3.3. STABILITY PROPERTIES OF IDENTIFICATION ALGORITHMS

We study now the stability properties of the gradient algorithm

$$\dot{\phi} = \dot{\theta} = -g e_I w \qquad (G)$$

and normalised gradient algorithm

$$\dot{\phi} = \dot{\theta} = -g \frac{e_I w}{1+\gamma w^T w} \qquad (NG)$$

with error equation

$$e_I = \phi^T w \qquad (E)$$

Theorem 3.3.1 Consider (G) with (E) and w be piecewise continuous.

a) $e_I \in \mathcal{L}_2$

b) $\phi \in \mathcal{L}_\infty$

Proof Choose a Lyapunov function

$$v = \phi^T \phi \Rightarrow \dot{v} = -2g e_I^2 \leq 0$$

Hence $0 \leq v(t) \leq v(0) \quad \forall t \geq 0 \Rightarrow$ $v, \phi \in \mathcal{L}_\infty$. Since $v(t)$ is positive, monotonically decreasing, $\lim_{t \to +\infty} v(t)$ is well defined and

$$\underbrace{-\frac{1}{2g} \int_0^\infty \dot{v}(t)dt}_{\frac{1}{2g}\left(v(0) - \lim_{t \to \infty} v(t)\right) < \infty} = \int_0^\infty e_I^2(t)dt$$

$\Rightarrow e_I \in \mathcal{L}_2$ ◀

Theorem 3.3.2. Consider (NG) with (E) and w be piecewise continuous.

a) $\dfrac{e_I}{\sqrt{1+\gamma w^T w}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$

b) $\phi \in \mathcal{L}_\infty$, $\dot{\phi} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$

c) $\beta = \dfrac{\phi^T w}{1+\|w_t\|_\infty} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$

Proof. With $v = \phi^T \phi$:

$$\dot{v} = -\frac{2g e_I^2}{1+\gamma w^T w} \leq 0$$

Hence $0 \leq v(t) \leq v(0) \quad \forall t \geq 0 \Rightarrow$

$v, \phi, \dfrac{e_I}{\sqrt{1+\gamma w^T w}}, \beta \in \mathcal{L}_\infty$ (since $e_I = \phi^T w$)

Moreover,

$$|\dot{\phi}| \leq \frac{g}{\gamma} |\phi| \Rightarrow \dot{\phi} \in \mathcal{L}_\infty$$

Since $v(t)$ is positive definite,

monotonically decreasing

$\lim_{t\to\infty} v(t)$ is well defined so that

$$-\int_0^\infty \dot{v}(t)\,dt = \int_0^\infty 2g \frac{e_I^2(t)}{1+\gamma w^T w(t)}\,dt$$

$\Rightarrow \dfrac{e_I}{\sqrt{1+\gamma w^T w}} \in \mathcal{L}_2$. Also,

$$\beta = \underbrace{\frac{e_I}{\sqrt{1+\gamma w^T w}}}_{\in \mathcal{L}_2} \cdot \underbrace{\frac{\sqrt{1+\gamma w^T w}}{1+\|w_t\|_\infty}}_{\in \mathcal{L}_\infty}$$

$\Rightarrow \beta \in \mathcal{L}_2$. Since

$$|\dot{\phi}|^2 \leq \frac{g^2}{\gamma} \frac{e_I^2}{1+\gamma w^T w} \Rightarrow \dot{\phi} \in \mathcal{L}_2 \quad \blacktriangleleft$$

3.3.1. Effect of initial conditions and projection

The equation error is in practice

$$e_I(t) = \phi^T(t)\,w(t) + \varepsilon(t) \quad (EE)$$

where $\varepsilon(t) \to 0$ exponentially as $t \to \infty$. Moreover, the gradient algorithm may be replaced with the algorithm with projection.

Theorem 3.3.3. If (E) is replaced with (EE) and/or the gradient algorithm is replaced by the algorithm with projection the conclusions of theorems 3.3.1 remain valid.

Proof.

I) Effect of initial conditions.

