Notes on Linear Control Systems: Module II

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***Abstract*—State and output solutions of a linear differential model: the matrix exponential. Natural modes and modal decomposition. Structural properties of natural modes: observability from the outputs and excitability with impulsive inputs.**

I. Solutions of linear models

From now on we will consider the class of differential models

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t)\end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ the input vector and $\mathbf{y}(t) \in \mathbb{R}^p$ the output vector. In this chapter we will characterize the solution $\mathbf{x}(t) := \mathbf{x}(t, x_0, \mathbf{u})$ of (1) with initial value x_0 and piecewise continuous input function $\mathbf{u} : \mathbb{R}_\geqslant \to \mathbb{R}^m$. Using the matrix exponential (see appendix D), we want to prove the following result.

Theorem *1.1: The solution of $\mathbf{x}(t, x_0, \mathbf{u})$ of (1) is continuous and unique over $\mathbb{R}_\geqslant$ and*

$$\mathbf{x}(t, x_0, \mathbf{u}) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \tag{2}$$

Proof. By well-known facts from mathematical analysis the solution of $\mathbf{x}(t, x_0, \mathbf{u})$ of (1) is a function of time, defined over $\mathbb{R}_\geqslant$ and uniquely determined from the initial value x_0 and the input function $\mathbf{u}$. For proving the claim, it is sufficient to prove that (2) satisfies (1) for $t \geqslant 0$. Therefore, on account of (140) and by differentiating the left and right-hand parts of (2)

$$\begin{aligned}&\frac{d}{dt}\mathbf{x}(t, x_0, \mathbf{u}) = Ae^{At}x_0 + B\mathbf{u}(t) + \int_0^t Ae^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\\ &= Ae^{At}x_0 + B\mathbf{u}(t) + A\int_0^\top e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\\ &= A[e^{At}x_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau] + B\mathbf{u}(t)\\ &= A\mathbf{x}(t, x_0, \mathbf{u}) + B\mathbf{u}(t)\end{aligned} \tag{3}$$

which is the first equation of (1). This proves that the solution $\mathbf{x}(t, x_0, \mathbf{u})$ of (1) is (2). $\lhd$

Similarly, we can prove the following result on the output $\mathbf{y}(t) := \mathbf{y}(t, x_0, \mathbf{u})$ of (1). Let $\boldsymbol{\delta}^{(0)}(t)$ denote the Dirac impulse at $t = 0$.

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Theorem *1.2: The output function $\mathbf{y}(t, x_0, \mathbf{u})$ of (1) is continuous and unique over $\mathbb{R}_\geqslant$ and*

$$\mathbf{y}(t, x_0, \mathbf{u}) = Ce^{At}x_0 + \int_0^t (Ce^{A(t-\tau)}B + \boldsymbol{\delta}^{(0)}(t-\tau)D)\mathbf{u}(\tau)d\tau \tag{4}$$

Proof. On account of (2) and from the second equation of (1)

$$\begin{aligned}&\mathbf{y}(t, x_0, \mathbf{u}) = C\mathbf{x}(t, x_0, \mathbf{u}) + D\mathbf{u}(t)\\ &= C(e^{At}x_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau) + Du(t)\\ &= Ce^{At}x_0 + \int_0^t (Ce^{A(t-\tau)}B + \boldsymbol{\delta}^{(0)}(t-\tau)D)\mathbf{u}(\tau)d\tau. \lhd\end{aligned}$$

Note that both (2) and (4) consist of two summands: the first one is a linear function of x_0, the initial value of the state, and the second one is a function of the input $\mathbf{u}$. We will denote

$$\mathbf{x}^{(0)}(t, x_0) := e^{At}x_0 \tag{5}$$

the *unforced (or free) state response*, stressing the fact that it is the solution (2) under null input. On the other hand, we will denote

$$\mathbf{x}^{(u)}(t, \mathbf{u}) := \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \tag{6}$$

the *forced state response*, stressing the fact that it is the state response under null initial state. Therefore, for each $t \geqslant 0$

$$\mathbf{x}(t, x_0, \mathbf{u}) = \mathbf{x}^{(0)}(t, x_0) + \mathbf{x}^{(u)}(t, \mathbf{u}) \tag{7}$$

The solution of $\mathbf{x}(x_0, \mathbf{u})$ of (1) can be obtained as the *sum* of the solution of (1) with initial state x_0 and null input ($\mathbf{x}^{(0)}(t, x_0)$) and the solution of (1) with null initial state x_0 and input $\mathbf{u}$ ($\mathbf{x}^{(u)}(t, \mathbf{u})$). Also, we denote

$$\mathbf{y}^{(0)}(t, x_0) := Ce^{At}x_0 \tag{8}$$

the *unforced (or free) output response* and

$$\mathbf{y}^{(u)}(t, \mathbf{u}) := \int_0^t (Ce^{A(t-\tau)}B + \boldsymbol{\delta}^{(0)}(t-\tau)D)\mathbf{u}(\tau)d\tau \tag{9}$$

the *forced output response*. For each $t \geqslant 0$

$$\mathbf{y}(t, x_0, \mathbf{u}) = \mathbf{y}^{(0)}(t, x_0) + \mathbf{y}^{(u)}(t, \mathbf{u}) \tag{10}$$

The $(n \times n)$ matrix

$$\boldsymbol{\Phi}(t) := e^{At} \tag{11}$$

is known as the *state transition* matrix while the $(n \times m)$ matrix

$$\mathbf{H}(t) := e^{At}B = \boldsymbol{\Phi}(t)B \tag{12}$$

is known as the *state impulsive response* matrix, since the i-th column $[\mathbf{H}(t)]_i$ of $\mathbf{H}(t)$ is obtained as state response of (1) by applying the input

$$\mathbf{u}(t) = \boldsymbol{\delta}^{(0)}(t)\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} \leftarrow \text{(i-th component)} \qquad (13)$$

Indeed, by direct calculations

$$\begin{aligned}\mathbf{x}^{(u)}(t,\mathbf{u}) &= \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\\ &= \int_0^t e^{A(t-\tau)}[B]_i\boldsymbol{\delta}^{(0)}(\tau)d\tau\\ &= \int_0^t [e^{A(t-\tau)}B]_i\boldsymbol{\delta}^{(0)}(\tau)d\tau = [e^{At}B]_i = [\mathbf{H}(t)]_i.\end{aligned}$$

