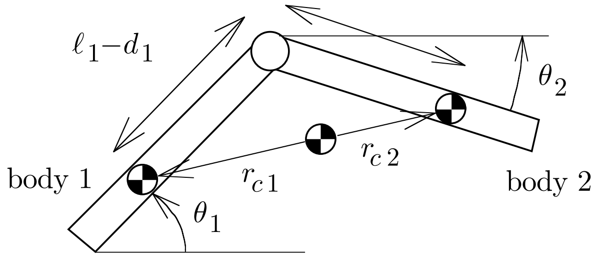
$$p_i = b_i^T(q)\dot{q}$$

represents an angular momentum along the z axis (orthogonal to the xy plane); thus, conservation of (zero) angular momentum can be expressed as a Pfaffian constraint:

$$\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} b_i^T(q)\dot{q} = \mathbf{1}^T B(q)\dot{q} = A^T(q)\dot{q} = 0,$$

where $\mathbf{1} = (1, 1, \ldots, 1)$

2-body space robot



from the two vector equations

$$\begin{aligned}
\begin{pmatrix} r_{c1x} \\ r_{c1y} \end{pmatrix} + (\ell_1 - d_1) \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + d_2 \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} &= \begin{pmatrix} r_{c2x} \\ r_{c2y} \end{pmatrix} \\
m_1 \begin{pmatrix} r_{c1x} \\ r_{c1y} \end{pmatrix} + m_2 \begin{pmatrix} r_{c2x} \\ r_{c2y} \end{pmatrix} &= 0
\end{aligned}$$

one solves for

$$\begin{pmatrix} r_{c1} \\ r_{c2} \end{pmatrix} = \begin{pmatrix} r_{c1x} \\ r_{c1y} \\ r_{c2x} \\ r_{c2y} \end{pmatrix} = \begin{pmatrix} k_{11} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + k_{12} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \\ k_{21} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + k_{22} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \end{pmatrix}$$

where (setting $m_t = m_1 + m_2$)

$$\begin{aligned}
k_{11} &= -\frac{m_2(\ell_1 - d_1)}{m_t} \qquad k_{12} = -\frac{m_2 d_2}{m_t} \\
k_{21} &= \frac{m_1(\ell_1 - d_1)}{m_t} \qquad k_{22} = \frac{m_1 d_2}{m_t}
\end{aligned}$$

- kinetic energy of the system $T = T_1 + T_2$, with

$$T_i = \frac{1}{2} m_i \, \dot{r}_{ci}^T \dot{r}_{ci} + \frac{1}{2} I_{zzi} \, \dot{\theta}_i^2 \qquad\qquad i = 1, 2$$

so that

$$T = \frac{1}{2} \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{pmatrix} \begin{pmatrix} \bar{I}_1 & b_{12}(\theta_2 - \theta_1) \\ b_{12}(\theta_2 - \theta_1) & \bar{I}_2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

where

$$\begin{aligned} \bar{I}_i &= I_{zzi} + m_1 k_{1i}^2 + m_2 k_{2i}^2 \qquad\qquad i = 1, 2 \\ b_{12} &= (m_1 k_{11} k_{12} + m_2 k_{21} k_{22}) \cos(\theta_2 - \theta_1) \end{aligned}$$

- since T is only a function of $\phi_1 = \theta_2 - \theta_1$, the conservation of momentum can be written as the differential constraint

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{I}_1 & b_{12}(\phi_1) \\ b_{12}(\phi_1) & \bar{I}_2 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{\phi}_1 \right) = 0$$

from which

$$\dot{\theta}_1 = -\frac{\bar{I}_2 + b_{12}(\phi_1)}{\bar{I}_1 + \bar{I}_2 + 2 b_{12}(\phi_1)} \, \dot{\phi}_1$$

- taking the single joint velocity $\dot{\phi}_1 = u$ as input and using as generalized coordinates

$$q = \begin{pmatrix} \theta_1 \\ \phi_1 \end{pmatrix} \qquad \begin{array}{l} \text{base angle (\textbf{absolute orientation})} \\ \text{relative angle (\textbf{shape})} \end{array}$$

  the kinematic model describing all the system feasible trajectories is

$$\dot{q} = g(q)u = \begin{pmatrix} -\frac{\bar{I}_2 + b_{12}(\phi_1)}{\bar{I}_1 + \bar{I}_2 + 2b_{12}(\phi_1)} \\ 1 \end{pmatrix} u$$

- it may be shown (see later) that such system is **not controllable**; thus, the constraint expressing conservation of the angular momentum is in this case **integrable**

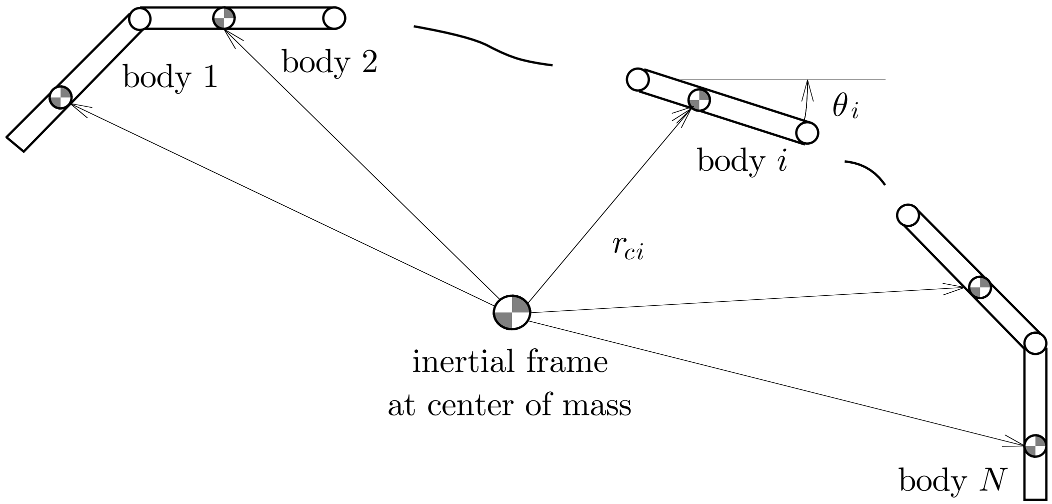
  in particular, if $\bar{I}_1 = \bar{I}_2$

$$\dot{\theta}_1 = -\frac{1}{2}\dot{\phi}_1 \qquad \Rightarrow \qquad \theta_1 = -\frac{1}{2}\phi_1 + k$$

- angular momentum conservation is a **holonomic** constraint for a planar space robot with $N = 2$ bodies

- this mechanical system **cannot be controlled** through u so as to achieve **an arbitrary pair** of absolute orientation **and** internal shape

N-body space robot



- follow the same steps as before, with the inertial reference frame placed at the system center of mass and θ_i = absolute angle of i-th body

- position of center of mass of i-th body

$$\begin{pmatrix} r_{cix} \\ r_{ciy} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N} k_{ij} \cos\theta_j \\ \sum_{j=1}^{N} k_{ij} \sin\theta_j \end{pmatrix}$$

where

$$k_{ij} = \begin{cases} \frac{1}{m_t}\left(\ell_j \sum_{h=1}^{j-1} m_h + (\ell_j - d_j) m_j\right) & (j < i) \\ \frac{1}{m_t}\left(d_i \sum_{h=1}^{i-1} m_h - (\ell_i - d_i) \sum_{k=i+1}^{N} m_k\right) & (j = i) \\ \frac{1}{m_t}\left(-\ell_j \sum_{h=j+1}^{N} m_h - d_j m_j\right) & (j > i) \end{cases}$$

- kinetic energy of i-th body

$$\begin{aligned} T_i &= \frac{1}{2} m_i \, \dot{r}_{ci}^T \dot{r}_{ci} + \frac{1}{2} I_{zzi} \, \dot{\theta}_i^2 \\ &= \frac{1}{2} \, m_i \left(\sum_{h=1}^{N} \sum_{j=1}^{N} k_{ij} k_{ih} \cos(\theta_h - \theta_j) \dot{\theta}_h \dot{\theta}_j \right) + \frac{1}{2} I_{zzi} \dot{\theta}_i^2 \end{aligned}$$

- kinetic energy of the system

$$T = \sum_{i=1}^{N} T_i = \frac{1}{2} \dot{\theta}^T B(\theta) \dot{\theta}$$

with elements of inertia matrix $B = \{b_{ij}(\theta_i - \theta_j)\}$

$$b_{ij} = \begin{cases} \sum_{h=1}^{N} m_h k_{hi} k_{hj} \cos(\theta_i - \theta_j) & i \neq j \\ \sum_{h=1}^{N} m_h k_{hh}^2 + I_{zzi} & i = j \end{cases}$$

depending only on **relative** angles between bodies

- let

$$\begin{aligned} \phi_i &= \theta_{i+1} - \theta_i \qquad i = 1, \ldots, N-1 \\ &\Rightarrow \qquad \phi = P\theta \end{aligned}$$

where P is a $(N-1) \times N$ matrix

- redefine generalized coordinates as $q = (\theta_1, \phi)$

$$q = \begin{pmatrix} 1 & \mathbf{0}^T \\ & P \end{pmatrix} \theta = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots \\ -1 & 1 & 0 & \dots & \dots \\ 0 & -1 & 1 & 0 & \dots \\ & & \dots & & \\ \dots & \dots & 0 & -1 & 1 \end{pmatrix} \theta$$

with the inverse mapping

$$\theta = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots \\ 1 & 1 & 0 & \dots & \dots \\ 1 & 1 & 1 & 0 & \dots \\ & & \dots & & \\ 1 & \dots & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \phi \end{pmatrix} = \begin{pmatrix} \mathbf{1} & S \end{pmatrix} \begin{pmatrix} \theta_1 \\ \phi \end{pmatrix}$$

where S is a $N \times (N-1)$ matrix

- conservation of angular momentum becomes

$$\mathbf{1}^T B(\phi) \left(\mathbf{1} \dot{\theta}_1 + S \dot{\phi} \right) = 0$$

from which

$$\dot{\theta}_1 = -\frac{\mathbf{1}^T B(\phi) S}{\mathbf{1}^T B(\phi) \mathbf{1}} v$$

where $\dot{\phi} = v$ are the robot joint velocities

- the **kinematic model** of the N-body space robot is then

$$\dot{q} = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} s_1(\phi) & s_2(\phi) & \dots & s_{N-1}(\phi) \\ & \mathbf{I}_{N-1} & & \end{pmatrix} v$$

in which

$$s_i(\phi) = -\frac{s_i'(\phi)}{\mathbf{1}^T B(\phi)\mathbf{1}}$$

where the positive denominator is given by

$$\sum_{j=1}^{N} \bar{I}_j + \sum_{j=1}^{N} \sum_{\substack{h=1 \\ h \neq j}}^{N} \sum_{l=1}^{N} m_l \, k_{lj} \, k_{lh} \cos\left(\sum_{r=h}^{j-1} \phi_r\right)$$

with

$$\bar{I}_j = I_{zzj} + \sum_{h=1}^{N} m_h \, k_{hj}^2$$

and the numerator is

$$s_i'(\phi) = \sum_{j=i+1}^{N} \left(\bar{I}_j + \sum_{h=1}^{N} \sum_{l=1}^{N} m_l \, k_{lj} \, k_{lh} \cos\left(\sum_{r=h}^{j-1} \phi_r\right)\right)$$