Modify the Lyapunov function as

$$v = \phi^T\phi + \frac{g}{2}\underbrace{\int_t^\infty \varepsilon^2(\tau)\,d\tau}_{\substack{\text{this term is bounded}\\ \text{and tends to 0 as}\\ t\to\infty}}$$

Hence

$$\dot{v} = -2g(\phi^T w)^2 - 2g(\phi^T w)\varepsilon - \frac{g}{2}\varepsilon^2$$
$$= -2g\left(\phi^T w + \frac{\varepsilon}{2}\right)^2$$

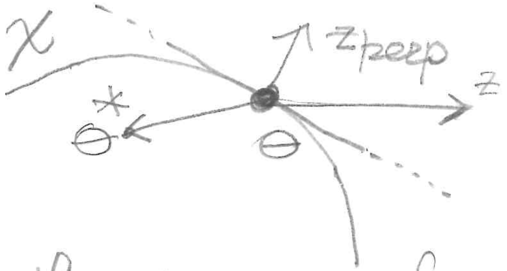
Therefore, as in the proof of theorem 3.3.1 and since $\varepsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, we get a) and b) of theorem 3.3.1 ◀

II) effect of projection

Denote $z \triangleq -g e_I w$. When $\theta \in \partial \mathcal{X}$ and z is directed outside $\mathcal{X}$; z is replaced by $\Pr[z]$ according to the algorithm with projection. Let z_{perp} the component of z perpendicular to the tangent plane of $\mathcal{X}$ at $\theta \Rightarrow z = \Pr[z] + z_{perp}$.

Since $\theta^* \in \mathcal{X}$ and $\mathcal{X}$ is convex,

$$(\theta - \theta^*)^T \cdot z_{perp} = \phi^T z_{perp} \geq 0 .$$



$$\underbrace{(\theta^* - \theta)^T z_{perp}}_{= \langle \theta^* - \theta, z_{perp} \rangle} \leq 0$$

Using the Lyapunov function $v = \phi^T \phi$ we find that with no projection

$$\dot{v} = 2 \phi^T z \leq 0$$

On the other hand, with projection

$$\dot{v}\Big|_{P_z} = 2\phi^T P_z\{z\} = \dot{v} - 2\phi^T z_{perp}$$

$$\leq \dot{v} \leq 0$$

3.3.2 Stability of the identifier

Convergence of the identifier error e_I can be guaranteed under the additional following condition.

A3) Assume that the plant is either stable or located in a control loop such that r and y_p are bounded ∎

Theorem 3.3.4 Consider the identifiers with gradient and normalized gradient.

1) $e_I \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $e_I \to 0$ as $t \to \infty$ and $\phi, \dot{\phi} \in \mathcal{L}_\infty$;

2) $\dot{\phi} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{\phi} \to 0$ as $t \to \infty$.

Proof Since r and y_p are bounded and

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_1 r$$
$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_1 y_p$$

with Hurwitz Λ, then w and $\dot{w}$ are bounded. By theorems 3.3.1–3.3.2 ϕ and $\dot{\phi}$ are bounded so that also e_I and $\dot{e}_I$ are bounded. By theorems 3.3.1 - 3.3.2 also $e_I \in \mathcal{L}_2$ and by Barbalat lemma $e_I \to 0$ as $t \to \infty$.

Similar conclusions for $\dot{\phi}$

We can relax (A3) into a <u>regularity</u> condition of w.

<u>Definition</u> (Regular signals)

Let $z : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ such that $z, \dot{z} \in \mathcal{L}_{\infty e}$. z is <u>regular</u> if for some $k_1, k_2 \geq 0$

$$|\dot{z}(t)| \leq k_1 \|z_t\|_\infty + k_2 \quad \forall\, t \geq 0$$

Signals with increasing <u>frequency</u> (as $z(t) = \sin e^t$) are not regular.

Regular signals may be unbounded.

Theorem 3.3.5

Let $\phi, w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ be such that $w, \dot{w} \in \mathcal{L}_{\infty e}$ and $\phi, \dot{\phi} \in \mathcal{L}_\infty$. If

a) w is regular

b) $$\beta = \frac{\phi^T w}{1 + \|w_t\|_\infty} \in \mathcal{L}_2$$

then $\beta, \dot{\beta} \in \mathcal{L}_\infty$ and $\beta \to 0$ as $t \to \infty$.