On the other hand, the $(p \times p)$ matrix

$$\mathbf{W}(t) := Ce^{At}B + \boldsymbol{\delta}^{(0)}(t)D \qquad (14)$$

is known as *output impulsive response* matrix, since its i-th column $[\mathbf{W}(t)]_i$ is obtained as output response of (1) by applying the input (13). Indeed, by direct calculations

$$\begin{aligned}\mathbf{y}^{(u)}(t,\mathbf{u}) &= \int_0^t (Ce^{A(t-\tau)}B + \boldsymbol{\delta}^{(0)}(t-\tau)D)\mathbf{u}(\tau)d\tau\\ &= \int_0^t [Ce^{A(t-\tau)}B + \boldsymbol{\delta}^{(0)}(t-\tau)D]_i\boldsymbol{\delta}^{(0)}(\tau)d\tau\\ &= [Ce^{At}B + \boldsymbol{\delta}^{(0)}(t)D]_i = [\mathbf{W}(t)]_i.\end{aligned}$$

II. State and output response: time domain analysis

In this chapter we study the properties of the state and output responses of a linear model. In particular, we will see that the state and output responses are superposition of *modes* (modal decomposition). These modes are characterized by some specific time functions which uniquely characterize the behaviour of the state and output responses versus time.

III. Modal decomposition of the unforced state response

We will study the modal decomposition for the unforced state response in two simple cases. First, we assume that the eigenvalues of the matrix A are all real and distinct (aperiodic modes). Secondly, we assume that the eigenvalues of the matrix A are all complex conjugate and pairwise distinct (pseudoperiodic modes). Finally, we will show through an example the structure of the modes when at least one eigenvalue of the matrix A has multiplicity greater than one.

A. Case of distinct eigenvalues

1) Aperiodic modes: In this section we assume that the eigenvalues of the matrix A are all real and distinct (i.e. with algebraic multiplicity 1) and denote these eigenvalues by $\lambda_1, \ldots, \lambda_n$, i.e. the roots of the characteristic polynomial $\det(A-\lambda_i I)$ of A are $\lambda_1, \ldots, \lambda_n$. Therefore, the characteristic polynomial of A factors out as

$$\det(A - \lambda_i I) = \prod_{i=1}^{n}(\lambda - \lambda_i) \qquad (15)$$

Let z_i be an eigenvector associated to the eigenvalue λ_i, i.e. a non-zero vector such that

$$(A - \lambda_i I)z_i = 0 \qquad (16)$$

Also

$$(A - \lambda_i I)^j z_i = 0,\ j \geqslant 1. \qquad (17)$$

Since $\lambda_1, \ldots, \lambda_n$ are real and distinct, the eigenvectors $z_1, \ldots, z_n$ associated to $\lambda_1, \ldots, \lambda_n$ are real and linearly independent. It follows the existence of unique reals $c_1, \ldots, c_n$ such that

$$x_0 = \sum_{i=1}^{n} c_i z_i \qquad (18)$$

Since the matrices $\lambda_i I$ and $A - \lambda_i I$ commute, on account of (141) and (142) we have

$$e^{(A-\lambda_i I)t+\lambda_i It} \equiv e^{\lambda_i t}e^{(A-\lambda_i I)t} \qquad (19)$$

On account of (17), the unforced state response is given by

$$\begin{aligned}\mathbf{x}^{(0)}(t,x_0) &= e^{At}x_0 = \sum_{i=1}^{n} e^{(A-\lambda_i I)t+\lambda_i It}c_i z_i\\ &= \sum_{i=1}^{n} e^{(A-\lambda_i I)t}e^{\lambda_i It}c_i z_i = \sum_{i=1}^{n} c_i e^{\lambda_i t}e^{(A-\lambda_i I)t}z_i\\ &= \sum_{i=1}^{n} c_i e^{\lambda_i t}(\sum_{j=0}^{\infty}\frac{t^j}{j!}(A-\lambda_i I)^j)z_i = \sum_{i=1}^{n} c_i e^{\lambda_i t}z_i\end{aligned}$$

We sum up our result as follows.

Theorem *3.1: Assume that A has all real eigenvalues with algebraic multiplicity 1. The unforced state response can be decomposed as follows: for each $t \geqslant 0$*

$$\mathbf{x}^{(0)}(t,x_0) = \sum_{i=1}^{n} c_i e^{\lambda_i t}z_i \qquad (20)$$

The i-th term of the sum on the left of (20) is called *aperiodic mode* and (20) is the *modal decomposition* of the unforced state response. The modal decomposition (20) can be interpreted from a geometric point of view as follows. The vector

$$\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$$

is the coordinate vector of x_0 in the axis frame $\{z_1, \ldots, z_n\}$ (see (18)) and

$$\begin{pmatrix}c_1 e^{\lambda_1 t}\\ \vdots\\ c_n e^{\lambda_n t}\end{pmatrix}$$

is at each time $t \geqslant 0$ the coordinate vector of $\mathbf{x}^{(0)}(t, x_0)(x_0)$ in the axis frame $\{z_1, \ldots, z_n\}$. Therefore, the i-th aperiodic mode is the (time-dependent) coordinate of $\mathbf{x}^{(0)}(t, x_0)(x_0)$ in the direction of the eigenvector z_i associated to the real eigenvalue λ_i of A.

