Control Systems

2/7/2019

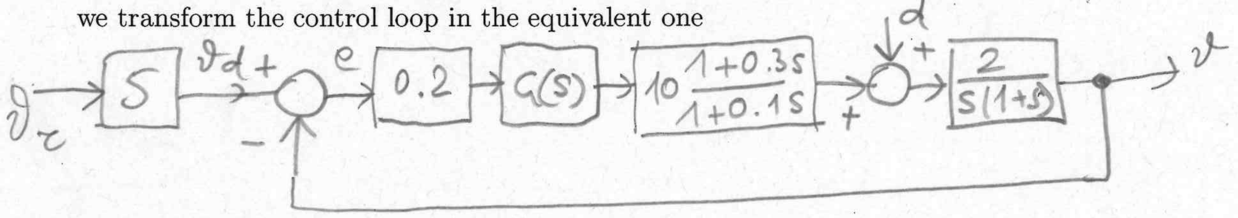
Exercise 1 For the control scheme in figure we have

$$P(s) = 20\frac{1+0.3s}{s(1+0.1s)(1+s)}$$

for which the corresponding gain is

$$K_P = P(0) = 20$$

Notice that the feedback is not unitary. In order to turn to a unitary feedback control loop, we transform the control loop in the equivalent one



(i) requires that the angular velocity $\dot{\theta}$ be constant which corresponds to a reference ramp input for the angular position. In other words

$$\dot{\theta}_d(t) = 2 \Rightarrow \theta_d(t) = 2t \Rightarrow \theta_r(t) = 2 * 0.2t = 0.4t$$

To have null steady state error to an input $\theta_r(t) = 0.4t$ it is necessary that the closed-loop system be of type 1. This is guaranteed by the presence of a pole at $s = 0$ in $P(s)$. As a consequence the controller will have the form

$$G(s) = K_{G,1}G_2(s)$$

in which $G_2(s)$ has unitary gain. To meet the requirement on the error, we must have that

$$|e_1| = \left|0.4 * 5 * \frac{W_e(s)}{s}\Big|_{s=0}\right| = 0.4 * 5\frac{1}{0.2|K_P||K_{G,1}|} \Rightarrow |K_{G,1}| \geq 12.5.$$

The disturbance-to-output transfer function is

$$W_d(s) = \frac{2(1+0.1s)}{s(1+s)(1+0.1s)+4G(s)(1+0.3s)}$$

The steady state response for a unitary constant disturbance is given by

$$W_d(0) = \frac{1}{2K_{G,1}}$$

Therefore for the limitation on this response

$$\frac{1}{|2K_{G,1}|} \leq 0.02 \Rightarrow |K_{G,1}| \geq 25$$

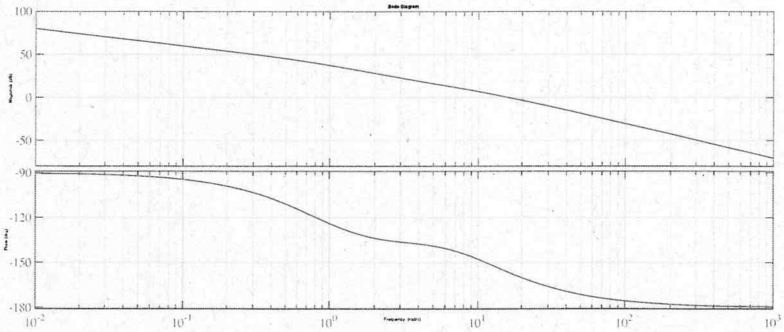


Figure 1: Bode plots of $\hat{F}(s)$

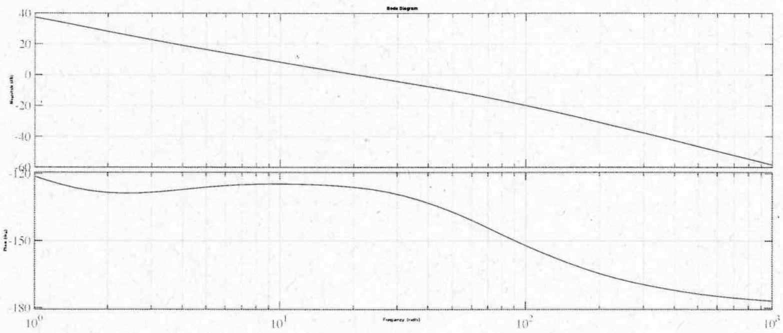


Figure 2: Bode plots of $F(s)$

Set $K_{G,1} = 25$.