e.g., in the case of $N = 3$ bodies

$$
\begin{aligned}
s_1' &= \bar{I}_2 + \bar{I}_3 + h_{12}\cos\phi_1 + 2h_{23}\cos\phi_2 + h_{13}\cos(\phi_1+\phi_2) \\
s_2' &= \bar{I}_3 + h_{23}\cos\phi_2 + h_{13}\cos(\phi_1+\phi_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{1}^T B(\phi)\mathbf{1} &= \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \\
&\quad 2\left(h_{12}\cos\phi_1 + h_{23}\cos\phi_2 + h_{13}\cos(\phi_1+\phi_2)\right)
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{I}_1 &= m_1 k_{11}^2 + m_2 k_{21}^2 + m_3 k_{31}^2 + I_{zz1} \\
\bar{I}_2 &= m_1 k_{12}^2 + m_2 k_{22}^2 + m_3 k_{32}^2 + I_{zz2} \\
\bar{I}_3 &= m_1 k_{13}^2 + m_2 k_{23}^2 + m_3 k_{33}^2 + I_{zz3} \\
h_{12} &= m_1 k_{11} k_{12} + m_2 k_{21} k_{22} + m_3 k_{31} k_{32} \\
h_{13} &= m_1 k_{11} k_{13} + m_2 k_{21} k_{23} + m_3 k_{31} k_{33} \\
h_{23} &= m_1 k_{12} k_{13} + m_2 k_{22} k_{23} + m_3 k_{32} k_{33}
\end{aligned}
$$

with the k_{ij}'s and m_t depending on the inertial parameters

- the dynamic model of the N-body space robot is

$$B(\theta)\ddot{\theta} + n(\theta, \dot{\theta}) = P^T \tau$$

where $\tau =$ **torques** at the $N-1$ robot joints, with

$$\mathbf{1}^T B(\theta)\dot{\theta} = 0$$

- the **reduced dynamic model** (in the 'shape space') consists of $2N-1$ first-order differential equations

$$\begin{aligned}
\dot{\theta}_1 &= -\frac{\mathbf{1}^T B(\phi) S}{\mathbf{1}^T B(\phi)\mathbf{1}} v \\
\dot{\phi} &= v \\
\dot{v} &= M^{-1}(\phi)\left(-m(\phi, v) + \tau\right)
\end{aligned}$$

where

$$\begin{aligned}
M(\phi) &= P B(\phi) P^T \\
m(\phi, v) &= \dot{M}(\phi) v - \frac{1}{2}\frac{\partial}{\partial \phi}\left(v^T M(\phi) v\right)
\end{aligned}$$

- the right hand side of the above is **independent** of θ_1

$$\Downarrow$$

in this case, the mechanical system is referred to as a nonholonomic **Čaplygin** system

TOOLS FROM DIFFERENTIAL GEOMETRY

- a smooth **vector field** $f : I\!R^n \mapsto T_q I\!R^n$ is a smooth mapping from each point of $I\!R^n$ to the tangent space $T_q I\!R^n$

- if f defines the rhs of a differential equation

$$\dot{q} = f(q)$$

  the **flow** $\phi_t^f(q)$ of the vector field f is the mapping which associates to each q the solution evolving from q, i.e., it satisfies

$$\frac{d}{dt}\phi_t^f(q) = f(\phi_t^f(q))$$

  with the **group** property $\phi_t^f \circ \phi_s^f = \phi_{t+s}^f$

  in linear systems, $f(q) = Aq$, the flow is $\phi_t^f = e^{At}$

- considering two vector fields g_1 and g_2 as in

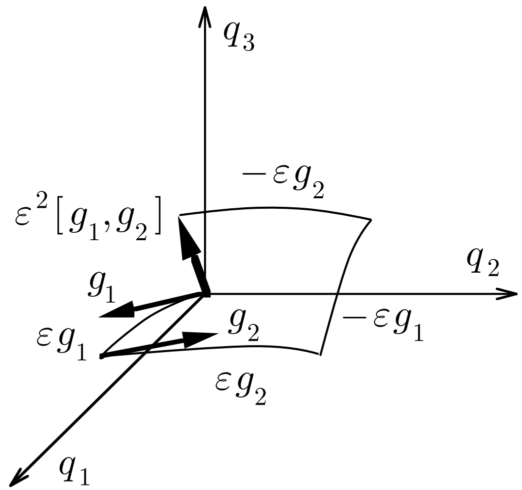
$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

  the composition of their flows (obtained by taking $u_1 = \{1, 0\}$ and $u_2 = \{0, 1\}$) is **non-commutative**

$$\phi_t^{g_1} \circ \phi_s^{g_2} \neq \phi_s^{g_2} \circ \phi_t^{g_1}$$

- starting at q_0, an infinitesimal flow of time ϵ along g_1, then g_2, then $-g_1$, and finally $-g_2$, yields (R. Brockett: *'a computation everybody should do once in his life'*)

$$q(4\epsilon) = \phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(q_0) = q_0 + \epsilon^2 \left(\frac{\partial g_2}{\partial q} g_1(q_0) - \frac{\partial g_1}{\partial q} g_2(q_0) \right) + O(\epsilon^3)$$



- **Lie bracket** of two vector fields g_1, g_2

$$\boxed{[g_1, g_2](q) = \frac{\partial g_2}{\partial q} g_1(q) - \frac{\partial g_1}{\partial q} g_2(q)}$$

- g_1 and g_2 **commute** if $[g_1, g_2] = 0$; moreover,

$$[g_1, g_2] = 0 \quad \Rightarrow \quad q(4\epsilon) = q_0 \;\; \text{(zero net flow)}$$

- **properties** of Lie brackets

$$[f,g] = -[g,f] \qquad \textbf{skew-symmetry}$$
$$[f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0 \qquad \textbf{Jacobi identity}$$

and the **chain rule**

$$[\alpha f, \beta g] = \alpha\beta[f,g] + \alpha(L_f\beta)g - \beta(L_g\alpha)f$$

with $\alpha, \beta\colon \mathbb{R}^n \mapsto \mathbb{R}$ and the **Lie derivative** of α along g defined as

$$L_g\alpha(q) = \frac{\partial\alpha}{\partial q}\, g(q)$$

in linear single input systems, $f(q) = Aq$, $g(q) = b$,

$$\begin{aligned} [f,g] &= -Ab & \qquad [f,[f,g]] = A^2 b \\ [f,[f,[f,g]]] &= -A^3 b & \qquad \ldots \end{aligned}$$

- a smooth **distribution** Δ associated with a set of smooth vector fields $\{g_1, \ldots, g_m\}$ assigns to each point q a subspace of its tangent space defined as

$$\begin{aligned} \Delta &= \text{span}\,\{g_1, \ldots, g_m\} \\ & \Updownarrow \\ \Delta_q &= \text{span}\,\{g_1(q), \ldots, g_m(q)\} \subset T_q\mathbb{R}^n \end{aligned}$$

- a distribution is **regular** if dim $\Delta_q = \text{const}, \ \forall q$

- a distribution is **involutive** if it is closed under the Lie bracket operation

$$\Delta \text{ involutive} \quad \Longleftrightarrow \quad \forall g_i, g_j \in \Delta \quad [g_i, g_j] \in \Delta$$

- the **involutive closure** $\bar{\Delta}$ of a distribution Δ is its closure under the Lie bracket operation

- the set of smooth vector fields on $I\!R^n$ with the Lie bracket operation is a **Lie algebra**

- a Lie algebra is **nilpotent** if all Lie brackets of order $\geq k$ (finite integer) vanish

- a regular distribution Δ on $I\!R^n$ of dimension k is **integrable** when there exist $n-k$ independent functions h_i such that, $\forall q$ and $\forall g_j \in \Delta$

$$L_{g_j} h_i = \frac{\partial h_i}{\partial q} g_j(q) = 0 \qquad i = 1, \ldots, n-k$$

- the hypersurfaces defined as the level sets

$$\{q : h_1(q) = c_1, \ldots, h_{n-k}(q) = c_{n-k}\}$$

are **integral manifolds** of Δ

Frobenius Theorem

a regular distribution is integrable if and only if it is involutive

- $\Rightarrow$ a distribution of dimension 1 (i.e., associated to a single vector field) is **always** integrable

- the proof of sufficiency is constructive

- if the distribution Δ of dimension k is involutive, then its integral manifolds (level sets of functions h_i) are **leaves** of a **foliation** of $\mathbb{R}^n$

e.g. the distribution $\Delta = \text{span}\{g_1, g_2\}$ with

$$g_1(q) = \begin{pmatrix} 1 \\ q_2 \\ 0 \end{pmatrix} \quad g_2(q) = \begin{pmatrix} 1 \\ 0 \\ q_3 \end{pmatrix}$$

is involutive, because

$$[g_1, g_2](q) = 0$$

it induces a foliation of $\mathbb{R}^3$ according to

$$q_1 - \log(q_2 q_3) = c \qquad c \in \mathbb{R}$$

Integrability of Pfaffian Constraints

- a smooth **one-form** is a mapping $a^T \colon I\!R^n \mapsto T_q^* I\!R^n$, the dual space of linear forms on $T_q I\!R^n$

NB: one forms are represented in local coordinates as **row vectors** (hence the transpose notation!)

$$a^T(q) = \begin{pmatrix} a_1(q) & a_2(q) & \dots & a_n(q) \end{pmatrix}$$

- an **exact one-form** ω^T is the differential of a smooth function h

$$\omega^T = \frac{\partial h}{\partial q} = \begin{pmatrix} \frac{\partial h}{\partial q_1} & \frac{\partial h}{\partial q_2} & \dots & \frac{\partial h}{\partial q_n} \end{pmatrix}$$

- a smooth **codistribution** A^T assigns to each point q a subspace of the dual of its tangent space and can be defined by a set of smooth one-forms a_i^T

$$\begin{aligned} A^T &= \text{span}\,\{a_1^T, \dots, a_k^T\} \\ &\Updownarrow \\ A_q^T &= \text{span}\,\{a_1^T(q), \dots, a_k^T(q)\} \subset T_q^* I\!R^n \end{aligned}$$

- **distribution annihilating a codistribution**

  given a set of smooth independent one-forms

$$a_i^T(q) \qquad i =, \ldots, k$$

  which define a codistribution A^T, there exist smooth independent vector fields

$$g_j(q) \qquad j = 1, \ldots, n-k = m$$

  defining a distribution $\Delta = \left(A^T\right)^\perp$ such that

$$a_i^T(q) \cdot g_j(q) = 0 \qquad \forall i, j$$

  i.e., distribution Δ annihilates codistribution A^T

$$\Downarrow$$

A set of Pfaffian constraints is integrable if and only its annihilating distribution is involutive

Controllability of Nonholonomic Systems

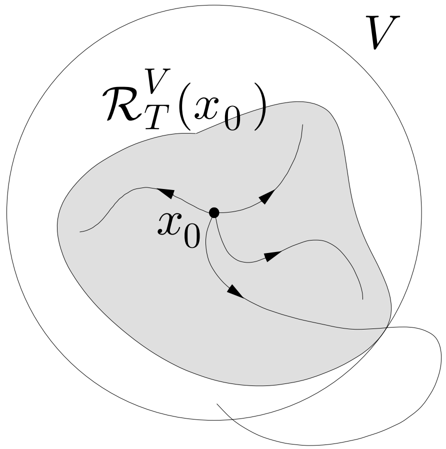
consider a nonlinear control system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j \qquad \text{(NCS)}$$

with state $x \in \mathcal{M} \simeq I\!R^n$, and input in the class $\mathcal{U}$ of piecewise-continuous time functions

- denote its unique solution at time $t \geq 0$ by $x(t, 0, x_0, u)$, with input $u(\cdot)$, and $x(0) = x_0$
- (NCS) is **controllable** if $\forall x_1, x_2 \in \mathcal{M}$, $\exists T < \infty$, $\exists u\colon [0, T] \rightarrow \mathcal{U} \;:\; x(T, 0, x_1, u) = x_2$
- the set of states **reachable** from x_0 **within** time $T > 0$, with trajectories contained in a neighborhood V of x_0, is denoted by