Notice that if $\{z_1', \ldots, z_n'\}$ are a different set of eigenvectors associated to $\lambda_1, \ldots, \lambda_n$ and

$$\begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}$$

is the coordinate vector of of x_0 in the axis frame $\{z_1', \ldots, z_n'\}$, i.e.

$$x_0 = \sum_{i=1}^{n} c_i' z_i' \tag{21}$$

then

$$\mathbf{x}^{(0)}(t, x_0) = \sum_{i=1}^{n} c_i e^{\lambda_i t} z_i = \sum_{i=1}^{n} c_i' e^{\lambda_i t} z_i' \tag{22}$$

In other words, the modes do not depend on how we choose the set of eigenvectors, i.e. coordinate-free. Indeed, any two distinct eigenvectors z_i, z_i' associated to the same λ_i are necessarily parallel, i.e. $z_i' = \alpha_i z_i$ for some real α_i, therefore $c_i' = \frac{c_i}{\alpha_i}$ and

$$\sum_{i=1}^{n} c_i' e^{\lambda_i t} z_i' = \sum_{i=1}^{n} c_i' \alpha_i e^{\lambda_i t} z_i = \sum_{i=1}^{n} c_i e^{\lambda_i t} z_i \tag{23}$$

As it is clear from (20), the unforced state response is a superposition of modes. We want to answer the following question: is it possible to reconstruct each mode separately, given the unforced state response? In principle, this is possible by choosing the initial state x_0 appropriately. If we want to reconstruct the i-th aperiodic mode, choose x_0 in such a way that for some nonzero real c_i

$$x_0 = c_i z_i \tag{24}$$

i.e. x_0 is along the direction of the i-th eigenvector and with coordinate c_i. With this choice of x_0 the sum in (20) reduces to one term

$$\mathbf{x}^{(0)}(t, x_0) = c_i e^{\lambda_i t} z_i \tag{25}$$

i.e. its i-th mode. This kind of operation is known as *mode isolation* and we say that *the i-th aperiodic mode has been isolated from the unforced state response.*

An aperiodic mode behaves in time according to the value of the corresponding eigenvalue. A classification of the modes comes natural according to the value of the corresponding eigenvalue.

Definition *3.1: An aperiodic mode is said to be convergent, divergent or constant according if its associated eigenvalue satisfies $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$, respectively.*

Therefore, a convergent mode tends to zero as $t \to +\infty$, a divergent mode tends to infinity as $t \to +\infty$ and a constant modes is constant for all times.

We can associate to a convergent/divergent aperiodic mode a *time constant* τ defined as

$$\tau := -\frac{1}{\lambda} \tag{26}$$

Note that $\tau > 0$ if and only if $\lambda < 0$. The inverse formula are given

$$\lambda = -\frac{1}{\tau} \tag{27}$$

In other words, for convergent modes smaller is τ more slowly the mode converges to zero as $t \to +\infty$. The time constant is the time for which the mode amplitude reduces by a factor $e \approx 2.7$. Moreover, in terms of time constants the characteristic polynomial (15) can be recast as

$$\begin{aligned} \det(A - \lambda I) &= \prod_{i=1}^{n} (\lambda - \lambda_i) \\ &= \prod_{i=1}^{n} \lambda_i \prod_{i=1}^{n} (1 + \tau_i \lambda) = K \prod_{i=1}^{n} (1 + \tau_i \lambda) \end{aligned} \tag{28}$$

where λ_i is an eigenvalue, $\tau_i := -\frac{1}{\lambda_i}$ and $K := \prod_{i=1}^{n} \lambda_i$.

We explain the theoretical setting with an example.

Exercize *3.1: Given*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 2 & 2 & 0 \end{pmatrix} \tag{29}$$

calculate the unforced state response ensuing from

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

at $t = 0$.

Let us calculate the eigenvalues of A. We have

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - 1 & 0 & -1 \\ 2 & \lambda + 1 & 1 \\ -2 & -2 & \lambda \end{pmatrix} \\ &= \lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1) \end{aligned}$$

Therefore, the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 0$ and we have one divergent aperiodic mode, one convergent aperiodic mode and a costant mode.

Calculate an eigenvector z_1 associated to λ_1, i.e. a nonzero z_1 such that

$$(A - \lambda_1 I) z_1 = \begin{pmatrix} 0 & 0 & 1 \\ -2 & -2 & -1 \\ 2 & 2 & -1 \end{pmatrix} z_1 = 0 \tag{30}$$

Note that the rank of

$$(A - \lambda_1 I) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & -2 & -1 \\ 2 & 2 & -1 \end{pmatrix} \tag{31}$$

is 2. Therefore, it is sufficient to solve

$$\begin{pmatrix} 0 & 0 & 1 \\ -2 & -2 & -1 \end{pmatrix} z_1 = 0 \tag{32}$$

for z_1, i.e. we discard the third row of $(A-\lambda_1)$ which linearly depends from its first two rows. Choose

$$z_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{33}$$

Calculate an eigenvector z_2 associated to λ_2, i.e. a nonzero z_2 such that

$$(A-\lambda_2 I)z_2 = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} z_2 = 0 \tag{34}$$

Choose

$$z_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{35}$$

Finally, calculate an eigenvector z_3 associated to λ_3, i.e. a nonzero z_3 such that

$$(A-\lambda_3 I)z_3 = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 2 & 2 & 0 \end{pmatrix} z_3 = 0 \tag{36}$$

Choose

$$z_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \tag{37}$$

Next, find the coordinates of x_0 in the axis framework $\{z_1, z_2, z_3\}$, i.e. find the unique reals c_1, c_2, c_3 such that

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sum_{j=1}^{3} c_i z_i \tag{38}$$

This can be easily done as follows. Note that (38) can be written as

$$x_0 = Zc \tag{39}$$

where

$$Z := \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}, \; c := \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

But Z is nonsingular by construction since $\{z_1, z_2, z_3\}$ is a basis of $\mathbb{R}^3$. Therefore

$$c = Z^{-1} x_0 \tag{40}$$

Therefore,

$$c = Z^{-1} x_0 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{41}$$

The unforced state response ensuing from x_0 at $t=0$ is given by

$$\begin{aligned} \mathbf{x}^{(0)}(t, x_0) &= e^{At} x_0 = \sum_{j=1}^{3} e^{\lambda_i t} c_i z_i \\ &= 2e^t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^t + e^{-t} - 2 \\ -2e^t + 2 \\ -2e^{-t} + 2 \end{pmatrix} . \lhd \end{aligned} \tag{42}$$

2) Pseudoperiodic modes: In this section we assume that the eigenvalues of the matrix A are all complex conjugate and distinct (i.e. with algebraic multiplicity 1) and denote these eigenvalues by $\mu_1, \mu_1^* \ldots, \mu_{n/2}, \mu_{n/2}^*$. The characteristic polynomial of A factors out as

$$\det(A - \lambda I) = \prod_{i=1}^{n/2} (\lambda - \mu_i)(\lambda - \mu_i^*) \tag{43}$$