As to the requirement (iii), if $G(s) := K_{G,1}G_2(s)$, the open loop transfer function is

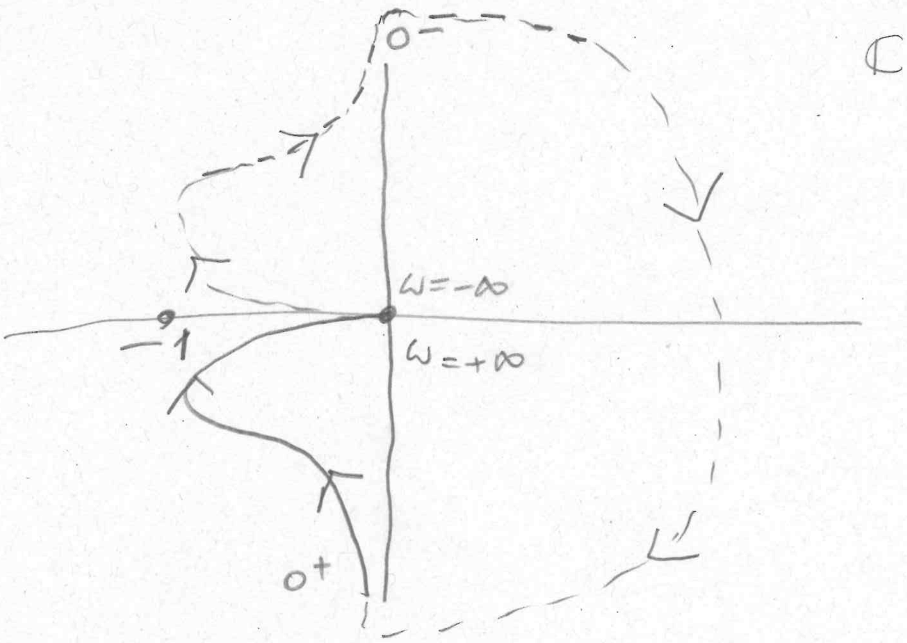
$$F(s) = G_2(s)\hat{F}(s), \; \hat{F}(s) := 0.2 * K_{G,1} * 10\frac{(1+0.3s)}{s(1+0.1s)(1+s)} = \frac{100(1+0.3s)}{s(1+0.1s)(1+s)}$$

The choice of $G_2(s)$ can be done on the inspection of the Bode plots of $\hat{F}(s)$ in Fig. 1. Since $\omega_t \approx 16$ rad/sec and $m_\phi \approx 23°$, we must increase the phase with $G_2(s)$. Since there is no requirement on the crossover frequency, we can do this either with an attenuative action $G_2(s)$ or an anticipative action $G_2(s)$. Due to the low rate of phase bode plot we should use an attentive action with large magnitude which require multiple functions to be used. For this reason we use an anticipative action. Inspection of the compensating function diagram shows that for $m_a = 4$ it is possible to obtain a phase increase of approximately 31° at the normalized frequency $\omega_N = \omega\tau_a = 1$ rad/sec. Consequently, adding this phase increase at the crossover frequency ω_t we obtain a phase value $-157° + 31° = -126°$ which guarantees the required phase margin m_ϕ^*. We set $16\tau_a = 1$ obtaining $\tau_a = \frac{1}{16}$. The controller is finally

$$G_2(s) = 25\frac{1 + \frac{1}{16}s}{1 + \frac{1}{64}s}$$

The Bode plots of $F(s) = G_2(s)\hat{F}(s)$ (Fig. 2) show that we obtained a crossover frequency of approximately 20 rad/sec and a phase margin of approximately 54°.