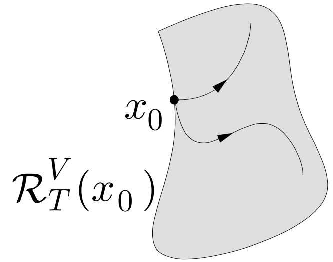
$$\mathcal{R}_T^V(x_0) = \bigcup_{\tau \leq T} \mathcal{R}^V(x_o, \tau)$$



where $\mathcal{R}^V(x_0, \tau) = \{x \in \mathcal{M} \mid x(\tau, 0, x_0, u) = x, \forall t \in [0, \tau],\, x(t, 0, x_0, u) \in V\}$

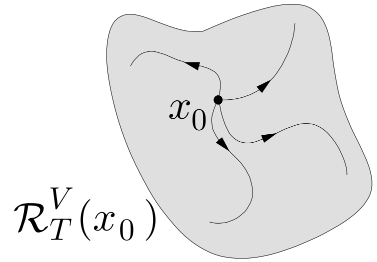
- (NCS) is **locally accessible** (LA) from x_0 if $\forall V$, a neighborhood of x_0, and $\forall T > 0$

$$\mathcal{R}_T^V(x_o) \supset \Omega, \quad \text{with } \Omega \text{ some non-empty open set}$$



- (NCS) is **small-time locally controllable** (STLC) from x_0 if $\forall V$, a neighborhood of x_0, and $\forall T > 0$

$$\mathcal{R}_T^V(x_o) \supset \Psi, \quad \text{with } \Psi \text{ some neighborhood of } x_0$$



- STLC $\Rightarrow$ controllability $\Rightarrow$ LA (not vice versa)

- LA is checked through an algebraic test
  - let $\bar{\mathcal{C}}$ be the involutive closure of the distribution associated with $\{f, g_1, g_2, \ldots, g_m\}$
  - **Chow Theorem** (1939): (NCS) is LA from x_0 if and only if

$$\dim \bar{\mathcal{C}}(x_0) = n \qquad \textbf{accessibility rank condition}$$

  - an algorithmic test:

$$\bar{\mathcal{C}} = \text{span}\left\{v \in \bigcup_{k \geq 0} \mathcal{C}^k\right\} \quad \text{with} \quad \begin{cases} \mathcal{C}^0 = \text{span}\,\{f, g_1, \ldots, g_m\} \\ \mathcal{C}^k = \mathcal{C}^{k-1} + \text{span}\,\{[f, v], [g_j, v], j = 1, .., m : v \in \mathcal{C}^{k-1}\} \end{cases}$$

- only **sufficient** conditions exists for STLC , e.g., [Sussmann 1987]

- however, for driftless control systems:

$$\text{LA} \iff \text{controllability} \iff \text{STLC}$$

- this equivalence holds also whenever

$$f(x) \in \text{span}\,\{g_1(x), \ldots, g_m(x)\} \qquad \forall x \in \mathcal{M}$$

  ('trivial' drift)

- if the driftless control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

  is controllable, then its **dynamic extension**

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{m} g_i(x) v_i \\ \dot{v}_i &= u_i \qquad i = 1, \ldots, m \end{aligned}$$

  is also controllable (and vice versa)

- in the linear case $\dot{x} = Ax + \sum_{j=1}^{m} b_j u_j = Ax + Bu$, all controllability definitions are equivalent and the associated tests reduce to the well-known Kalman rank condition:

$$\text{rank}\ (\, B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B\,) = n$$

- a controllability test is a nonholonomy test!

> a set of k Pfaffian constraints $A(q)\dot{q} = 0$ is nonholonomic if and only if the associated kinematic model
>
> $$\dot{q} = G(q)u = \sum_{i=1}^{m} g_i(q)u_i \qquad m = n - k$$
>
> is controllable, that is
>
> $$\dim \bar{\mathcal{C}} = n$$
>
> being $\bar{\mathcal{C}}$ the involutive closure of the distribution associated with $g_1, \dots, g_m$

$$\Downarrow$$

for a nonholonomic system, it is always possible to design **open-loop** commands that drive the system from any state to any other state (**nonholonomic path planning**)

Stabilizability of Nonholonomic Systems

for a nonlinear control system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j = f(x) + g(x) u$$

one would like to build a **feedback control** law of the form

$$u = \alpha(x) + \beta(x) v$$

in such a way that either

a) a desired closed-loop equilibrium point x_e is made asymptotically stable, or

b) a desired feasible closed-loop trajectory $x_d(t)$ is made asymptotically stable

- feedback laws are essential in motion control to counteract the presence of disturbances as well as modeling inaccuracies

- in linear systems, controllability directly implies asymptotic (actually, exponential) stabilizability at x_e by **smooth** (actually, linear) state feedback

$$\alpha(x) = K(x - x_e)$$

- if the linear approximation of the system at x_e

$$\dot{\delta x} = A\delta x + B\delta u \qquad \delta x = x - x_e,\ \delta u = K\delta x$$

  is controllable, then the original system can be locally smoothly stabilized at x_e (a **sufficient** condition)

- in the presence of **uncontrollable eigenvalues at zero**, nothing can be concluded (except that smooth exponential stability is not achievable)

- for kinematic models of nonholonomic systems $\dot{q} = G(q)u$, the linear approximation around x_e has **always** uncontrollable eigenvalues at zero since

$$A \equiv 0 \quad \text{and} \quad \text{rank}\, B = \text{rank}\, G(q_e) = m < n$$

- however, there are **necessary** conditions for the existence of a C^0-stabilizing state feedback law (next slide)

- whenever these conditions fail, two alternatives are left:

  a) **discontinuous feedback** $\quad u = \alpha(x),\ \alpha \in \bar{C}^0$

  b) **time-varying feedback** $\quad u = \alpha(x,t),\ \alpha \in C^1$

Brockett stabilization theorem (1983)

if the system

$$\dot{x} = f(x, u)$$

is locally asymptotically C^1-stabilizable at x_e, then the image of the map

$$f : \mathcal{M} \times \mathcal{U} \rightarrow I\!R^n$$

contains some **neighborhood** *of x_e* (a **necessary** condition)

a special case: the **driftless** system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

with linearly independent vectors $g_i(x_e)$, i.e.,

$$\text{rank}\ (\, g_1(x_e) \quad g_2(x_e) \quad \ldots \quad g_m(x_e)\,) = m$$

is locally asymptotically C^1-stabilizable at x_e **if and only if** $m \geq n$

$$\Downarrow$$

nonholonomic mechanical systems
(either in kinematic or dynamic form)
cannot be stabilized at a point by smooth feedback

Classification of Nonholonomic Distributions

- the equivalence between a set of Pfaffian constraints

$$a_i^T(q)\dot{q} = 0 \qquad i = 1, \dots, k$$

and the associated kinematic model

$$\dot{q} = \sum_{j=1}^{m} g_j(q) u_j \qquad m = n - k$$

i.e., in matrix format

$$A^T(q)\dot{q} = 0 \qquad \Longleftrightarrow \qquad \dot{q} = G(q)u$$

in the light of controllability (LA) conditions gives

$\dim \bar{\mathcal{C}} = n$	$\Longleftrightarrow$	completely nonholonomic constraints (distribution)
$m < \dim \bar{\mathcal{C}} < n$	$\Longleftrightarrow$	partially nonholonomic constraints (distribution)
$\dim \bar{\mathcal{C}} = m$	$\Longleftrightarrow$	holonomic constraints (distribution)

- Frobenius Theorem $\Rightarrow$

if $\bar{\mathcal{C}}$ is regular of dimension $n - p$, there exist p functions h_j such that

$$h_j(q) = c_j \ \ (j = 1, \dots, p) \ \Leftrightarrow \ a_i^T(q)\dot{q} = 0 \ \ (i = 1, \dots, k)$$

- one may show that the **complexity** of the path planning problem is related to the level of Lie bracketing needed to span $\mathbb{R}^n$

$$\Downarrow$$

classify nonholonomic systems accordingly

- the **filtration** $\{\mathcal{C}_i\}$ generated by the distribution $\mathcal{C} = \text{span}\,\{g_1, \ldots, g_m\}$ is defined as

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{C} \\ \mathcal{C}_i &= \mathcal{C}_{i-1} + [\mathcal{C}_1, \mathcal{C}_{i-1}] \quad i > 2 \end{aligned}$$

  where

$$[\mathcal{C}_1, \mathcal{C}_{i-1}] = \text{span}\,\{[g_j, v] : g_j \in \mathcal{C}_1, v \in \mathcal{C}_{i-1}\}$$

- a filtration is **regular** in a neighborhood $V(q_0)$ if $\dim \mathcal{C}_i(q) = \dim \mathcal{C}_i(q_0), \quad \forall q \in V(q_0)$

- if $\{\mathcal{C}_i\}$ is regular, the **degree of nonholonomy** of $\mathcal{C}$ is the smallest integer κ such that

$$\dim\ \mathcal{C}_{\kappa+1} = \dim\ \mathcal{C}_\kappa$$

- $\Rightarrow$ nonholonomy conditions in terms of κ: a set of k Pfaffian constraints is
  1. completely nonholonomic if $\dim\ \mathcal{C}_\kappa = n$
  2. partially nonholonomic if $m < \dim\ \mathcal{C}_\kappa < n$
  3. holonomic if $\dim\ \mathcal{C}_\kappa = m$ $(m = n - k)$

Examples of Classification

- **unicycle kinematics** ($n = 3$)

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad g_3 = [g_1, g_2] = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

degree of nonholonomy $\kappa = 2$, $\dim \bar{\mathcal{C}} = 3$ for all q

- **unicycle dynamics** ($n = 5$)

$$f = \begin{pmatrix} \cos\theta\, v_1 \\ \sin\theta\, v_1 \\ v_2 \\ 0 \\ 0 \end{pmatrix} \quad g_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/m \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/I \end{pmatrix}$$

$$[g_1, f] = \begin{pmatrix} \cos\theta/m \\ \sin\theta/m \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad [g_2, f] = \begin{pmatrix} 0 \\ 0 \\ 1/I \\ 0 \\ 0 \end{pmatrix} \quad [g_2, [f, [g_1, f]]] = \begin{pmatrix} -\sin\theta/mI \\ \cos\theta/mI \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

degree of nonholonomy $\kappa = 3$; satisfies both the LA and STLC conditions since

$$g_1 \quad g_2 \quad [g_1, f] \quad [g_2, f] \quad [g_2, [f, [g_1, f]]]$$

span $\mathbb{R}^5$, and the sequence is 'good' [Sussmann]

- **car-like robot (RD)** $(n = 4)$

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ -1/\ell\cos^2\phi \\ 0 \end{pmatrix}$$

$$g_4 = [g_1, g_3] = \begin{pmatrix} -\sin\theta/\ell\cos^2\phi \\ \cos\theta/\ell\cos^2\phi \\ 0 \\ 0 \end{pmatrix}$$

degree of nonholonomy $\kappa = 3$, dim $\bar{\mathcal{C}} = 4$ away from the singularity at $\phi = \pm\pi/2$ of g_1

- **car-like robot (FD)** $(n = 4)$

$$g_1 = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ -\cos\phi/\ell \\ 0 \end{pmatrix}$$

$$g_4 = [g_1, g_3] = \begin{pmatrix} -\sin\theta/\ell \\ \cos\theta/\ell \\ 0 \\ 0 \end{pmatrix}$$

degree of nonholonomy $\kappa = 3$, $\dim\bar{\mathcal{C}} = 4$ for all q

- **N-trailer system** $(n = N + 4)$

for a slightly modified version of this mobile robot the degree of nonholonomy is n

- **all** the previous WMRs are STLC; **none** of these is smoothly stabilizable

- **3-body space robot** $(n = 3)$

$$g_1 = \begin{pmatrix} s_1(\phi) \\ 1 \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} s_2(\phi) \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} \frac{\partial s_2(\phi)}{\partial \phi_1} - \frac{\partial s_1(\phi)}{\partial \phi_2} \\ 0 \\ 0 \end{pmatrix}$$

  but $g_3 = 0$ for some combinations of ϕ_1 and ϕ_2

  – the filtration is not regular: thus, the degree of nonholonomy is not well defined