Let q_i be an eigenvector associated to the eigenvalue μ_i, i.e.

$$(A - \mu_i I) q_i = 0 \tag{44}$$

Denote by α_i and ω_i the real and, respectively, imaginary part of μ_i, i.e.

$$\mu_i := \alpha_i + j\omega_i$$

and by v_i and w_i the real and, respectively, imaginary part of q_i, i.e.

$$q_i := v_i + j w_i$$

By considering the real and imaginary part of $(A - \mu_i I) q_i$ separately, we obtain from (44)

$$\begin{aligned} (A - \alpha_i I) v_i + \omega_i w_i &= 0 \\ (A - \alpha_i I) w_i - \omega_i v_i &= 0 \end{aligned} \tag{45}$$

By induction we can prove that for all $j = 0, 1, \ldots$

$$\begin{aligned} (A - \alpha_i I)^{2j+1} v_i &= -(-1)^j \omega_i^{2j+1} w_i \\ (A - \alpha_i I)^{2j} v_i &= (-1)^j \omega_i^{2j} v_i \\ (A - \alpha_i I)^{2j+1} w_i &= (-1)^j \omega_i^{2j+1} w_i \\ (A - \alpha_i I)^{2j} w_i &= (-1)^j \omega_i^{2j} v_i \end{aligned} \tag{46}$$

Since $\mu_1, \ldots, \mu_n$ are complex conjugate and distinct, $\{v_1, w_1, \ldots, v_{n/2}, w_{n/2}\}$ is a set of real independent vectors and a basis for $\mathbb{R}^n$. Therefore, there exist unique reals $g_1, h_1, \ldots, g_{n/2}, h_{n/2}$ such that

$$x_0 = \sum_{i=1}^{n/2} (g_i v_i + h_i w_i) \tag{47}$$

On account of (46) and since $A - \alpha_i I$ and $\alpha_i I$ commute, for each $t \geqslant 0$ we have

$$\begin{aligned} \mathbf{x}^{(0)}(t, x_0) &= e^{At} x_0 \\ &= \sum_{i=1}^{n/2} e^{(A - \alpha_i I)t + \alpha_i I t} (g_i v_i + h_i w_i) \\ &= \sum_{i=1}^{n/2} e^{(A - \alpha_i I)t} e^{\alpha_i I t} (g_i v_i + h_i w_i) \\ &= \sum_{i=1}^{n/2} e^{\alpha_i t} e^{(A - \alpha_i I)t} (g_i v_i + h_i w_i) \\ &= \sum_{i=1}^{n/2} e^{\alpha_i t} \Big(\sum_{j=0}^{\infty} \frac{t^j}{j!} (A - \alpha_i I)^j\Big) (g_i v_i + h_i w_i) \end{aligned} \tag{48}$$

and after some computations

$$\sum_{i=1}^{n/2} e^{\alpha_i t}(\sum_{j=0}^{\infty}\frac{t^j}{j!}(A-\alpha_i I)^j)(g_i v_i + h_i w_i)$$
$$= \sum_{i=1}^{n/2} e^{\alpha_i t}(v_i(g_i \cos(\omega_i t) + h_i \sin(\omega_i t))$$
$$+ w_i(h_i \cos(\omega_i t) - g_i \sin(\omega_i t)))$$

In order to write the last expression in a more picturesque form, let us define m_i as

$$m_i := \sqrt{g_i^2 + h_i^2} \tag{49}$$

and ϕ_i such that

$$\sin\phi_i := \frac{g_i}{m_i},\ \cos\phi_i := \frac{h_i}{m_i} \tag{50}$$

Using the fact that for any pair of angles (ψ, ϕ)

$$\sin(\psi+\phi) := \sin(\psi)\cos(\phi) + \sin(\phi)\cos(\psi),$$
$$\cos(\psi+\phi) = \cos(\psi)\cos(\phi) - \sin(\psi)\sin(\phi) \tag{51}$$

we obtain the final result.

Theorem *3.2: Assume that A has all complex conjugate eigenvalues with algebraic multiplicity 1. The unforced state response can be decomposed as follows: for each $t \geqslant 0$*

$$\mathbf{x}^{(0)}(t, x_0) = \sum_{i=1}^{n/2} m_i e^{\alpha_i t}(v_i \sin(\omega_i t + \phi_i)$$
$$+ w_i \cos(\omega_i t + \phi_i)) \tag{52}$$

The i-th term of the sum in the last expression of (65) is called *pseudoperiodic mode* and (65) is the *modal decomposition* of the unforced state response. The modal decomposition (65) can be interpreted from a geometric point of view as follows: if

$$\begin{pmatrix} g_1 \\ h_1 \\ \vdots \\ g_{n/2} \\ h_{n/2} \end{pmatrix}$$

are the coordinates of x_0 in the axis frame $\{v_1, w_1, \ldots, v_{n/2}, w_{n/2}\}$, then

$$\begin{pmatrix} m_1 e^{\alpha_1 t}\sin(\omega_1 t + \phi_1) \\ m_1 e^{\alpha_1 t}\cos(\omega_1 t + \phi_1) \\ \vdots \\ m_{\frac{n}{2}} e^{\alpha_{\frac{n}{2}} t}\sin(\omega_{\frac{n}{2}} t + \phi_{\frac{n}{2}}) \\ m_{\frac{n}{2}} e^{\alpha_{\frac{n}{2}} t}\cos(\omega_{\frac{n}{2}} t + \phi_{\frac{n}{2}})] \end{pmatrix}$$

are at each time $t \geqslant 0$ the coordinates of $\mathbf{x}^{(0)}(t, x_0)(x_0)$ in the same axis frame $\{v_1, w_1, \ldots, v_{n/2}, w_{n/2}\}$. Therefore, an aperiodic mode is the (time-dependent) component of $\mathbf{x}^{(0)}(t, x_0)(x_0)$ in the plane spanned by the vectors $\{v_i, w_i\}$.

