Knowledge representation and semantic technologies
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Exercises on Description Logics

Exercise 1 Given the following TBox:

\[
\begin{align*}
A & \sqsubseteq B \\
B & \sqsubseteq C \\
C & \sqsubseteq \exists R.D \\
D & \sqsubseteq \neg A
\end{align*}
\]

1. tell whether the TBox $T$ is satisfiable, and if so, show a model for $T$;
2. tell whether the concept $D$ is satisfiable with respect to $T$, and if so, show a model for $T$ where the interpretation of $D$ is non-empty;
3. tell whether the concept expression $A \cap D$ is satisfiable with respect to $T$, and if so, show a model for $T$ where the interpretation of $A \cap D$ is non-empty.

Solution

1. Let $I$ be the interpretation over the domain $\Delta^I = \{d\}$ such that $A^I = B^I = C^I = D^I = R^I = \emptyset$. It is immediate to see that all the axioms of $T$ are satisfied in $I$: e.g., since $A^I$ is empty, it is obviously true that $A^I \subseteq B^I$, hence the first axiom of $T$ is satisfied by $I$. Consequently, $I$ is a model for $T$, which implies that $T$ is satisfiable.

2. To prove that the concept $D$ is satisfiable with respect to $T$ we have to show a model for $T$ where the interpretation of $D$ is non-empty. Now, the above model $I$ does not show that $D$ is satisfiable with respect to $T$, because $D^I$ is empty. So, we define a new interpretation $J$, over the domain $\Delta^J = \{d\}$, such that $A^J = B^J = C^J = R^J = \emptyset$ and $D^J = \{d\}$. Again, it is immediate to verify that all the axioms of $T$ are satisfied in $J$. In particular, $D \sqsubseteq \neg A$ is satisfied since $(-\neg A)^J = \Delta^J = \{d\}$. Consequently, $J$ is a model for $T$.

3. Since the TBox $T$ contains the axiom $D \sqsubseteq \neg A$, it follows that every model $I$ for $T$ is such that $D^I \subseteq (-\neg A)^I$, i.e., $D^I \cap A^I = \emptyset$. Consequently, no model $I$ for $T$ exists such that $(A \cap D)^I$ is non-empty.

Exercise 2 Given the knowledge base (KB) $K = \langle T, A \rangle$, where $T$ is the following TBox:

\[
\begin{align*}
(Ax1) & & A \sqsubseteq B \\
(Ax2) & & B \sqsubseteq C \\
(Ax3) & & C \sqsubseteq \exists R.D \\
(Ax4) & & D \sqsubseteq \neg A \\
(Ax5) & & A \sqsubseteq \forall R.A
\end{align*}
\]
and \( \mathcal{A} \) is the following ABox:

\[
\{ A(a), D(c), R(a, b), R(b, c) \}
\]

1. using the tableau method, tell whether the KB \( \mathcal{K} \) is satisfiable (i.e., consistent), and if so, show a model for \( \mathcal{K} \);

2. now consider the KB \( \mathcal{K}' \) obtained from \( \mathcal{K} \) by deleting axiom (Ax1) in the TBox. Tell whether the concept assertion \( \neg A \sqcup \neg D(c) \) is entailed by \( \mathcal{K}' \), using the tableau method.

**Solution, point 1**

We start by considering point 1 of the exercise. First, \( \top \text{GCI} \) for the given TBox is the following concept expression:

\[
\top \text{GCI} = (\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A)
\]

Now, we start the tableau from the initial ABox \( \mathcal{A}_0 = \mathcal{A} \):

\[
\mathcal{A}_0 = \{ A(a), D(c), R(a, b), R(b, c) \}
\]

We then apply the tableau \( \top \text{GCI} \)-rule to the individual \( a \), obtaining

\[
\mathcal{A}_1 = \mathcal{A}_0 \cup \{ (\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A))(a) \}
\]

We then apply the tableau and-rule to the last assertion, obtaining

\[
\mathcal{A}_2 = \mathcal{A}_1 \cup \{ (\neg A \sqcup B)(a), (\neg B \sqcup C)(a), (\neg C \sqcup \exists R.D)(a), (\neg D \sqcup \neg A)(a), (\neg A \sqcup \forall R.A)(a) \}
\]