  – using higher order brackets, $\dim \bar{\mathcal{C}} = 3$ for all q and the system is controllable

- ***N*-body space robot dynamics** $(n = 2N - 1)$

  the system satisfies the conditions for LA, STLC, but not the necessary condition for stabilizability via C^1-feedback

NONHOLONOMIC MOTION PLANNING

- the objective is to build a sequence of **open-loop** input commands that steer the system from q_i to q_f satisfying the nonholonomic constraints

- the degree of nonholonomy gives a good measure of the complexity of the steering algorithm

- there exist **canonical** model structures for which the steering problem can be solved efficiently
  - chained form
  - power form
  - Caplygin form

- interest in the **transformation** of the original model equation into one of these forms

- such model structures allow also a simpler design of feedback stabilizers (necessarily, non-smooth or time-varying)

- we limit the analysis to the case of systems **with two inputs**, where the three above forms are equivalent (via a coordinate transformation)

Chained Forms [Murray and Sastry 1993]

- a $(2, n)$ **chained form** is a two-input driftless control system

$$\dot{z} = g_1(z)v_1 + g_2(z)v_2$$

in the following form

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ \dot{z}_4 &= z_3 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1 \end{aligned}$$

- denoting repeated Lie brackets as $\text{ad}_{g_1}^k g_2$

$$\text{ad}_{g_1} g_2 = [g_1, g_2] \qquad \text{ad}_{g_1}^k g_2 = [g_1, \text{ad}_{g_1}^{k-1} g_2]$$

one has

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \end{pmatrix} \; g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Rightarrow\; \text{ad}_{g_1}^k g_2 = \begin{pmatrix} 0 \\ \vdots \\ (-1)^k \\ \vdots \\ 0 \end{pmatrix}$$

in which $(-1)^k$ is the $(k+2)$-th entry

- a one-chain system is **completely nonholonomic (controllable)** since the n vectors

$$\{g_1, g_2, \ldots, \text{ad}^i_{g_1} g_2, \ldots\} \quad i = 1, \ldots, n-2$$

are independent

- its degree of nonholonomy is $\kappa = n - 1$

- v_1 is called the **generating** input, z_1 and z_2 are called **base variables**

- if v_1 is (piecewise) constant, the system in chained form behaves like a (piecewise) linear system

- chained systems are a generalization of first- and second-order controllable systems for which sinusoidal steering from z_i to z_f minimizes the integral norm of the input

- different input commands can be used, e.g.
  - sinusoidal inputs
  - piecewise constant inputs
  - polynomial inputs

steering with sinusoidal inputs

- it is a two-phase method:

  I. steer the base variables z_1 and z_2 to their desired values z_{f1} and z_{f2} (in finite time)

  II. for each z_{k+2}, $k \geq 1$, steer z_{k+2} to its final value $z_{f,k+2}$ using

  $$v_1 = \alpha \sin \omega t \qquad v_2 = \beta \cos k\omega t$$

  over one period $T = 2\pi/\omega$, where α, β are such that

  $$\frac{\alpha^k \beta}{k!(2\omega)^k} = z_{f,k+2}(T) - z_{k+2}(0)$$

  this guarantees $z_i(T) = z_i(0) = z_{fi}$ for $i < k$

  in phase II, this step-by-step procedure adjusts one variable at a time by exploiting the closed-form integrability of the system equations under sinusoidal inputs

- phase II can be executed also all at once, choosing

  $$\begin{aligned} v_1 &= a_0 + a_1 \sin \omega t \\ v_2 &= b_0 + b_1 \cos \omega t + \ldots + b_{n-2} \cos(n-2)\omega t \end{aligned}$$

  and solving numerically for the $n+1$ unknowns in terms of the desired variation of the $n-2$ states

steering with piecewise constant inputs

- an idea coming from multirate digital control, with the total travel time T divided in subintervals of length δ over which constant inputs are applied

$$\begin{aligned} v_1(\tau) &= v_{1,k} \\ v_2(\tau) &= v_{2,k} \end{aligned} \qquad \tau \in [(k-1)\delta, k\delta)$$

- it is convenient to keep v_1 always constant and take $n-1$ subintervals so that

$$T = (n-1)\delta \qquad v_1 = \frac{z_{f1} - z_{01}}{T}$$

and the $n-1$ constant values of input v_2

$$v_{2,1},\ v_{2,2},\ \ldots,\ v_{2,n-1}$$

are obtained solving a triangular linear system coming from the closed-form integration of the model equations

- if $z_{f1} = z_{01}$, an intermediate point must be added

- for small δ, a fast motion but with large inputs

steering with polynomial inputs

- idea similar to piecewise constant input, but with improved **smoothness** properties w.r.t. time (remember that kinematic models are controlled at the (pseudo)velocity level)

- the controls are chosen as

$$\begin{aligned} v_1 &= \text{sign}(z_{f1} - z_{01}) \\ v_2 &= c_0 + c_1 t + \ldots + c_{n-2} t^{n-2} \end{aligned}$$

  with $T = z_{f1} - z_{01}$ and $c_0, \ldots, c_n$ obtained solving the linear system coming from the closed-form integration of the model equations

$$M(T) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{pmatrix} + m(z_i, T) = \begin{pmatrix} z_{f2} \\ z_{f3} \\ \vdots \\ z_{fn} \end{pmatrix}$$

  with $M(T)$ nonsingular for $T \neq 0$

- if $z_{f1} = z_{01}$, an intermediate point must be added

- for small T, a fast motion but with large inputs

transformation into chained form

- there exist necessary and sufficient conditions for transforming a control system

$$\dot{q} = g_1(q)u_1 + \ldots + g_m(q)u_m \qquad q \in I\!R^n$$

into chained form via input transformation and change of coordinates

$$v = \beta(q)u \qquad z = T(q)$$

- for $m = 2$, $\mathcal{C} = \text{span}\{g_1, g_2\}$, define the filtrations

$$\begin{aligned} E_1 &= \mathcal{C} & F_1 &= \mathcal{C} \\ E_2 &= E_1 + [E_1, E_1] & F_2 &= F_1 + [F_1, F_1] \\ &\vdots & &\vdots \\ E_{i+1} &= E_i + [E_i, E_i] & F_{i+1} &= F_i + [F_i, F_1] \end{aligned}$$

- the system can be transformed in chained form if and only if

$$\dim E_i = \dim F_i = i + 1 \qquad i = 1, \ldots, n-1$$

nonholonomic systems up to order $n = 4$ can be **always** be put in chained form!

- a simpler constructive sufficient condition: define the distributions

$$\begin{aligned}\Delta_0 &= \text{span}\,\{g_1, g_2, \text{ad}_{g_1} g_2, \ldots, \text{ad}_{g_1}^{n-2} g_2\}\\ \Delta_1 &= \text{span}\,\{g_2, \text{ad}_{g_1} g_2, \ldots, \text{ad}_{g_1}^{n-2} g_2\}\\ \Delta_2 &= \text{span}\,\{g_2, \text{ad}_{g_1} g_2, \ldots, \text{ad}_{g_1}^{n-3} g_2\}\end{aligned}$$

  if, for some open set, one has *(i)* $\dim \Delta_0 = n$ *(ii)* Δ_1, Δ_2 are involutive *(iii)* there exists a function h_1 such that

$$dh_1 \cdot \Delta_1 = 0 \qquad dh_1 \cdot g_1 = 1$$

  then the system can be transformed into chained form

- the change of coordinates is given by

$$\begin{aligned}z_1 &= h_1\\ z_2 &= L_{g_1}^{n-2} h_2\\ &\vdots\\ z_{n-1} &= L_{g_1} h_2\\ z_n &= h_2\end{aligned}$$

  with h_2 independent from h_1 and such that

$$dh_2 \cdot \Delta_2 = 0$$

- the input transformation is given by

$$\begin{aligned}v_1 &= u_1\\ v_2 &= \left(L_{g_1}^{n-1} h_2\right) u_1 + \left(L_{g_2} L_{g_1}^{n-2} h_2\right) u_2\end{aligned}$$

WMRs in Chained Form

- **unicycle**

  the change of coordinates

$$\begin{aligned} z_1 &= x \\ z_2 &= \tan\theta \\ z_3 &= y \end{aligned}$$

  and input transformation

$$\begin{aligned} u_1 &= v_1 / \cos\theta \\ u_2 &= v_2 \cos^2\theta \end{aligned}$$

  yield

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \end{aligned}$$

  other, globally defined transformations are possible

- **unicycle with N trailers**

  the sufficient conditions are not satisfied but an 'ad hoc' transformation can be found (it starts using as (x, y) the position of the **last trailer** instead of the position of the trailing car)

- **car-like robot (RD)**

scaling first u_1 by $\cos\theta$

$$
\begin{aligned}
\dot{x} &= u_1 \\
\dot{y} &= u_1 \tan\theta \\
\dot{\theta} &= \frac{1}{\ell} u_1 \sec\theta \tan\phi \\
\dot{\phi} &= u_2
\end{aligned}
$$

then setting

$$
\begin{aligned}
z_1 &= x \\
z_2 &= \frac{1}{\ell} \sec^3\theta \tan\phi \\
z_3 &= \tan\theta \\
z_4 &= y
\end{aligned}
$$

and

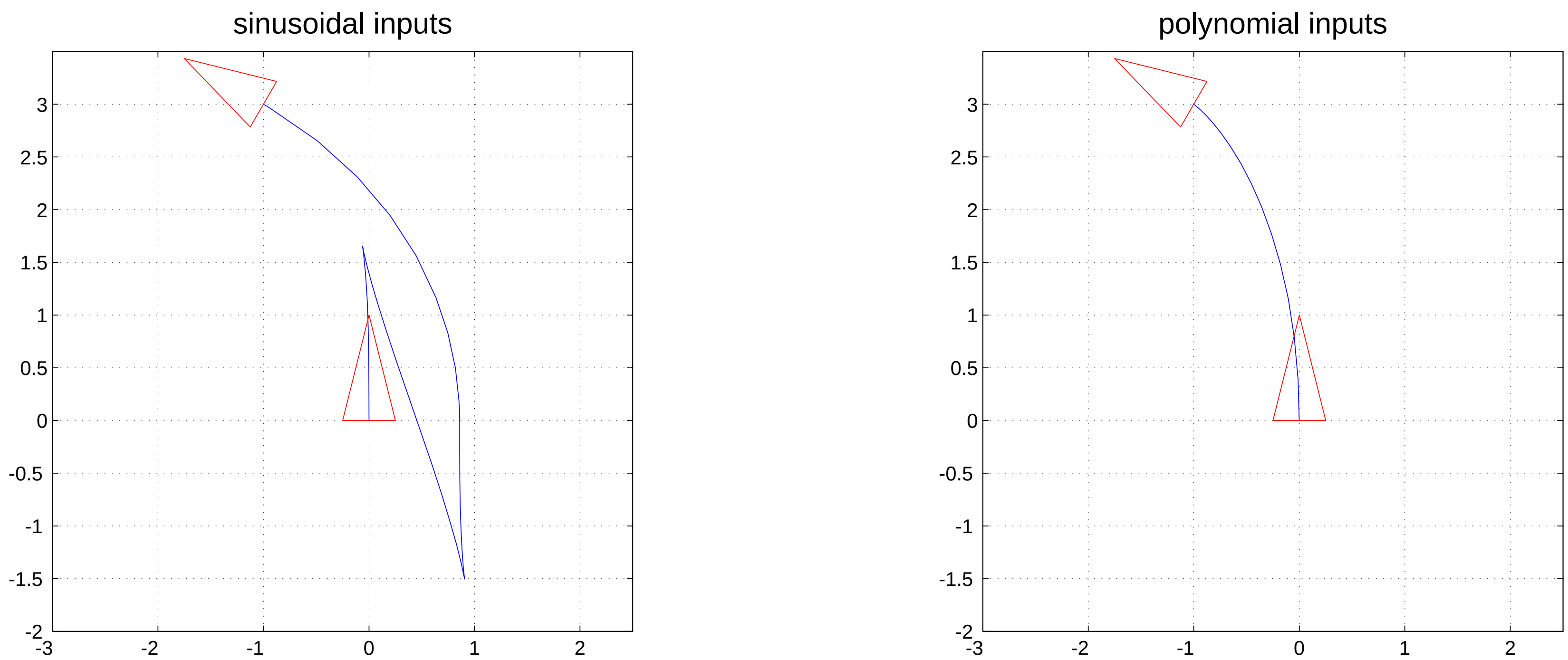
$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= -\frac{3}{\ell} v_1 \sec\theta \sin^2\phi + \frac{1}{\ell} v_2 \cos^3\theta \cos^2\phi
\end{aligned}
$$