Note that if $\{v_1', w_1', \ldots, v_{n/2}', w_{n/2}'\}$ are a different set of real independent vectors defining the eigenvectors $q_1', \ldots q_{n/2}'$ associated to the eigenvalues $\mu_1, \ldots, \mu_{n/2}$ and the reals $g_1', h_1', \ldots, g_{n/2}', h_{n/2}'$ are the coordinates of x_0 in the axis frame $\{v_1', w_1', \ldots, v_{n/2}', w_{n/2}'\}$ and if

$$m_i' := \sqrt{(g_i')^2 + (h_i')^2}, \tag{53}$$

and ϕ_i' is such that

$$\sin\phi_i' := \frac{g_i'}{m_i'},\ \cos\phi_i := \frac{h_i'}{m_i'} \tag{54}$$

it is easy to see that

$$\mathbf{x}^{(0)}(t, x_0) = \sum_{i=1}^{n/2} m_i e^{\alpha_i t}[v_i \sin(\omega_i t + \phi_i)$$
$$+ w_i \cos(\omega_i t + \phi_i))$$
$$= \sum_{i=1}^{n/2} m_i' e^{\alpha_i t}(v_i' \sin(\omega_i t + \phi_i') + w_i' \cos(\omega_i t + \phi_i')) \tag{55}$$

In other words, the modes do not depend on how we choose the eigenvector basis, i.e. coordinate-free. Indeed, $q_i' = (\alpha_i + j\beta_i)q_i$ for some reals α_i, β_i since q_i' and q_i are eigenvectors associated to the same μ_i. Therefore

$$v_i' + jw_i' = q_i = (\alpha_i + j\beta_i)q_i = (\alpha_i + j\beta_i)(v_i + jw_i)$$
$$= \alpha_i v_i - \beta_i w_i + j(\alpha_i w_i + \beta_i v_i) \tag{56}$$

so that

$$\begin{pmatrix} v_i' \\ w_i' \end{pmatrix} = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}\begin{pmatrix} v_i \\ w_i \end{pmatrix}$$

It follows that

$$\begin{pmatrix} g_i' \\ h_i' \end{pmatrix} = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}^{-1}\begin{pmatrix} g_i \\ h_i \end{pmatrix}$$
$$= \frac{1}{\alpha_i^2 + \beta_i^2}\begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}\begin{pmatrix} g_i \\ h_i \end{pmatrix}$$

and

$$m_i' := \sqrt{(g_i')^2 + (h_i')^2} = \sqrt{g_i^2 + h_i^2} = m_i \tag{57}$$

and

$$\begin{pmatrix} \sin\phi_i' \\ \cos\phi_i' \end{pmatrix} := \begin{pmatrix} \frac{g_i'}{m_i'} \\ \frac{h_i'}{m_i'} \end{pmatrix} = \frac{1}{m_i(\alpha_i^2 + \beta_i^2)}\begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}\begin{pmatrix} g_i \\ h_i \end{pmatrix} \tag{58}$$

It follows that

$$\begin{pmatrix} v_i' & w_i' \end{pmatrix}\begin{pmatrix} \sin\phi_i' \\ \cos\phi_i' \end{pmatrix}$$
$$= \frac{1}{m_i(\alpha_i^2 + \beta_i^2)}\begin{pmatrix} v_i & w_i \end{pmatrix}\begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}\begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}\begin{pmatrix} g_i \\ h_i \end{pmatrix}$$
$$= \frac{1}{m_i}\begin{pmatrix} g_i \\ h_i \end{pmatrix} = \begin{pmatrix} v_i & w_i \end{pmatrix}\begin{pmatrix} \sin\phi_i \\ \cos\phi_i \end{pmatrix} \tag{59}$$

which implies (55).

As it is clear from (20), the unforced state response is a sum of modes. Is it possible to reconstruct each mode separately given the unforced state response? In principle, this is possible by choosing the initial state x_0 appropriately. If we want to reconstruct the i-th pseudoperiodic mode, choose x_0 in such a way that for some (at least one nonzero) reals c_i, d_i

$$x_0 := g_i v_i + h_i w_i \tag{60}$$

i.e. on the plane spanned by the vectors $\{v_i, w_i\}$ with coordinates (c_i, d_i). Therefore, the sum in (65) boils down to one term

$$\begin{aligned}\mathbf{x}^{(0)}(t, x_0) = m_i e^{\alpha_i t}(v_i \sin(\omega_i t + \phi_i)\\ + w_i \cos(\omega_i t + \phi_i))\end{aligned} \tag{61}$$

i.e. its i-th mode. Also in this case we say that *the i-th pseudoperiodic mode has been isolated from the unforced state response.*

A pseudoperiodic mode behaves differently in time according to the value of the real part of its corresponding eigenvalue. A similar classification to aperiodic modes is possible for pseudoperiodic modes.

Definition *3.2: A pseudoperiodic is said to be convergent, divergent or periodic according if the real part of its associated pair of complex conjugate eigenvalues satisfies $\alpha < 0$, $\alpha > 0$ or $\alpha = 0$, respectively.*

Therefore, the magnitude of a convergent mode tends to zero as $t \to +\infty$, in a divergent mode tends to infinity as $t \to +\infty$ and in a periodic mode remain constant for all times. Moreover, the oscillation frequency is proportional to the imaginary part of its corresponding eigenvalue.