<u>Proof</u>. Clearly $\beta \in \mathcal{L}_\infty$ (since $\phi \in \mathcal{L}_\infty$) If we prove that $\dot{\beta} \in \mathcal{L}_\infty$ then, since $\beta \in \mathcal{L}_\infty$ and $\beta \in \mathcal{L}_2$ (by (b)), it follows $\beta \to 0$ as $t \to \infty$ by Barbalat Lemma.

We have

$$|\dot{\beta}| \le \left| \dot{\phi}^T \frac{w}{1 + \|w_t\|_\infty} \right| + \left| \phi^T \frac{\dot{w}}{1 + \|w_t\|_\infty} \right|$$

$$+ \left| \frac{\phi^T \bar{w}}{1+\|w_t\|_\infty} \; \frac{\frac{d}{dt}\|w_t\|_\infty}{1+\|w_t\|_\infty} \right|$$

but

$$\left| \frac{d}{dt}\|w_t\|_\infty \right| = \left| \frac{d}{dt} \sup_{\tau \le t} |w(\tau)| \right|$$

$$\leq \left| \frac{d}{dt} |w(t)| \right| \leq \left| \frac{d}{dt} w(t) \right| \quad (*)$$

$$\leq K_1 \|w_t\|_\infty + K_2$$

(by regularity of $\bar{w}$)

and hence $\dot{\beta} \in \mathcal{L}_\infty$

$(*)$ Indeed for any signal x:

$$\left| \frac{d}{dt} |x|^2 \right| = 2|x| \left| \frac{d}{dt}|x| \right| =$$

$$= 2 \left| x^T \frac{d}{dt} x \right| \leq 2|x| \left| \frac{dx}{dt} \right|$$

$$\Rightarrow \left| \frac{d}{dt}|x| \right| \leq \left| \frac{dx}{dt} \right|$$

<u>Theorem 3.3.6</u> Consider the identifier with normalised gradient (under assumptions A1, A2). The normalised error $\beta = \dfrac{\phi^T w}{1 + \|w_t\|_\infty}$ $\in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\beta \to 0$ as $t \to \infty$ and $\phi, \dot{\phi} \in \mathcal{L}_\infty$.

<u>Proof</u> It is sufficient to see that w is regular and use theorem 3.3.2 followed by theorem 3.3.5. Indeed,

$$\dot{w} = \begin{pmatrix} \Lambda & 0 \\ b_\lambda c^{*T} & \Lambda + b_\lambda c^{*T} \end{pmatrix} w + \begin{pmatrix} b_\lambda \\ 0 \end{pmatrix} r \quad (*)$$

Since r is bounded by (A2) $\Rightarrow$ w is regular

(*) recall $y_p = \theta^{*T} w \ (+\ \epsilon)$

3.3.3. Persistent excitation and parameter convergence.

We study now the parameter error convergence to zero (i.e. $\phi \to 0$).

As we know

$$\underbrace{\dot{\phi} = -g w w^T \phi}, \quad g > 0 \qquad (S)$$

$$\dot{\phi} = A(t)\phi$$

where $A(t) \in \mathbb{R}^{2n \times 2n}$ is a positive semidefinite matrix $\forall t$. With the Lyapunov function $v = \phi^T \phi$:

$$\dot{v} = -\phi^T (A + A^T)\phi$$

If $A(t)$ is uniformly positive definite with $\lambda_{min}(A + A^T) \geq 2\alpha > 0$ then

$$\dot{v} \leq -2\alpha v$$

and (S) is exponentially stable.

However, this is not the case since any $z \neq 0$: $w^T z = 0$ is also such that $A(t) z = 0 \Rightarrow$ $A(t)$ is singular. However, $A(t)$ is time-varying we may still hope that $\phi \to 0$ as $t \to \infty$. We introduce the following definition.

<u>Definition</u>. <u>Persistency of Excitation (PE)</u>

A vector $w: \mathbb{R}_+ \to \mathbb{R}^{2n}$ is Persistently exciting if there exist $\beta_1, \beta_2 > 0$ and $\delta > 0$ such that

$$\beta_2 I \geqslant \int_{t_0}^{t_0+\delta} w(\tau) w^T(\tau) d\tau \geq \beta_1 I$$

$$\forall t_0 \geq 0$$

Although $w(t) w^T(t)$ is not positive definite, $\int_t^{t+\delta} w(\tau) w^T(\tau)$ may be positive definite.