We then apply the tableau or-rule to the assertion \( (\neg A \sqcup B)(a) \), obtaining

\[
\mathcal{A}_3 = \mathcal{A}_2 \cup \{ \neg A(a) \}
\]

\[
\mathcal{A}_4 = \mathcal{A}_2 \cup \{ B(a) \}
\]

Now, \( \mathcal{A}_3 \) contains the clash \( \{ A(a), \neg A(a) \} \) (since \( A(a) \in \mathcal{A}_0 \)), so it is closed. We then consider \( \mathcal{A}_4 \) and apply the tableau or-rule to the assertion \( (\neg B \sqcup C)(a) \), obtaining

\[
\mathcal{A}_5 = \mathcal{A}_4 \cup \{ \neg B(a) \}
\]

\[
\mathcal{A}_6 = \mathcal{A}_4 \cup \{ C(a) \}
\]

Now, \( \mathcal{A}_5 \) contains the clash \( \{ B(a), \neg B(a) \} \) (since \( B(a) \in \mathcal{A}_4 \)), so it is closed. We then consider \( \mathcal{A}_6 \) and apply the tableau or-rule to the assertion \( (\neg C \sqcup \exists R.D)(a) \), obtaining

\[
\mathcal{A}_7 = \mathcal{A}_6 \cup \{ \neg C(a) \}
\]

\[
\mathcal{A}_8 = \mathcal{A}_6 \cup \{ \exists R.D(a) \}
\]

Now, \( \mathcal{A}_7 \) contains the clash \( \{ C(a), \neg C(a) \} \) (since \( C(a) \in \mathcal{A}_6 \)), so it is closed. We then consider \( \mathcal{A}_8 \) and apply the tableau \( \exists \)-rule to the assertion \( \exists R.D(a) \), obtaining

\[
\mathcal{A}_9 = \mathcal{A}_8 \cup \{ R(a, x), D(x) \}
\]
We now apply the tableau or-rule to the assertion \((\neg D \sqcup \neg A)(a)\), obtaining

\[ \mathcal{A}_{10} = \mathcal{A}_0 \cup \{ \neg D(a) \} \]
\[ \mathcal{A}_{11} = \mathcal{A}_0 \cup \{ \neg A(a) \} \]

Now, \(\mathcal{A}_{11}\) contains the clash \(\{A(a), \neg A(a)\}\) (since \(A(a) \in \mathcal{A}_0\)), so it is closed. We then consider \(\mathcal{A}_{10}\) and apply the tableau or-rule to the assertion \((\neg A \sqcup \forall R.A)(a)\), obtaining

\[ \mathcal{A}_{12} = \mathcal{A}_{10} \cup \{ \neg A(a) \} \]
\[ \mathcal{A}_{13} = \mathcal{A}_{10} \cup \{ \forall R.A(a) \} \]

Again, \(\mathcal{A}_{12}\) contains the clash \(\{A(a), \neg A(a)\}\) (since \(A(a) \in \mathcal{A}_0\)), so it is closed. We now consider \(\mathcal{A}_{13}\) and apply the tableau \(\forall\)-rule to the assertion \(\forall R.A(a)\) (notice that \(A(a)\) and \(R(a, x)\) belong to \(\mathcal{A}_{13}\)), obtaining

\[ \mathcal{A}_{14} = \mathcal{A}_{13} \cup \{ A(x) \} \]

We then apply the tableau \(\top\)-rule to the individual \(x\), obtaining

\[ \mathcal{A}_{15} = \mathcal{A}_{14} \cup \{ ((\neg A \sqcup B) \cap (\neg B \sqcup C) \cap (\neg C \sqcup \exists R.D) \cap (\neg D \sqcup \neg A) \cap (\neg A \sqcup \forall R.A))(x) \} \]