We can associate to a convergent/divergent pseudoperiodic mode the *natural frequency* ω_n and *damping* ζ defined as

$$\omega_n := \sqrt{\alpha^2 + \omega^2},\ \zeta := -\frac{\alpha}{\omega_n} \tag{62}$$

Clearly $|\zeta| \leqslant 1$ and $\omega_n > 0$ (the limit values $\zeta = \pm 1$ correspond to a pair of coincident real eigenvalues). Note that $\zeta \in (0, 1)$ if and only if $\alpha < 0$. The inverse formulas are given

$$\alpha = -\zeta\omega_n,\ \omega = \omega_n\sqrt{1 - \zeta^2} \tag{63}$$

In other words, for convergent modes by decreasing ζ and keeping ω_n constant the oscillation frequency increases and the convergence to zero is slowed down. On the other hand, by increasing ω_n and keeping ζ constant the convergence to zero speeds up while the oscillation frequency decreases. Moreover, in terms of the parameters ω_n and ζ the characteristic polynomial (43) can be recast as

$$\begin{aligned}\det(A - \lambda I) &= \prod_{i=1}^{n/2}(\lambda - \mu_i)(\lambda - \mu_i^*)\\ &= \prod_{i=1}^{\frac{n}{2}}\omega_{n,i}^2 \prod_{i=1}^{\frac{n}{2}}(1 + 2\lambda\frac{\zeta_i}{\omega_{n,i}} + \frac{\lambda^2}{\omega_{n,i}^2})\\ &= K\prod_{i=1}^{\frac{n}{2}}(1 + 2\lambda\frac{\zeta_i}{\omega_{n,i}} + \frac{\lambda^2}{\omega_{n,i}^2})\end{aligned} \tag{64}$$

where $K := \prod_{i=1}^{\frac{n}{2}} \omega_{n,i}^2$.

B. Aperiodic and pseudoperiodic modes

By combining the result of the previous two sections we obtain the modal decomposition of the unforced state response in the case of distinct (either real or complex conjugate) eigenvalues of A. Let $\lambda_1, \dots, \lambda_r$ ($r \leqslant n$) be the distinct real eigenvalues and $\mu_1, \mu_1^*, \dots, \mu_{\frac{n-r}{2}}, \mu^*_{\frac{n-r}{2}}$ be the distinct conjugate complex eigenvectors of A. Moreover, let $z_1, \dots, z_r$ be the eigenvectors associated to $\lambda_1, \dots, \lambda_r$ and $q_1, \dots, q_{\frac{n-r}{2}}$ the eigenvectors associated to $\mu_1, \dots, \mu_{\frac{n-r}{2}}$ (assuming $n - r$ even), with $\mu_i := \alpha_i + j\omega_i$ and $q_i := v_i + jw_i$. With the usual notations we have

$$\begin{aligned}\mathbf{x}^{(0)}(t, x_0) &= \sum_{i=1}^{r} c_i e^{\lambda_i t} z_i\\ &+ \sum_{i=1}^{\frac{n-r}{2}} m_i e^{\alpha_i t}(v_i \sin(\omega_i t + \phi_i) + w_i \cos(\omega_i t + \phi_i))\end{aligned} \tag{65}$$

Exercize *3.2: Given*

$$A = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -2 & -2 \end{pmatrix} \tag{66}$$

calculate the unforced state response ensuing from

$$x_0 = \begin{pmatrix} 1\\ 0\\ 2 \end{pmatrix}.$$

at $t = 0$.

Let us calculate the eigenvalues of A. We have

$$\begin{aligned}\det(\lambda I - A) &= \det\begin{pmatrix} \lambda - 1 & 0 & 0\\ 0 & \lambda & -1\\ 0 & 2 & \lambda + 2 \end{pmatrix}\\ &= (\lambda - 1)(\lambda^2 + 2\lambda + 2)\end{aligned}$$

Therefore, the eigenvalues of A are $\lambda_1 = 1$ and

$$\begin{aligned}\mu_1 &= \alpha_1 + j\beta_1 = -1 + j\\ \mu_2 &= \alpha_2 + j\beta_2 = \mu_1^* = -1 - j\end{aligned} \tag{67}$$

and we have one divergent aperiodic mode and one convergent pseudoperiodic mode. Clearly, to represent μ_1 as $\alpha_1 + j\omega_1$ we set $\alpha_1 := -1$ and $\beta_1 := 1$. Calculate an eigenvector z_1 associated to λ_1, i.e. a nonzero z_1 such that

$$(A - \lambda_1 I)z_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & -1 & 1\\ 0 & -2 & -3 \end{pmatrix} z_1 = 0 \tag{68}$$

Choose

$$z_1 = \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix} \tag{69}$$

Calculate an eigenvector q_1 associated to μ_1, i.e. a nonzero q_1 such that

$$(A - \lambda_1 I)q_1 = \begin{pmatrix} 2 - j & 0 & 0\\ 0 & 1 - j & 1\\ 0 & -2 & -1 - j \end{pmatrix} q_1 = 0 \tag{70}$$

Note that the rank of

$$(A - \lambda_1 I) = \begin{pmatrix} 0 & 0 & 0\\ 0 & -1 & 1\\ 0 & -2 & -3 \end{pmatrix} \tag{71}$$

is 2 (over the set of complex numbers $\mathbb{C}$). Indeed, the third row is equal to the second multiplied by $-1-j$. Therefore, it is sufficient to solve

$$\begin{pmatrix} 2-j & 0 & 0 \\ 0 & 1-j & 1 \end{pmatrix} z_2 = 0 \tag{72}$$

for q_1, i.e. we discard the third row of $(A-q_1 I)$ which linearly depends from the second row. Choose

$$q_1 = \begin{pmatrix} 0 \\ 1 \\ -1+j \end{pmatrix} \tag{73}$$

The eigenvector q_1 can be written as

$$q_1 = v_1 + jw_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + j \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{74}$$

Finally, calculate an eigenvector q_2 associated to μ_2, i.e. a nonzero q_2 such that

$$(A-\lambda_2 I)q_2 = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 2 & 2 & 0 \end{pmatrix} q_2 = 0 \tag{75}$$

Since $q_2 = q_1^*$, this can be always done by choosing $\mu_2 = \mu_1^*$,

$$q_2 = q_1^* = \begin{pmatrix} 0 \\ 1 \\ -1-j \end{pmatrix} \tag{76}$$

Next, find the coordinates of x_0 in the basis $\{z_1, v_1, w_1\}$, i.e. find the unique reals c_1, g_1, h_1 such that

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = c_1 z_1 + g_1 v_1 + h_1 w_1 \tag{77}$$

This can be easily done as follows. Note that (77) can be written as

$$x_0 = Zc \tag{78}$$

where

$$Z = \begin{pmatrix} z_1 & v_1 & w_1 \end{pmatrix}, \; c = \begin{pmatrix} c_1 \\ g_1 \\ h_1 \end{pmatrix}$$