The stability of the closed-loop system is met as shown by the Nyquist plot in Fig. 3.



Exercise 2 It is convenient to design $G_1(s)$ in such a way to stabilize the internal loop and then $G_2(s)$ to stabilize the external loop and guarantee the steady state performances for the error.

First, design $G_1(s)$. Consider the root locus of $P_1(s) = \frac{1}{s(s-2)}$ (Figs. 4 and 5). This shows that we must use a zero-pole action for moving the asymptote center into the negative complex half-plane. In this way the dimension of $G_1(s)$ is one. For example, we can choose

$$G_1(s) := K_{G,1}\frac{s+3}{s+7}$$

The asymptote center becomes

$$s_0 = \frac{2-7+3}{2} = -1$$

and $K_{G,1}$ will be chosen in such a way that the internal loop is stable. The denominator of the internal loop is

$$NUM(1 + P_1(s)G_1(s)) = s^3 + 5s^2 + (K_{G,1} - 14)s + 3K_{G,1}$$

Computing the corresponding Routh table one sees that it is sufficient to choose any $K_{G,1} \geq$

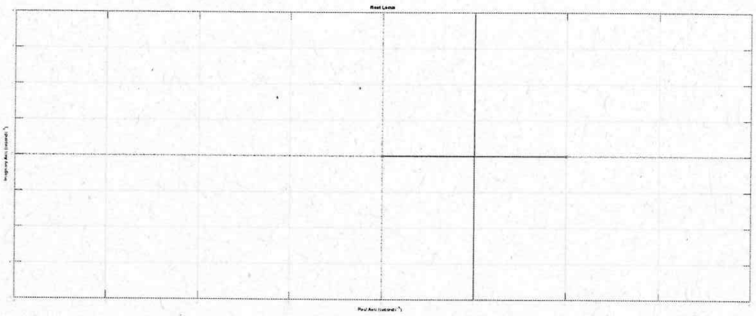


Figure 3: Positive root locus of $P_1(s)$

35. For example, select $K_{G,1} = 72$ for which the poles of the internal loop $F_1(s) = \frac{P_1(s)G_1(s)}{1+P_1(s)G_1(s)}$ are -4 and $-0.5 \pm j7.33$. This value of $K_{G,1}$ can be also obtained by imposing that

$$NUM(1 + P_1(s)G_1(s)) = s^3 + 5s^2 + (K_{G,1} - 14)s + 3K_{G,1} = (s+4)(s+0.5-j7.33)(s+0.5+j7.33)$$

This is helpful in view of the fact that the knowledge of the poles of the internal loop is needed for the subsequent computations. Therefore,

$$F_1(s) = \frac{P_1(s)G_1(s)}{1 + P_1(s)G_1(s)} = 72\frac{s+3}{(s+4)(s+0.5-j7.33)(s+0.5+j7.33)}$$

Next, the choice of $G_2(s)$ has to be done in such a way to include a pole at $s = 0$ (for the requirement on the steady state error). Notice that the pole at $s = 0$ in the internal loop is not effective since this loop moves this poles away from the origin. Therefore, since $G_2(s)$ must be one dimensional, our choice of $G_2(s)$ will be

$$G_2(s) = \frac{K_{G,2}}{s}$$

The choice of $K_{G,2}$ is obtained from the Routh table corresponding to the external loop

$$NUM(1 + G_2(s)F_1(s)\frac{s-2}{s+3}) = s^4 + 5s^3 + 58s^2 + (216 + KG, 2)s - 2K_{G,2}$$

We have $-180.54 < /72 < K_{G,2} < 0$ for stability. For example, $G_2(s) = -\frac{1}{s}$.

Exercise 3. First, let us study the observability and controllability of the open loop $\dot{x} = Ax + Bu$, $y = Cx$.