yields

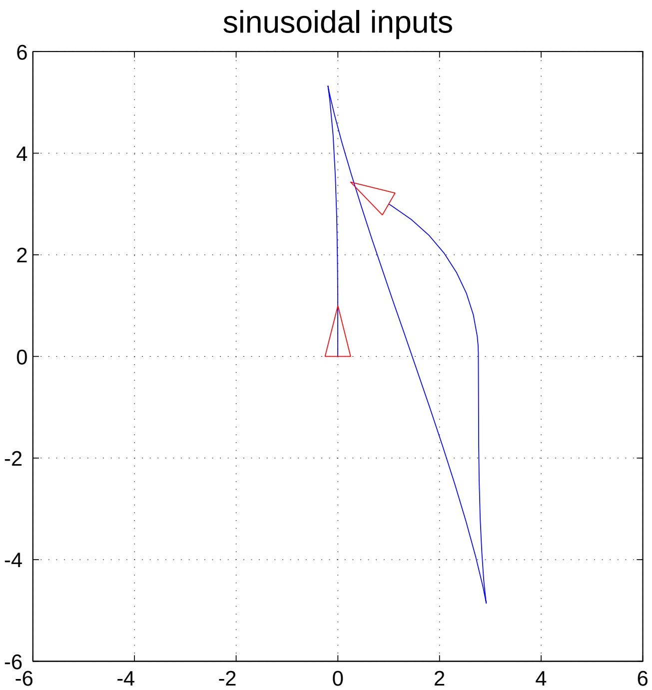
$$
\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1 \\
\dot{z}_4 &= z_3 v_1
\end{aligned}
$$

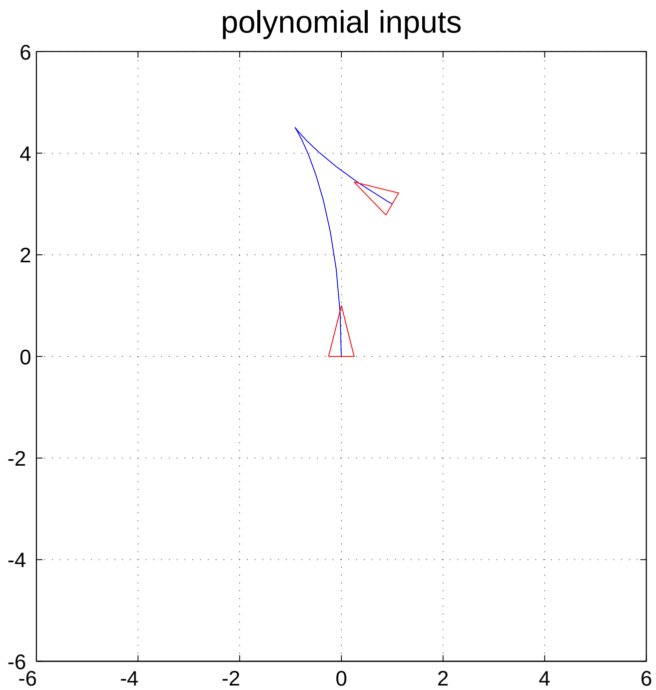
Path Planning for the Unicycle

simulation 1: $q_i = (-1, 3, 150°)$, $q_f = (0, 0, 90°)$



simulation 2: $q_i = (1, 3, 150°)$, $q_f = (0, 0, 90°)$





A General Viewpoint: Differential Flatness [Fliess *et al.* 1995]

- a nonlinear control system $\dot{z} = f(z) + G(z)v$ is **differentially flat** if there exists a set of outputs y (**flat outputs**) such that the state and the input can be expressed **algebraically** in terms of y and a certain number r of its derivatives

$$
\begin{aligned}
z &= z(y, \dot{y}, \ddot{y}, \ldots, y^{[r]}) \\
v &= v(y, \dot{y}, \ddot{y}, \ldots, y^{[r]})
\end{aligned}
$$

- for driftless systems, flatness is equivalent to chained-form transformability; the flat outputs of a chained form are z_1, z_n (i.e., the x, y coordinates of the robot for a WMR)

- for example, for the (2,3) chained form equivalent to a unicycle, the flat outputs are z_1, z_3; one has

$$
z_2 = \frac{\dot{z}_3}{\dot{z}_1} \qquad \text{and} \qquad v_1 = \dot{z}_1, \quad v_2 = \frac{\dot{z}_1 \ddot{z}_3 - \ddot{z}_1 \dot{z}_3}{\dot{z}_1^2}
$$

- for systems with drift, flatness is equivalent to dynamic feedback linearizability

- flatness is particularly useful for **path planning**: once the flat outputs are identified (a nontrivial task), any interpolation scheme can be used to join their initial and final values (with the appropriate boundary conditions); the evolution of the other variables as well as the control inputs are then computed through the algebraic transformations

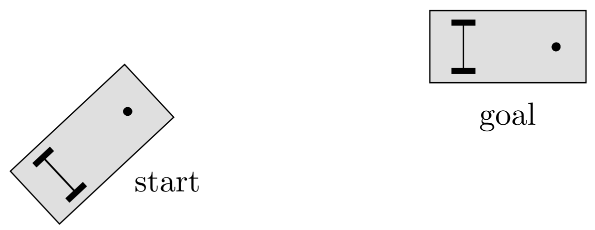
FEEDBACK CONTROL OF NONHOLONOMIC SYSTEMS

Basic Problems

- target system: **unicycle**
  - the kinematic models of most single-body WMRs can be reduced to a unicycle
  - most of the presented design techniques can be systematically extended to chained-form transformable systems

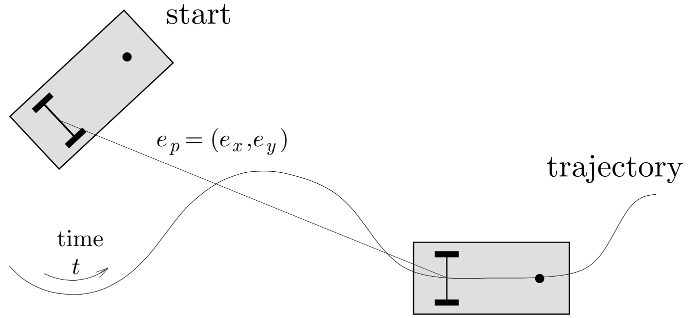
- basic motion tasks

(a) point-to-point motion (PTPM)



(a)

(b) trajectory following (TF)



- PTPM via feedback: **posture stabilization**
  - w.l.o.g., the origin $(0,0,0)$ is assumed to be the desired posture
  - a **nonsquare** ($q \in I\!R^3$, $u \in I\!R^2$) state regulation problem
  - need to use discontinuous/time-varying feedback in view of Brockett Theorem
  - poor, erratic transient performance is often obtained (inefficient, unsafe in the presence of obstacles)

- TF via feedback: **asymptotic tracking**
  - the desired trajectory $q_d(t)$ must be feasible, i.e., comply with the nonholonomic constraints
  - a **square** ($e_p \in I\!R^2$, $u \in I\!R^2$) error zeroing problem
  - in this case, smooth feedback can be used because the linear approximation along a nonvanishing trajectory is controllable (see later)

$$\Downarrow$$

asymptotic tracking is easier (and more useful) **than posture stabilization for nonholonomic systems**

Asymptotic Tracking

- a reference output trajectory $(x_d(t), y_d(t))$ is given
- control action: **feedforward** + **error feedback**

  error may be defined w.r.t. the reference output (**output error**) or the associated reference state (**state error**)

- given an initial posture and a desired trajectory $(x_d(t), y_d(t))$ there is a **unique** associated state trajectory $q_d(t) = (x_d(t), y_d(t), \theta_d(t))$ which can be computed in a purely algebraic way as

$$\theta_d(t) = \text{ATAN2}\left(\dot{y}_d(t), \dot{x}_d(t)\right) + k\pi \qquad k = 0, 1$$

  this is due to the fact that (x, y) is a **flat** output for the unicycle

- **feedforward command generation**: being $\theta = \text{ATAN2}\left(\dot{y}, \dot{x}\right) + k\pi$, $k = 0, 1$, we get

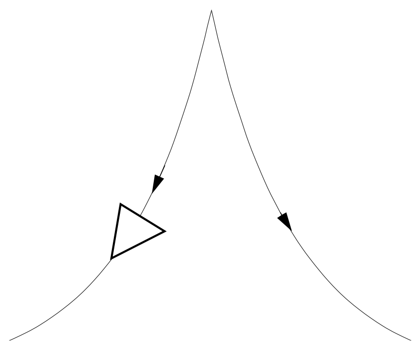
$$\begin{aligned} u_{d1}(t) &= \pm\sqrt{\dot{x}_d^2(t) + \dot{y}_d^2(t)} \\ u_{d2}(t) &= \frac{\ddot{y}_d(t)\dot{x}_d(t) - \ddot{x}_d(t)\dot{y}_d(t)}{\dot{x}_d^2(t) + \dot{y}_d^2(t)} \end{aligned}$$

– the choice of sign for $u_{d1}(t)$ produces forward or backward motion

– to be exactly reproducible, $(x_d(t), y_d(t))$ should be twice differentiable

– $\theta_d(t)$ may be computed off-line and used in order to define a state error

– if $u_{d1}(\bar{t}) = 0$ for some $\bar{t}$ (e.g., at a cusp)



neither $u_{d2}(\bar{t})$ nor $\theta_d(\bar{t})$ are defined

$\Rightarrow$ a continuous motion is guaranteed by keeping the same orientation attained at $\bar{t}^-$

asymptotic tracking: controllability

linear approximation along $q_d(t) = (x_d(t), y_d(t), \theta_d(t))$

- define:
  u_{d1}, u_{d2} the inputs associated to $q_d(t)$
  $\tilde{q} = q - q_d$ the state tracking error
  $\tilde{u}_1 = u_1 - u_{d1}$ and $\tilde{u}_2 = u_2 - u_{d2}$ the input variations

- the linear approximation along $q_d(t)$ is

$$\dot{\tilde{q}} = \begin{pmatrix} 0 & 0 & -u_{d1} \sin\theta_d \\ 0 & 0 & u_{d1} \cos\theta_d \\ 0 & 0 & 0 \end{pmatrix} \tilde{q} + \begin{pmatrix} \cos\theta_d & 0 \\ \sin\theta_d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$$

  a time-varying system

  $\Rightarrow$ the N&S controllability condition is that the controllability Gramian is nonsingular

- a simpler analysis can be performed by 'rotating' the state tracking error

$$\tilde{q}_R = \begin{pmatrix} \cos\theta_d & \sin\theta_d & 0 \\ -\sin\theta_d & \cos\theta_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{q}$$

  according to the reference orientation θ_d

- we get

$$\dot{\tilde{q}}_R = \begin{pmatrix} 0 & u_{d2} & 0 \\ -u_{d2} & 0 & u_{d1} \\ 0 & 0 & 0 \end{pmatrix} \tilde{q}_R + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$$