On account of the fact that Z is nonsingular by construction since $\{z_1, v_1, w_1\}$ is a basis of $\mathbb{R}^3$, we readily have

$$c = Z^{-1} x_0 \tag{79}$$

Therefore,

$$c = Z^{-1} x_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \tag{80}$$

Next, define

$$m_1 := \sqrt{g_1^2 + h_1^2} = 2 \tag{81}$$

and φ_1 as the angle such that

$$\sin\varphi_1 := \frac{g_1}{m_1} = 2, \; \cos\varphi_1 := \frac{h_1}{m_1} = 1$$

Since $\tan\varphi_1 = \frac{g_1}{h_1} = 0$ we get $\varphi_1 = \arctan 0 = 0$ or $\varphi_1 = \arctan 0 + \pi = \pi$ according if $g_1 \geqslant 0$ or $g_1 < 0$ (arctan denotes the principal arc tangent function and its argument ranges in $(-\frac{\pi}{2}, \frac{\pi}{2}]$). In this case, since $g_1 = 0$ we have $\varphi_1 = 0$.

The unforced state response ensuing from x_0 at $t = 0$ is

$$\begin{aligned} &\mathbf{x}^{(0)}(t, x_0) = e^{At} x_0 = e^{\lambda_1 t} c_1 z_1 \\ &+ m_1 e^{\alpha_1 t}(v_1 \sin(\omega_1 t + \phi_1) + w_1 \cos(\omega_1 t + \phi_1)) \\ &= e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2e^{-t}(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \sin(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(t)) \\ &= \begin{pmatrix} e^t \\ 2e^{-t}\sin(t) \\ 2e^{-t}(-\sin(t) + \cos(t)) \end{pmatrix} . \lhd \end{aligned} \tag{82}$$

C. *Eigenvalues with multiplicity greater than one: the case of the pendulum*

Consider the linearized simple pendulum around null angular position and velocity (see (13), Module I) under the condition that

$$\frac{k^2}{m^2} = 4\frac{g}{l} \tag{83}$$

and calculate the unforced state response. In this case the two eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{pmatrix} \tag{84}$$

are both equal to $\lambda := -\frac{k}{2m}$. Notice that $\dim \mathrm{Ker}(A-\lambda I) = 1$ since

$$A - \lambda I = \begin{pmatrix} \frac{k}{2m} & 1 \\ -\frac{g}{l} & -\frac{k}{2m} \end{pmatrix} \tag{85}$$

and det $(A - \lambda I) = 0$ on account of (83). We can construct one chain of two *generalized* eigenvectors as follows. An eigenvector $z^{(1)}$ of order one is obtained as usual from the equation

$$(A - \lambda I) z^{(1)} = 0, \tag{86}$$

for example

$$z^{(1)} = \begin{pmatrix} 1 \\ -\frac{k}{2m} \end{pmatrix} \tag{87}$$

An eigenvector $z^{(2)}$ of order two, independent from $z^{(1)}$, is obtained from the equation

$$(A - \lambda_i I) z^{(2)} = z^{(1)}, \tag{88}$$

for example

$$z^{(2)} = \begin{pmatrix} 1 \\ 1 - \frac{k}{2m} \end{pmatrix} \tag{89}$$

We recall that an eigenvector $z^{(k)}$ of order $k \geqslant 1$ is a nonzero vector such that

$$\begin{aligned} (A - \lambda_i I)^{k-1} z^{(k)} &\neq 0 \\ (A - \lambda_i I)^k z^{(k)} &= 0. \end{aligned}$$

The set $\{z^{(1)}, z^{(2)}\}$ form a basis of $\mathbb{R}^2$. If $c^{(1)}$ and $c^{(2)}$ are reals such that

$$x_0 = c^{(1)}z^{(1)} + c^{(2)}z^{(2)}, \tag{90}$$

by similar calculations as for distinct eigenvalues the unforced state response is

$$\begin{aligned}
&\mathbf{x}^{(0)}(t, x_0) = e^{At}x_0 \\
&= e^{\lambda t}[c^{(1)}z^{(1)} + c^{(2)}z^{(2)} + tc^{(2)}z^{(1)}] \\
&= e^{-\frac{k}{2m}t}[(c^{(1)} + tc^{(2)})\begin{pmatrix}1\\ -\frac{k}{2m}\end{pmatrix} + c^{(2)}\begin{pmatrix}1\\ 1-\frac{k}{2m}\end{pmatrix}] \\
&= e^{-\frac{k}{2m}t}[tc^{(2)}\begin{pmatrix}1\\ -\frac{k}{2m}\end{pmatrix} + \begin{pmatrix}c^{(1)} + c^{(2)}\\ -c^{(1)}\frac{k}{2m} + c^{(2)}[1-\frac{k}{2m}]\end{pmatrix}]
\end{aligned} \tag{91}$$

where the last expression represents the aperiodic (convergent) natural mode. If for instance

$$x_0 = \begin{pmatrix}1\\ 1\end{pmatrix} \tag{92}$$

we readily get

$$\begin{aligned}
&\begin{pmatrix}c^{(1)}\\ c^{(2)}\end{pmatrix} = \begin{pmatrix}z^{(1)} & z^{(2)}\end{pmatrix}^{-1}x_0 = \begin{pmatrix}1 & 1\\ -\frac{k}{2m} & 1-\frac{k}{2m}\end{pmatrix}^{-1}\begin{pmatrix}1\\ 1\end{pmatrix} \\
&= \begin{pmatrix}1-\frac{k}{2m} & -1\\ \frac{k}{2m} & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}-\frac{k}{2m}\\ 1+\frac{k}{2m}\end{pmatrix}
\end{aligned} \tag{93}$$

In general, for each eigenvalue we obtain a certain number of chains of generalized eigenvectors (with increasing order) which altogether form a basis of $\mathbb{R}^n$. The maximum length of the chains is equal to the multiplicity of the eigenvalue in the polynomial obtained as the m.c.m. of all the denominators of the entries of the matrix

$$(\lambda I - A)^{-1} \tag{94}$$

which is called *minimal polynomial* of A. In the above example we have

$$(\lambda I - A)^{-1} = \frac{1}{(\lambda + \frac{k}{2m})^2}\begin{pmatrix}\lambda + \frac{k}{m} & 1\\ -\frac{g}{l} & \lambda\end{pmatrix} \tag{95}$$

and the minimal polynomial is $(\lambda + \frac{k}{2m})^2$. Therefore, we have only one chain of two generalized eigenvectors, i.e. $z^{(1)}$ and $z^{(2)}$.