We then apply the tableau and-rule to the last assertion, obtaining

\[ \mathcal{A}_{16} = \mathcal{A}_{15} \cup \{ (\neg A \sqcup B)(x), (\neg B \sqcup C)(x), (\neg C \sqcup \exists R.D)(x), (\neg D \sqcup \neg A)(x), (\neg A \sqcup \forall R.A)(x) \} \]

We then apply the tableau or-rule to the assertion \((\neg D \sqcup A)(x)\), obtaining

\[ \mathcal{A}_{17} = \mathcal{A}_{16} \cup \{ \neg D(x) \} \]
\[ \mathcal{A}_{18} = \mathcal{A}_{16} \cup \{ \neg A(x) \} \]

Now, notice that \(D(x) \in \mathcal{A}_9\), therefore \(\mathcal{A}_{17}\) contains the clash \(D(x), \neg D(x)\). Moreover, notice that \(A(x) \in \mathcal{A}_{14}\), therefore \(\mathcal{A}_{18}\) contains the clash \(A(x), \neg A(x)\).

Consequently, all the ABoxes (branches) generated by the tableau are closed. We can thus conclude that the knowledge base \(\mathcal{K} = \langle T, \mathcal{A} \rangle\) of point 1 is inconsistent (unsatisfiable).

**Solution, point 2**

We now consider point 2 of the exercise. First, \(\top\)-rule for the given TBox is the following concept expression:

\[ \top = (\neg B \sqcup C) \cap (\neg C \sqcup \exists R.D) \cap (\neg D \sqcup \neg A) \cap (\neg A \sqcup \forall R.A) \]

Now, we start the tableau from the initial ABox \(\mathcal{A}_0\) obtained by the ABox \(\mathcal{A}\) of point 1 adding the negation of the assertion \(\neg A \sqcup \neg D(c)\):

\[ \mathcal{A}_0 = \mathcal{A} \cup \{ A \sqcap D(c) \} \]

We apply the tableau and-rule to the above assertion, obtaining

\[ \mathcal{A}_1 = \mathcal{A}_0 \cup \{ A(c), D(c) \} \]

We then apply the tableau \(\top\)-rule to the individual \(c\), obtaining

\[ \mathcal{A}_2 = \mathcal{A}_1 \cup \{ (\neg B \sqcup C) \cap (\neg C \sqcup \exists R.D) \cap (\neg D \sqcup \neg A) \cap (\neg A \sqcup \forall R.A))(c) \} \]
We then apply the tableau and-rule to the last assertion, obtaining
\[ A_3 = A_2 \cup \{ (\neg B \sqcup C)(a), (\neg C \sqcup \exists R.D)(a), (\neg D \sqcup \neg A)(a), (\neg A \sqcup \forall R.A)(c) \} \]
We then apply the tableau or-rule to the assertion \((\neg D \sqcup \neg A)(c)\), obtaining
\[ A_4 = A_3 \cup \{ \neg D(c) \} \]
\[ A_5 = A_3 \cup \{ \neg A(c) \} \]
Now, \(A_4\) contains the clash \(\{D(c), \neg D(c)\}\) (since \(D(c) \in A_1\)), so it is closed. Moreover, \(A_5\) contains the clash \(\{A(c), \neg A(c)\}\) (since \(A(c) \in A_1\)), so it is closed too.

Consequently, all the ABoxes (branches) generated by the tableau are closed. We can thus conclude that the knowledge base \(K'\) entails the assertion \(\neg A \sqcup \neg D(c)\).

**Exercise 3** Given the knowledge base (KB) \(K = \langle T, A \rangle\), where \(T\) is the following TBox:

\[
\begin{align*}
A & \sqsubseteq B \sqcup C \\
B & \sqsubseteq \exists R.D \\
C & \sqsubseteq \exists R.E \\
A & \sqsubseteq \forall R.F \\
D \sqcap F & \sqsubseteq G
\end{align*}
\]

and \(A\) is the following ABox:

\[ A(a) \]

1. using the tableau method, tell whether the concept assertion \(\exists R.(E \sqcap G)(a)\) is entailed by \(K\);

2. using the tableau method, tell whether the concept assertion \((\exists R.E) \sqcap (\exists R.G)(a)\) is entailed by \(K\).