- when the inputs u_{d1} and u_{d2} are constant, the linearization becomes time-invariant and controllable, since

$$\left(B \;\; AB \;\; A^2B \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & -u_{d2}^2 & u_{d1}u_{d2} \\ 0 & 0 & -u_{d2} & u_{d1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank 3 provided that either u_{d1} or u_{d2} is nonzero

$\Rightarrow$ the kinematic model of the unicycle can be locally asymptotically stabilized by linear feedback along trajectories consisting **of linear or circular paths** executed at a constant velocity

(actually: the same can be proven for **any** nonvanishing trajectory)

linear control design [Samson 1992]

- designed using a (slightly different) linear approximation along the reference trajectory

- define the state tracking error e as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{pmatrix}$$

- use a nonlinear transformation of velocity inputs

$$\begin{aligned} u_1 &= u_{d1} \cos e_3 - v_1 \\ u_2 &= u_{d2} - v_2 \end{aligned}$$

- the error dynamics becomes

$$\dot{e} = \begin{pmatrix} 0 & u_{d2} & 0 \\ -u_{d2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 \\ \sin e_3 \\ 0 \end{pmatrix} u_{d1} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

- linearizing around the reference trajectory, one obtains the same linear time-varying equations as before, now with state e and input (v_1, v_2)

- define the 'linear' feedback law

$$\begin{aligned} v_1 &= -k_1\, e_1 \\ v_2 &= -k_2\, \mathrm{sign}(u_{d1}(t))\, e_2 - k_3\, e_3 \end{aligned}$$

  with gains

$$k_1 = k_3 = 2\zeta a \qquad k_2 = \frac{a^2 - u_{d2}(t)^2}{|u_{d1}(t)|}$$

- the closed-loop characteristic polynomial is $(\lambda + 2\zeta a)(\lambda^2 + 2\zeta a \lambda + a^2)$, $\quad \zeta \in (0,1) \quad a > 0$

- a convenient **gain scheduling** is achieved letting

$$a = a(t) = \sqrt{u_{d2}^2(t) + b u_{d1}^2(t)} \quad \Longrightarrow \quad k_1 = k_3 = 2\zeta \sqrt{u_{d2}^2(t) + b u_{d1}^2(t)}, \quad k_2 = b\,|u_{d1}(t)|$$

  these gains go to zero when the state trajectory stops (and local controllability is lost)

- the actual controls are **nonlinear** and **time-varying**

- even if the eigenvalues are constant, local asymptotic stability is not guaranteed as the system is still time-varying

  $\Rightarrow$ a Lyapunov-based analysis is needed

nonlinear control design [Samson 1993]

for the previous error dynamics, define

$$
\begin{aligned}
v_1 &= -k_1(u_{d1}(t), u_{d2}(t))\, e_1 \\
v_2 &= -\bar{k}_2\, u_{d1}(t) \frac{\sin e_3}{e_3}\, e_2 - k_3(u_{d1}(t), u_{d2}(t))\, e_3
\end{aligned}
$$

with constant $\bar{k}_2 > 0$ and positive, continuous gain functions $k_1(\cdot,\cdot)$ and $k_3(\cdot,\cdot)$

theorem *if u_{d1}, u_{d2}, $\dot{u}_{d1}$ $\dot{u}_{d2}$ are bounded, and if $u_{d1}(t) \not\to 0$ or $u_{d2}(t) \not\to 0$ as $t \to \infty$, the above control globally asymptotically stabilizes the origin $e = 0$*

proof based on the Lyapunov function

$$
V = \frac{\bar{k}_2}{2}\left(e_1^2 + e_2^2\right) + \frac{e_3^2}{2}
$$

nonincreasing along the closed-loop solutions

$$
\dot{V} = -k_1 \bar{k}_2 e_1^2 - k_3 e_3^2 \leq 0
$$

$\Rightarrow$ $\|e(t)\|$ is bounded, $\dot{V}(t)$ is uniformly continuous, and $V(t)$ tends to some limit value

$\Rightarrow$ using Barbalat lemma, $\dot{V}(t)$ tends to zero

$\Rightarrow$ analyzing the system equations, one can show that $(u_{d1}^2 + u_{d2}^2)e_i^2$ $(i = 1, 2, 3)$ tends to zero so that, from the persistency of the trajectory, the thesis follows ∎

dynamic feedback linearization [Oriolo *et al.*, 2002]

- define the output as $\eta = (x, y)$; differentiation w.r.t. time yields

$$\dot{\eta} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\Rightarrow$ cannot recover u_2 from first-order differential information

- add an integrator on the linear velocity input

$$u_1 = \xi, \qquad \dot{\xi} = a \qquad \Rightarrow \qquad \dot{\eta} = \xi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

new input a is the unicycle **linear acceleration**

- differentiating further

$$\ddot{\eta} = \begin{pmatrix} \cos\theta & -\xi \sin\theta \\ \sin\theta & \xi \cos\theta \end{pmatrix} \begin{pmatrix} a \\ u_2 \end{pmatrix}$$

- **assuming** $\xi \neq 0$, we can let

$$\begin{pmatrix} a \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\xi \sin\theta \\ \sin\theta & \xi \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

obtaining

$$\ddot{\eta} = \begin{pmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

- the resulting dynamic compensator is

$$
\begin{aligned}
\dot{\xi} &= v_1 \cos\theta + v_2 \sin\theta \\
u_1 &= \xi \\
u_2 &= \frac{v_2 \cos\theta - v_1 \sin\theta}{\xi}
\end{aligned}
$$

- as the dynamic compensator is 1-dim, we have $n + 1 = 4$, equal to the total number of output differentiations

  $\Rightarrow$ in the new coordinates

$$
\begin{aligned}
z_1 &= x \\
z_2 &= y \\
z_3 &= \dot{x} = \xi \cos\theta \\
z_4 &= \dot{y} = \xi \sin\theta
\end{aligned}
$$

  the system is fully linearized and described by two chains of second-order input-output integrators

$$
\begin{aligned}
\ddot{z}_1 &= v_1 \\
\ddot{z}_2 &= v_2
\end{aligned}
$$

- the dynamic feedback linearizing controller has a potential singularity at $\xi = u_1 = 0$, i.e., when the unicycle is not rolling

  a singularity in the dynamic extension process is **structural** for nonholonomic systems

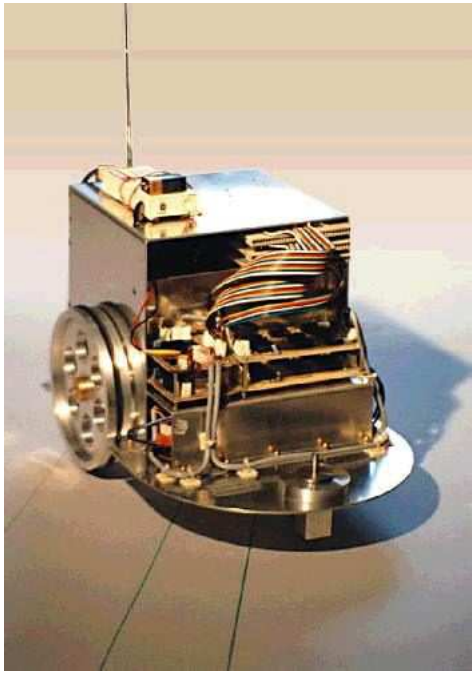
- for the (exactly) linearized system, a globally exponentially stabilizing feedback is

$$
\begin{aligned}
v_1 &= \ddot{x}_d(t) + k_{p1}(x_d(t) - x) + k_{d1}(\dot{x}_d(t) - \dot{x}) \\
v_2 &= \ddot{y}_d(t) + k_{p2}(y_d(t) - y) + k_{d2}(\dot{y}_d(t) - \dot{y})
\end{aligned}
$$

  with PD gains $k_{pi} > 0$, $k_{di} > 0$, for $i = 1, 2$

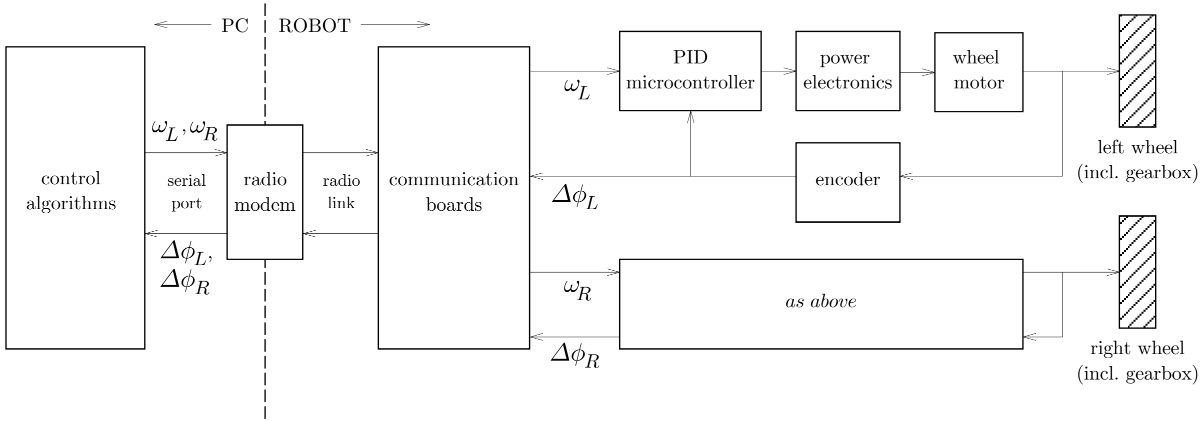
  - the desired trajectory $(x_d(t), y_d(t))$ must be smooth and **persistent**, i.e., $u_{d1}^2 = \dot{x}_d^2 + \dot{y}_d^2$ must never go to zero
  - cartesian transients are linear
  - $\dot{x}$ and $\dot{y}$ can be computed as a function ξ and θ; alternatively, one can use estimates of $\dot{x}$ and $\dot{y}$ obtained from odometric measurements
  - for **exact tracking**, one needs $q(0) = q_d(0)$ and $\xi(0) = u_{d1}(0)$ ($\Rightarrow$ pure feedforward)

experiments with SuperMARIO



- a two-wheel differentially-driven vehicle (with caster)
- the aluminum chassis measures $46 \times 32 \times 30.5$ cm (l/w/h) and contains two motors, transmission elements, electronics, and four 12 V batteries; total weight about 20 kg
- each wheel independently driven by a DC motor (peak torque ≈ 0.56 Nm); each motor equipped with an encoder (200 pulse/turn) and a gearbox (reduction ratio 20)
- typical nonidealities of electromechanical systems: friction, gear backlash, wheel slippage, actuator deadzone and saturation
- due to robot and motor dynamics, discontinuous velocity commands cannot be realized

two-level control architecture



- control algorithms (with reference generation) are written in C^{++} and run with a sampling time of $T_s = 50$ ms on a remote server

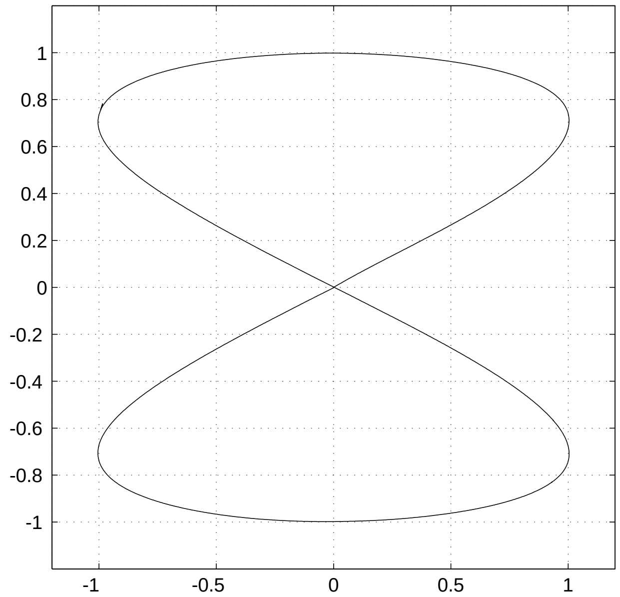
- the PC communicates through a radio modem with the serial communication boards on the robot