Theorem 3.3.7.

Let $w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ be piecewise continuous. If w is PE then

$\dot{\phi} = -g w w^T \phi$, $g > 0$, is exponentially stable. ∎

Proof Let $v = \phi^T \phi$ so that

$$\dot{v} = -2g (w^T \phi)^2 \leq 0.$$

$\forall\, t_0 \geq 0$

$$\int_{t_0}^{t_0+\delta} \dot{v}\, d\tau = -2g \int_{t_0}^{t_0+\delta} (w^T \phi)^2 d\tau$$

By the PE assumption the system

$$\begin{aligned} \dot{\phi} &= 0 \\ y &= w^T(t)\phi \end{aligned} \qquad (i)$$

is UCO. Indeed, if

$$W(t_0, t_0+\delta) \triangleq \int_{t_0}^{t_0+\delta} \Phi^T(\tau,t) C^T(\tau) C(\tau) \Phi(\tau,t)\, d\tau$$

where $\Phi(\tau, t)$ is the state transition matrix of $\dot{\phi} = 0$, which is $\Phi(\tau, t) = I$, and $C(\tau) = w^T(\tau)$, by PE assumption

$$\beta_2 I \geq W(t_0, t_0+\delta) \geq \beta_1 I \qquad \forall t_0 \geq 0$$

$\Rightarrow$ (i) is UCO. Under output injection with $K(t) = -g w(t)$, (i) becomes

$$\begin{aligned} \dot{\phi} &= -g w w^T \phi \\ y &= w^T \phi \end{aligned} \qquad \text{(ii)}$$

with

$$\int_{t_0}^{t_0+\delta} |g w(\tau)|^2 d\tau \quad \left(= \int_{t_0}^{t_0+\delta} |K(\tau)|^2 d\tau \right.$$

$$= g^2 \operatorname{Tr}\left(\int_{t_0}^{t_0+\delta} w(\tau) w^T(\tau)\, d\tau \right) \triangleq k_\delta$$

$\leq 2n g^2 \beta_2$ (the dimension of w is $2n$)

By the invariance of UCO property under output injection (pg. 65), (ii) is UCO.

Therefore $\forall\, t_0 \geq 0$

$$\int_{t_0}^{t_0+\delta} \dot{v}\, d\tau \leq \frac{-2g\beta_1 n}{(1+\sqrt{2n}\, g\beta_2)^2} |\phi(t_0)|^2 \qquad \text{(iii)}$$

Since the observability grammian W_k of (ii) is such that:

$$\phi^T(t_0) W_k(t_0, t_0+\delta)\phi(t_0) = \int_{t_0}^{t_0+\delta} \phi^T(\tau) w(\tau) w^T(\tau)\phi(\tau)\, d\tau$$

$$\geq \beta_1' |\phi(t_0)|^2$$

where $W_k(t_0, t_0+\delta) \geq \beta_1' I$

and $\beta_1' = \frac{\beta_1}{(1+\sqrt{k_\delta \beta_2}\,)^2}$ (see pg. 65)

But from (iii) and

$$\frac{2g\beta_1}{(1+\sqrt{2n}\; g\beta_2)^2} < 1 ,$$

by application of theorem 2.6.2 exponential convergence to 0 of $\dot{\phi} = -g w w^T \phi$ follows ◭

The following convergence result can be assessed.

<u>Theorem 3.3.8</u> Consider, under assumptions A1–A3, the gradient or normalized gradient identifiers. If w is PE then $\theta \to \theta^*$ exponentially as $t \to \infty$.

<u>Proof</u>. This is a direct consequence of theorem 3.3.7. When w is bounded

$$\bar{w} \text{ is PE} \iff \frac{\bar{w}}{\sqrt{1+\gamma w^T \bar{w}}} \text{ is PE}$$

and exponential convergence of θ to θ^* is guaranteed for gradient and normalized gradient algorithms