EXCERCISES

1. SET COVER

SHOW f-apx WHEN THE FRACTIONAL SOLUTION IS ROUNDED WITH :

IF $X_S^* > 0$ THEN $X_S = 1$

$X_S^* = 0$ THEN $X_S = 0$

USE PRIMAL COMPLEMENTARY SLACKNESS CONDITIONS :

EITHER $X_S^* = 0$ OR $\sum_{e \in S} y_e = c_S$

$\mathcal{C}$: ALGORITHM'S SOLUTION

$$\text{COST(ALG)} = \sum_{S \in \mathcal{C}} c_S = \sum_{S \in \mathcal{C}} \sum_{e \in S} y_e$$

$$= \sum_{e \in U} y_e \sum_{S \in \mathcal{C} : e \in S} 1 \leq f \sum_{e \in U} y_e$$

$\leq f \text{ OPT}$

MAX - SAT

FIND AN INTEGRALITY GAP FOR THE LP RELAXATION OF MAX-SAT

$$(X_1 \vee \bar{X}_2) \wedge (X_1 \vee X_2) \wedge (\bar{X}_1 \vee X_2) \wedge (\bar{X}_1 \vee \bar{X}_2)$$

- INTEGRAL SOLUTION:

  AT MOST THREE CLAUSES SATISFIED

  FRACTIONAL SOLUTION:

- $x_1 = x_2 = x_3 = x_4 = \frac{1}{2}$:

  4 CLAUSES FRACTIONALLY SATISFIED

- GIVES A $\frac{4}{3}$ INTEGRALITY GAP

MAX-CUT

TIGHT EXAMPLE FOR JOHNSON

$\alpha_k = 1 - 2^{-k}$

$c = (x_1 \vee x_2 \vee \bar{x}_3 \vee \dots \vee \bar{x}_k)$

CLAUSE SATISFIED WITH

pb $1 - 2^{-k}$

TIGHT EXAMPLE FOR GOEM. & WILLIAM.

SAME AS BEFORE

max $w_c z_c$

$x_1 + x_2 + (1 - x_3) + \dots + (1 - x_k) \geq z_c$

$0 \leq z_c \leq 1 \qquad 0 \leq x_i \leq 1$

$x_1^* = x_2^* = 1 - x_3^* = \dots = 1 - x_k^* = \frac{1}{k}$

CLAUSE SATISFIED WITH pb $1 - \left(1 - \frac{1}{k}\right)^k$

DE-RANDOMIZATION OF JOHNSON'S MAX-SAT

METHOD OF CONDITIONAL EXPECTATION

FORMULA W ON n VARIABLES

$x_1, x_2, \dots, x_n$

JOHNSON'S ALGORITHM:

$x_j = 1$ WITH PB $\frac{1}{2}$

$x_j = 0$ WITH PB $\frac{1}{2}$

TURN INTO A DETERMINISTIC ALGORITHM THAT FIX $x_1, \dots, x_n$

CONDITIONAL EXPECTATION :

$$E[W \mid X_1 = a_1, \ldots, X_i = a_i] \qquad a_j \in \{TRUE, FALSE\} \quad \{1, 0\}$$

EXPECTED TOTAL WEIGHT OF W

WITH $X_1 = a_1, \ldots, X_i = a_i$ AND

A RANDOM ASSIGNMENT FOR

$X_{i+1}, \ldots, X_n$

DECIDE HOW TO FIX X_{i+1}

$$\underbrace{E[W \mid X_1 = a_1, \ldots, X_i = a_i]}_{1} =$$

$$\frac{1}{2} \underbrace{E[W \mid X_1 = a_1, \ldots, X_i = a_i, X_{i+1} = TRUE]}_{2} +$$

$$\frac{1}{2} \underbrace{E[W \mid X_1 = a_1, \ldots, X_i = a_i, X_{i+1} = FALSE]}_{3}$$

IF 1 $\leq$ 2 THEN X_{i+1} = TRUE

IF 1 $\leq$ 3 THEN X_{i+1} = FALSE

THE PROCEDURE ENDS WITH A DETERMINISTIC ASSIGNMENT OF $X_1, \ldots, X_n$ WITH

VALUE $\geq E$ [JOHNSON'S ALGORITHM]

$= E[W]$

REQUIRES THE POLY TIME COMPUTATION OF $E[W \mid X_1 = a_1, \ldots, X_i = a_i]$

EASY!