- actual commands are the angular velocities ω_R and ω_L of right and left wheel (instead of driving and steering velocities u_1 and u_2):

$$u_1 = \frac{r\,(\omega_R + \omega_L)}{2} \qquad u_2 = \frac{r\,(\omega_R - \omega_L)}{d}$$

 with $d =$ axle length, $r =$ wheel radius

- reconstruction of the current robot state based on encoder data (**dead reckoning**) experiments on an eight-shaped trajectory



- the reference trajectory

$$x_d(t) = \sin \frac{t}{10} \qquad y_d(t) = \sin \frac{t}{20} \qquad t \in [0, T]$$

  starts from the origin with $\theta_d(0) = \pi/6$ rad

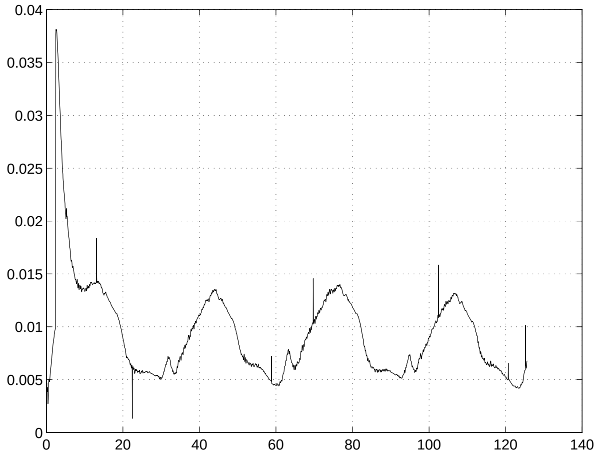
- a full cycle is completed in $T = 2\pi \cdot 20 \approx 125$ s

- the reference initial velocities are

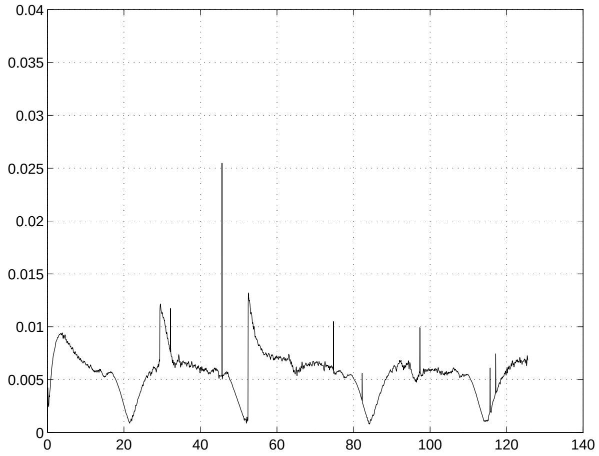
$$u_{d1}(0) \simeq 0.1118 \text{ m/s}, \qquad u_{d2}(0) = 0 \text{ rad/s}.$$

experiment 1: the robot initial state is **on** the reference trajectory

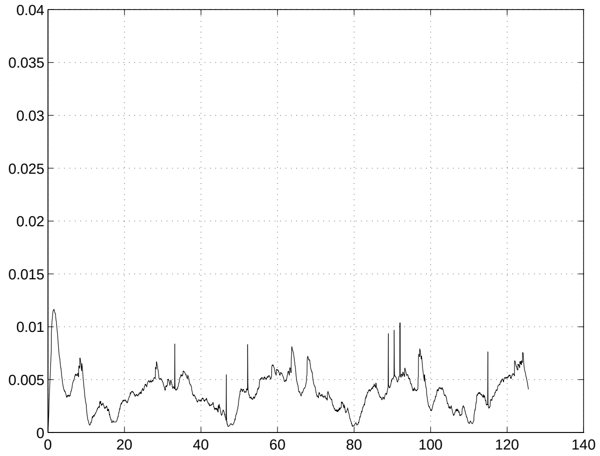
tracking error norm



linear design

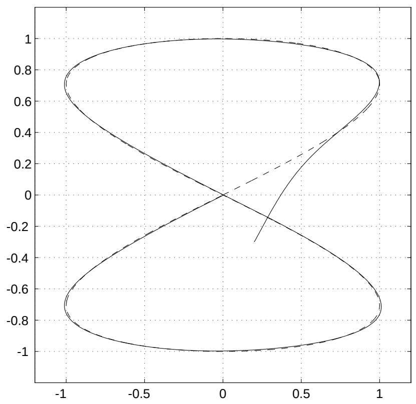


nonlinear design

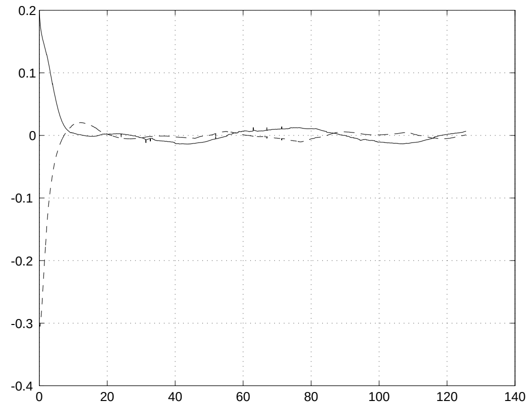


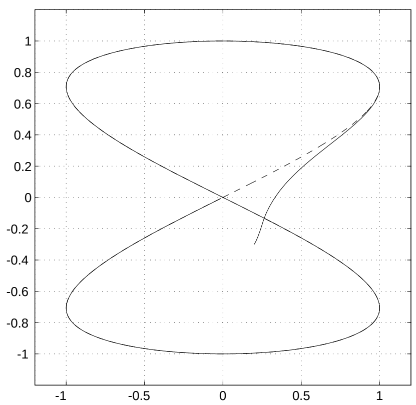
feedback linearization

experiment 2: the robot initial state is **off** the reference trajectory

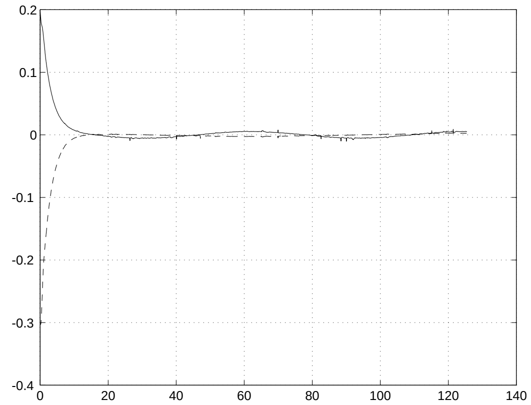


linear design





feedback linearization



Posture Stabilization: A Bird's Eye View

- the main obstruction is the non-smooth stabilizability of WMRs at a point
- two main approaches
  - **time-varying** stabilizers: an exogenous time-varying signal is injected in the controller [Samson 1991]
  - **discontinuous** stabilizers: the controller is time invariant but discontinuous at the origin [Sørdalen 1993]
- drawbacks: slow convergence (time-varying), oscillatory transient (both)
- improvements
  - **mixed time-varying/discontinuous** stabilizers [Pomet and Samson 1993; Murray and M'Closkey 1995, Lucibello and Oriolo 2001]
  - **non-Lyapunov, discontinuous** stabilizers: through coordinate transformations that circumvent Brockett's obstruction [Aicardi *et al.* 1995; Astolfi 1995] or via dynamic feedback linearization [Oriolo *et al.* 2002]

    ↪ excellent transient performance!

OPTIMAL TRAJECTORIES FOR WMRs
(by M. Vendittelli)

- the **main objective** is to determine an optimal control law steering the kinematic model of the nonholonomic system between any two points of the configuration space

- a **first step** is to obtain a family of trajectories containing an optimal solution to the steering problem

- Pontryagin's Maximum Principle (PMP) can be used to this end providing necessary conditions for trajectories to be optimal

- characterization of optimal trajectories is not easy essentially due to the local nature of PMP

- local information needs to be completed by global study based on geometric reasoning

Minimum-Time Problems

- **objective:** compute the control law (if it exists) that steers the nonholonomic system

$$\dot{q} = G(q)u, \quad q \in \mathcal{M} \simeq I\!R^n, \quad u \in U \subset I\!R^m$$

from q_i to q_f minimizing the functional

$$J = \int_{t_i}^{t_f} dt$$

- **theorem** (existence of optimal trajectories)

under the usual assumptions for existence and uniqueness of solution of an ordinary differential equation and the additional hypothesis

U **compact convex** subset of $I\!R^m$ $\qquad (\triangle)$

any two points $q_i, q_f \in \mathcal{M}$ that can be joined by an admissible trajectory can be joined by a time-optimal trajectory

- consider the **Hamiltonian**

$$H(\psi, q, u) = \langle \psi, G(q)u \rangle$$

where $\psi \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$

- if $u(t) : [t_i, t_f] \to U$ is an admissible control law and $q(t) : [t_i, t_f] \to \mathcal{M}$ the corresponding trajectory,

  a vector function $\psi : [t_i, t_f] \to \mathbb{R}^n$ is an **adjoint vector** for (q, u) if it satisfies

$$\dot{\psi}(t)^T = -\frac{\partial H}{\partial q}(\psi(t), q(t), u(t)) \quad \forall t \in [t_i, t_f]$$

- note that

  either $\psi(t) \neq 0 \;\; \forall t \in [t_i, t_f]$ (*nontrivial* ψ)

  or $\psi(t) \equiv 0 \;\; \forall t \in [t_i, t_f]$ (*trivial* ψ)

  due to the linearity of H ($\Rightarrow$ of $\dot{\psi}$) w.r.t. ψ

PMP for time-optimal control

consider an admissible control law $u(t)$ and the corresponding trajectory $q(t)$; a necessary condition for $q(t)$ to be time-optimal is that there exist a nontrivial adjoint vector $\psi(t)$ and a constant $\psi_0 \leq 0$ s.t.

$$H(\psi(t), q(t), u(t)) = \max_{v \in U}\{H(\psi(t), q(t), v)\} = -\psi_0 \quad (*)$$

$\forall t \in [t_i, t_f]$

- a control law $u(t)$ satisfying condition $(*)$ is called an **extremal control law**
- denoting by q, ψ the trajectory and the adjoint vector corresponding to the extremal control law u, the triple (q, u, ψ) is called **extremal**
- an extremal triple (q, u, ψ) s.t. $\psi_0 = 0$ is called **abnormal**
- a control law $u(t)$ is called **singular** if there exist a nonempty subset $W \subset U$ and a nonempty interval $I \subset [t_i, t_f]$ such that

$$H(\psi(t), q(t), u(t)) = H(\psi(t), q(t), w(t))$$

$\forall t \in I, \ \forall w(t) \in W$

Application to WMRs

- target system: unicycle

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 \quad (u_1, u_2) \in U \subset \mathbb{R}^2$$

with $q = (x, y, \theta)$ $\quad g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- new terminology based on **control domains**

$$U = [-k_1, k_1] \times [-k_2, k_2] \qquad \textit{unicycle}$$

$$U = \{-k_1, k_1\} \times [-k_2, k_2] \qquad \textit{Reeds and Shepp's car}$$

$$U = k_1 \times [-k_2, k_2] \qquad \textit{Dubins' car}$$

with $k_1, k_2 > 0$

unicycle and Reeds and Shepp's car are STLC
Dubins' car is controllable but not STLC

- unicycle and Dubins' car verify the conditions for existence of optimal trajectories
- Reeds and Shepp's car does not verify condition ($\triangle$); existence of optimal trajectories has been established as a byproduct of the analysis of the optimal control problem for the unicycle

unicycle

- $(u_1, u_2) \in U = [-1, 1] \times [-1, 1]$ (w.l.o.g.)