The controllability matrix

$$R = (B \quad AB) = \begin{pmatrix} 1 & -1 \\ \beta & 1-2\beta \end{pmatrix}$$

and $detR = 1 - \beta$. Therefore, the system is controllable for $\beta \neq 1$.

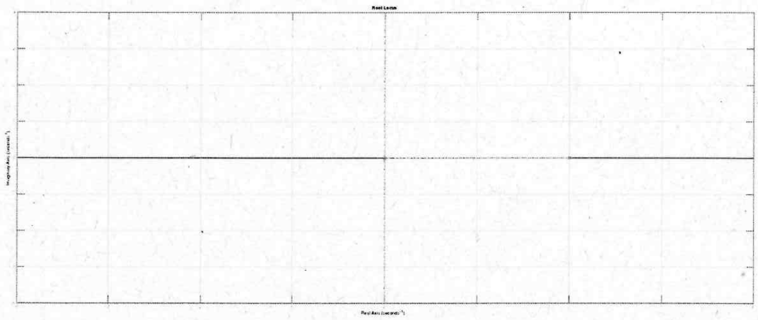


Figure 4: Negative root locus of $P_1(s)$

The observability matrix

$$O = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -1+\alpha & -2\alpha \end{pmatrix}$$

and $det = -\alpha(1+\alpha)$. Therefore, the system is controllable for $\beta \neq 1$.

We must discuss the values of α and β for which the eigenvalues of the controlled process can be moved (by state feedback) with real part ≤ -2 and the eigenvalues of the observer can be moved with real part ≤ -2 (i.e. state observation goes at least as e^{-2t}). Therefore, we must discuss the values of α and β for which the invariant spectrum $\mathcal{F}_R$ of $A+BF$ has real part ≤ -2 and the invariant spectrum $\mathcal{F}_O$ of $A-KC$ has real part ≤ -2. The cases for which we have no invariant spectrum is trivial, because we can move the eigenvalues wherever required. For this we use the Hautus tests. The iegnevalues of A are $\{-1,-2\}$.

1) Controllability. Case $\beta = 1$. Hautus test gives: for eigenvalue $\lambda = -1$

$$rank\,(A-\lambda I \quad B) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} = 2 \Rightarrow \lambda = -1 \notin \mathcal{F}_R$$

for eigenvalue $\lambda = -2$

$$rank\,(A-\lambda I \quad B) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 1 \Rightarrow \lambda = -2 \in \mathcal{F}_R$$

We conclude that for $\beta = 1$ the invariant spectrum of $A+BF$ satisfies the requirement that the real parts of the eigenvalues of $A+BF$ be ≤ -2.

1) Observability. Case $\alpha = 0$. Hautus test gives: for eigenvalue $\lambda = -1$

$$rank \begin{pmatrix} C \\ A-\lambda I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} = 2 \Rightarrow \lambda = -1 \notin \mathcal{F}_O$$

for eigenvalue $\lambda = -2$

$$rank \begin{pmatrix} C \\ A - \lambda I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = 1 \Rightarrow \lambda = -2 \in \mathcal{F}_O$$

We conclude that for $\alpha = 0$ the invariant spectrum of $A - KC$ satisfies the requirement that the real parts of the eigenvalues of $A - KC$ be ≤ -2.

1) Observability. Case $\alpha = -1$. Hautus test gives: for eigenvalue $\lambda = -1$

$$rank \begin{pmatrix} C \\ A - \lambda I \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} = 1 \Rightarrow \lambda = -1 \in \mathcal{F}_O$$

for eigenvalue $\lambda = -2$

$$rank \begin{pmatrix} C \\ A - \lambda I \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = 2 \Rightarrow \lambda = -2 \notin \mathcal{F}_O$$

We conclude that for $\alpha = -1$ the invariant spectrum of $A - KC$ does not satisfy the requirement that the real parts of the eigenvalues of $A - KC$ be ≤ -2.