MAX-SAT

CONSIDER A TRUTH ASSIGNMENT τ AND ITS COMPLEMENT τ'.

SHOW THAT THE BEST OF τ AND τ' IS A 2-APPROXIMATION.

EVERY CLAUSE IS SATISFIED EITHER IN τ OR IN τ', OR IN BOTH

$$SOL(\tau) + SOL(\tau') \geq \sum_i w_i$$

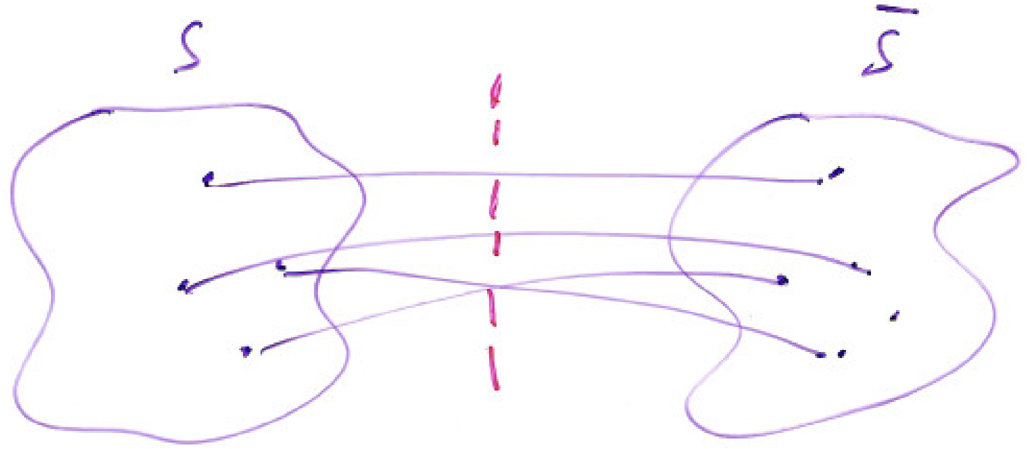
$$\geq OPT$$

MAX WEIGHTED CUT

$G = (V, E) \qquad \omega : E \to \mathbb{R}^+$

FIND A PARTITION $(S, \bar{S})$ OF V:

$$\sum_{e=(i,j) \in E \,:\, i \in S \,\wedge\, j \in \bar{S}} \omega(i\, j) \quad \text{IS MAXIMIZED}$$



2-APX ALGORITHM

ASSIGN i TO S WITH pb $\frac{1}{2}$

i TO $\bar{S}$ WITH pb $\frac{1}{2}$

$$PB[i \in S \wedge j \in \bar{S} \vee j \in S \wedge i \in \bar{S}] = \frac{1}{2}$$

$$E[ALG] = \frac{1}{2} \sum_{(i,j) \in E} w_{ij} \geq \frac{1}{2} OPT$$

USE VERY PESSIMISTIC BOUND ON OPT

QUADRATIC PROGRAMMING GIVES BETTER UPPER BOUND ON OPT

ALLOW 0.8785 APX [GOEMANS & WILLIAMSON, '94]

MAX-CUT GREEDY ALGORITHM

$w(v, A)$: total WEIGHT OF EDGES FROM v TO VERTICES OF A

1. $A = \{v_1\}$

   $B = \{v_2\}$

2. FOR $v \in V - \{v_1, v_2\}$ DO:

   IF $w(v, A) \geq w(v, B)$

   THEN $B = B \cup \{v\}$

   ELSE $A = A \cup \{v\}$

3. OUTPUT A AND B AS S AND $\bar{S}$

GREEDY IS A 2-APX FOR MAX-CUT

PROOF:

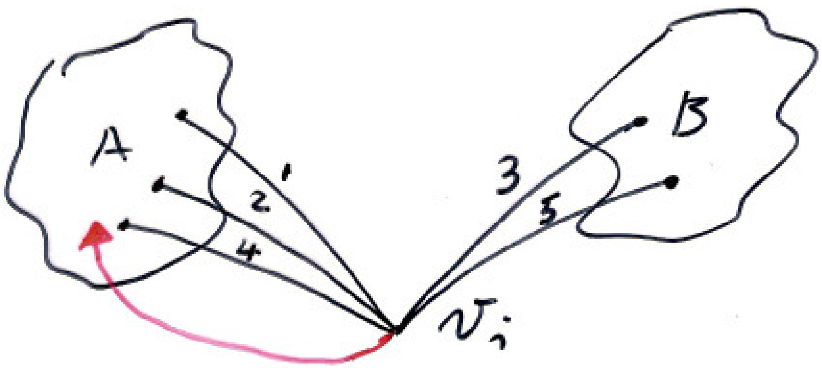
VERTICES ARE CONSIDERED IN ORDER $v_1, \dots, v_n$

$w(v_1, v_2)$ IS IN THE CUT

LET $w_<(v_i)$ BE THE TOTAL WEIGHT OF EDGES FROM v_i TO $v_1, \dots, v_{i-1}$

AT LEAST $\frac{w_<(v_i)}{2}$ WEIGHT IS SEPARATED BY THE CUT.

$$SOL \geq \frac{1}{2} \sum_{i=2}^{n} w_<(v_i) \geq \frac{OPT}{2}$$



DE-RANDOMIZATION OF MAX-CUT

START WITH $v_1 \in A$, $v_2 \in B$

$X_i = 1$ IF $v_i \in A$

$X_i = 0$ IF $v_i \in B$

W : INSTANCE OF MAX-CUT

$E[W]$: EXPECTED VALUE OF A RANDOM ASSIGNMENT

$$\underbrace{E[W \mid X_1 = a_1, \dots, X_i = a_i]}_{1} =$$

$$\frac{1}{2} \underbrace{E[W \mid X_1 = a_1, \dots, X_i = a_i, X_{i+1} = 1]}_{2} +$$

$$\frac{1}{2} \underbrace{E[W \mid X_1 = a_1, \dots, X_i = a_i, X_{i+1} = 0]}_{3}$$

2 $\geq$ 3 IF $w(v_{i+1}, B) \geq w(v_{i+1}, A)$

1 $\leq$ 2 OR 1 $\leq$ 3

DE-RAND MAX-CUT

COMPUTE $E[W \mid X_1 = Q_1, \dots, X_i = Q_i]$

IN POLYNOMIAL TIME:

- DECISIONS ON $X_1, \dots, X_i$ ARE FIXED

- $\forall j = i+1, \dots, n$

  ADD $w(v_j, A)$ WITH PB $\frac{1}{2}$

  $w(v_j, B)$ WITH PB $\frac{1}{2}$

- $\forall$ $j, k = i+1, \dots, n$, $j \neq k$

  IF $(v_j, v_k) \in E$, ADD $w(v_j, v_k)$

  WITH PB $\frac{1}{2}$

DIRECTED MAX CUT

$G=(V,E)$: DIRECTED GRAPH

FIND A PARTITION $(S,\bar{S})$ OF V:

$$\sum_{\substack{e=(i,j)\in E \\ i\in S \wedge j\in \bar{S}}} w(i,j) \quad \text{is maximized}$$

: ONLY EDGES DIRECTED FROM S TO $\bar{S}$

SHOW THAT THE RANDOMIZED ALGORITHM IS A 4-APX