- the corresponding Hamiltonian is

$$H = \psi_1 \cos\theta u_1 + \psi_2 \sin\theta u_1 + \psi_3 u_2$$

- it is convenient to define the **switching functions**

$$\phi_1 = \langle \psi, g_1 \rangle = \psi_1 \cos\theta + \psi_2 \sin\theta, \quad \phi_2 = \langle \psi, g_2 \rangle = \psi_3$$

  and write the Hamiltonian as

$$H = \phi_1 u_1 + \phi_2 u_2$$

- the switching functions determine the sign changes of u_1, u_2 (see later)

- applying PMP

$$-\psi_0 = H(\psi(t), q(t), u(t)) = \max_{v \in U}(H(\psi(t), q(t), v)) = \max_{(v_1, v_2) \in U}(\phi_1 v_1 + \phi_2 v_2) \quad (1)$$

  where

$$\dot{\psi}(t) = -\frac{\partial H}{\partial q}(\psi(t), q(t), u(t)) = -\frac{\partial}{\partial q}(\phi_1 u_1 + \phi_2 u_2)$$

extracting information from PMP

- maximization of the Hamiltonian (i.e. cond. (1)) implies that on extremal trajectories

$$u_1 = \text{sign}(\phi_1) \quad u_2 = \text{sign}(\phi_2) \tag{2}$$

  where

$$\text{sign}(s) = \begin{cases} \ \ 1 \text{ if } s > 0 \\ -1 \text{ if } s < 0 \\ \text{ any number in } [-1, 1] \text{ if } s = 0 \end{cases}$$

- on any subinterval of $[t_i, t_f]$ where $\phi_j \neq 0$ (j={1,2}) u_j is **bang** (i.e. maximal or minimal)

- a necessary condition for t to be a switching time for $u_j(t)$ is that $\phi_j(t) = 0$

- if $\phi_j(t) = 0$ on a nonempty interval $I \subset [t_i, t_f]$ the corresponding control $u_j(t)$ is singular on I

- to characterize the structure of extremals

  define

$$\phi_3 = \langle \psi, [g_2, g_1] \rangle$$

  compute

$$\begin{aligned} \dot{\phi}_1 &= u_2 \cdot \langle \psi, [g_2, g_1] \rangle = u_2 \phi_3 \\ \dot{\phi}_2 &= -u_1 \cdot \langle \psi, [g_2, g_1] \rangle = -u_1 \phi_3 \\ \dot{\phi}_3 &= -u_2 \phi_1 \end{aligned} \tag{3}$$

- from (1), (2)

$$| \phi_1 | + | \phi_2 | + \psi_0 = 0 \tag{4}$$

- from $\psi \neq 0$ + controllability

$$| \phi_1 | + | \phi_2 | + | \phi_3 | \neq 0 \tag{5}$$

- (2), (3), (4), (5) are called **Switching Structure Equations**

lemma 1 nontrivial abnormal extremals do not exist
proof: use (4), (5), (3)

lemma 2 for a nontrivial optimal extremal, ϕ_1 and ϕ_2 cannot have a common zero
proof: use (4)

lemma 3 along an extremal, $\kappa = \phi_1^2 + \phi_3^2$ is constant and $\kappa = 0 \iff \phi_1 \equiv 0$
proof: use lemma 2, (3)

lemma 4 along an extremal, either all the zeros of ϕ_1 are isolated and s.t. $\dot{\phi}_1$ exists and is $\neq 0$ or $\phi_1 \equiv 0$
proof: use lemma 3, lemma 2, (3)

$$\Downarrow$$

there exist two kinds of extremal trajectories

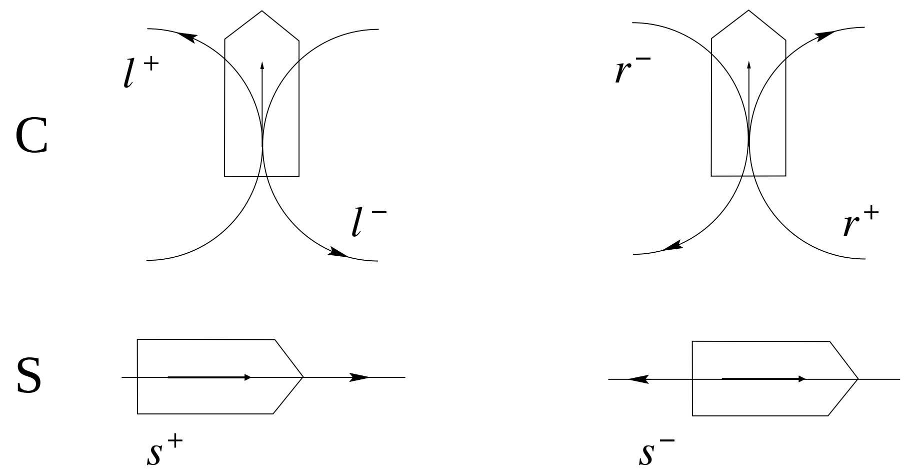
A trajectories with a finite number of switchings

B trajectories along which $\phi_1 \equiv 0$ and either $u_2 \equiv 1$ or $u_2 \equiv -1$

to simplify the geometric description of the extremals it is useful to introduce the following

notation

C_a	arc of circle of length a
S_a	straight line segment of length a
$C\|C$	arcs of circle connected by a cusp
$l_a^{+(-)}$	forward (backward) left motion along the arc of length a
$r_a^{+(-)}$	forward (backward) right motion along the arc of length a
$s_a^{+(-)}$	forward (backward) motion along the straight line segment of length a



type A trajectories

the integration of the adjoint system

$$\begin{cases} \dot{\psi}_1 = -\frac{\partial H}{\partial x} = 0 \\ \dot{\psi}_2 = -\frac{\partial H}{\partial y} = 0 \\ \dot{\psi}_3 = -\frac{\partial H}{\partial \theta} = \psi_1 \sin\theta u_1 - \psi_2 \cos\theta u_1 = \psi_1 \dot{y} - \psi_2 \dot{x} \end{cases}$$

implies (w.l.o.g. $x(t_i) = y(t_i) = 0$)

- ψ_1 and ψ_2 constant
- $\psi_3(t) = \psi_3(t_i) + \psi_1 y - \psi_2 x = \phi_2(t)$

- if $\phi_1 = 0$ (switch of u_1), (1) implies $\phi_2 u_2 + \psi_0 = \psi_3 u_2 + \psi_0 = 0$
  - if $u_2 = 1$ the **cusp** point is on the line

$$\mathcal{D}^+ : \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_i) + \psi_0 = 0$$

  - if $u_2 = -1$ the cusp point is on the line

$$\mathcal{D}^- : \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_i) - \psi_0 = 0$$

- if $\phi_2 = 0$ (switch of u_2) the **inflection** point lies on the line

$$\mathcal{D}_0 : \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_i) = 0$$

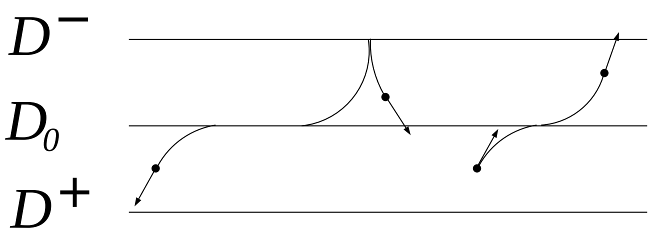
- if $\phi_2(t)$ vanishes on a nonempty interval $I \subset [t_i, t_f]$ from (1)

$$\psi_1 \cos(\theta(t)) + \psi_2 \sin(\theta(t)) + \psi_0 = 0$$

  from lemma 2, ψ_1 and ψ_2 cannot be both zero then θ must remain constant on I

summarizing:

- type A trajectories are sequences of
  - arcs of circle (C) of radius 1 corresponding to regular control laws ($u_1 = \pm 1, u_2 = \pm 1$)
  - straight segments (S) corresponding to the singularity of u_2 ($u_1 = \pm 1, u_2 = 0$)
- straight line segments and points of inflection are on $\mathcal{D}_0$
- cusp tangents are perpendicular to $\mathcal{D}^+$ and $\mathcal{D}^-$
- **lemma** trajectories of type A and with no cusps are necessarily of one of the following kinds
  - $C_a \quad 0 \leq a \leq \pi$
  - $C_a C_b \quad 0 < a \leq \frac{\pi}{2},\ 0 < b \leq \frac{\pi}{2}$
  - $C_a S_d C_b \quad d > 0,\ 0 < a \leq \frac{\pi}{2},\ 0 < b \leq \frac{\pi}{2}$



- to refine the large family of trajectories implied by type A a global geometric study would be needed

- a *boundary trajectory* is a trajectory $q : [t_i, t_f] \to \mathcal{M}$ such that $q(t_f)$ belongs to the boundary of the set of all reachable points from $q(t_i)$

PMP for boundary trajectories
if $q : [t_i, t_f] \to \mathcal{M}$ is a boundary trajectory, then it has a nontrivial adjoint vector $\psi(t)$ verifying $(*)$ with $\psi_0 = 0$

type B trajectories
type B trajectories correspond to the singularity of the control component u_1 and their characterization requires geometric reasoning plus the application of PMP for boundary trajectories

lemma the search for optimal trajectories of type B can be restricted to the sufficient family of path types

$$l_a^+ l_b^- l_e^+ \text{ or } r_a^+ r_b^- r_e^+ \qquad \text{with } 0 \le a, b, e \le \pi$$

in conclusion:
sufficient family of optimal trajectories for the unicycle (PMP + geometric reasoning)

(I)	$l_a^+ l_b^- l_e^+$ or $r_a^+ r_b^- r_e^+$	$0 \le a \le \pi$, $0 \le b \le \pi$, $0 \le e \le \pi$
$(II)(III)$	$C_a \| C_b C_e$ or $C_a C_b \| C_e$	$0 \le a \le b$, $0 \le e \le b$, $0 \le b \le \pi/2$
(IV)	$C_a C_b \| C_b C_e$	$0 \le a \le b$, $0 \le e \le b$, $0 < b \le \pi/2$
(V)	$C_a \| C_b C_b \| C_e$	$0 \le a \le b$, $0 \le e \le b$, $0 < b \le \pi/2$
(VI)	$C_a \| C_{\pi/2} S_d C_{\pi/2} \| C_b$	$0 \le a \le \pi/2$, $0 \le b < \pi/2$, $0 \le d$
$(VII)(VIII)$	$C_a \| C_{\pi/2} S_d C_b$ or $C_b S_l C_{\pi/2} \| C_a$	$0 \le a \le \pi$, $0 \le b \le \pi/2$, $0 \le d$
(IX)	$C_a S_d C_b$	$0 \le a \le \pi/2$, $0 \le b \le \pi/2$, $0 \le d$

- since $u_1 = \pm 1$ for all the path types contained in this family, they are admissible for the Reeds and Shepp's car; this implies that the family is also sufficient for the Reeds and Shepp's time-optimal control problem

- time-optimal trajectories for the Reeds and Shepp's car are paths of minimal length (recall that for Reeds and Shepp's car $u_1 = \pm 1$)

OPEN PROBLEMS

the techniques so far presented are fairly standard now, and the associated theoretical problems can be considered as solved

but: from an application viewpoint, many important issues deserve further research:

- **path planning in the presence of obstacles**: classical motion planning methods do not apply to WMRs because they ignore nonholonomic constraints

- **inclusion of dynamics**: for massive vehicles and/or at high speeds, consideration of robot dynamics is necessary for realistic control design

- **robust control design**: cope with disturbances and model perturbations (e.g., slipping)

- **use of exteroceptive feedback**: most control schemes require the measure of the WMR state; however, proprioceptive sensors, such as encoders, become unreliable in the long run $\Rightarrow$ close the feedback loop with exteroceptive sensors providing absolute information about the robot localization in its workspace (e.g., vision)

- **WMRs not transformable in chained form**: such as a unicycle towing two or more trailers hitched at some distance from the midpoint of the previous wheel axle