IV. Modal decomposition of the unforced output response

Using the modal decomposition of the unforced state response and assuming for simplicity distinct eigenvalues, we can decompose the unforced output response as follows

$$\begin{aligned}
&\mathbf{y}^{(0)}(t, x_0) = Ce^{At}x_0 = \sum_{i=1}^{r} c_i e^{\lambda_i t}Cz_i \\
&+ \sum_{i=1}^{\frac{n-r}{2}} m_i e^{\alpha_i t}(\sin(\omega_i t + \phi_i)Cv_i + \cos(\omega_i t + \phi_i)Cw_i)
\end{aligned}$$

By selecting in a suitable way the initial state x_0 we can reconstruct each mode separately from the observation of the unforced output response.

Definition *4.1: A mode which can be isolated from the unforced output response is said to be observable from the outputs.*

By inspection of the modal decomposition of the unforced output response we obtain the following necessary and sufficient condition for observability of a mode.

Proposition *4.1: The i-th aperiodic mode, $i = 1, \ldots, r$, is observable from the outputs if and only if*

$$Cz_i \neq 0$$

while the i-th pseudoperiodic mode, $i = 1, \ldots, (n-r)/2$, is observable from the outputs if and only if

$$C\begin{pmatrix}v_i & w_i\end{pmatrix} \neq \begin{pmatrix}0 & 0\end{pmatrix}.$$

Exercize *4.1: Let A be as in (66) and*

$$C = \begin{pmatrix}0 & 1 & 1\end{pmatrix}. \tag{96}$$

Calculate the unforced output response ensuing from

$$x_0 = \begin{pmatrix}1\\ 0\\ 2\end{pmatrix}$$

at $t = 0$. We have already calculated the unforced state response ensuing from the intial state x_0 at $t = 0$

$$\begin{aligned}
&\mathbf{x}^{(0)}(t, x_0) = e^{At}x_0 = e^{\lambda_1 t}c_1 z_1 \\
&+ m_1 e^{\alpha_1 t}(v_1 \sin(\omega_1 t + \phi_1) + w_1 \cos(\omega_1 t + \phi_1)) \\
&= e^t\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + 2e^{-t}(\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}\sin(t) + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\cos(t)) \\
&= \begin{pmatrix}e^t\\ 2e^{-t}\sin(t)\\ 2e^{-t}(-\sin(t) + \cos(t))\end{pmatrix}
\end{aligned}$$

The unforced output response ensuing from x_0 is

$$\begin{aligned}
&\mathbf{y}^{(0)}(t, x_0) = Ce^{At}x_0 = e^{\lambda_1 t}c_1 Cz_1 \\
&+ m_1 e^{\alpha_1 t}(Cv_1 \sin(\omega_1 t + \phi_1) + Cw_1 \cos(\omega_1 t + \phi_1)) \\
&= 2e^{-t}\cos(t)
\end{aligned}$$

Since $Cz_1 = 0$ and $C\begin{pmatrix}v_1 & w_1\end{pmatrix} = \begin{pmatrix}0 & 2\end{pmatrix}$ only the pseudoperiodic mode is observable from the outputs. $\lhd$

V. Modal decomposition of the forced state and output response (OPTIONAL)

In this section we will study the modal properties of the forced state response. For simplicity, we will study only the case of distinct eigenvalues of A. Let $\lambda_1, \ldots, \lambda_r$ ($r \leqslant n$) be the real eigenvalues and $\mu_1, \ldots, \mu_{\frac{n-r}{2}}$ (assuming $n-r$ even), with $\mu_i := \alpha_i + j\omega_i$, be the complex conjugate eigenvalues. Moreover, let $z_1, \ldots, z_r$ the eigenvectors associated to $\lambda_1, \ldots, \lambda_r$ and $q_1, \ldots, q_{\frac{n-r}{2}}$ the eigenvectors associated to $\mu_1, \ldots, \mu_{\frac{n-r}{2}}$, with $q_i := v_i + jw_i$. The vectors

$$\{z_1, \ldots, z_r, v_1, w_1, \ldots, v_{\frac{n-r}{2}}, w_{\frac{n-r}{2}}\}$$

form a basis for $\mathbb{R}^n$. Let $c_{i1}, \ldots, c_{ir}, g_{i1}, h_{i1}, \ldots, g_{i,\frac{n-r}{2}}, h_{i,\frac{n-r}{2}}$ be the unique reals such that

$$[B]_j = \sum_{i=1}^{r} c_{i,j} z_i + \sum_{i=1}^{\frac{n-r}{2}} (g_{i,j} v_i + h_{i,j} w_i) \tag{97}$$

where $[B]_j$ is the j-th column of B. Also, define $m_{i,j}$ as

$$m_{i,j} := \sqrt{g_{i,j}^2 + h_{i,j}^2} \tag{98}$$

and $\phi_{i,j}$ such that

$$\sin\phi_{i,j} := \frac{g_{i,j}}{m_{i,j}},\ \cos\phi_{i,j} := \frac{h_{i,j}}{m_{i,j}} \tag{99}$$

As for the unforced state response (with the roles of x_0 and B_j interchanged) we have

$$[\mathbf{H}(t)]_j = e^{At}[B]_j = \sum_{i=1}^{r} e^{\lambda_i t} c_{i,j} z_i + \sum_{i=1}^{\frac{n-r}{2}} m_{i,j} e^{\alpha_i t}(v_i \sin(\omega_i t + \phi_{i,j}) + w_i \cos(\omega_i t + \phi_{i,j}))$$

In conclusion, we can decompose the forced state response as follows

$$\begin{aligned}
\mathbf{x}^{(u)}(t,\mathbf{u}) &= \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau \\
&= \sum_{j=1}^{m} \int_0^t e^{A(t-\tau)} [B]_j [\mathbf{u}(\tau)]_j d\tau \\
&= \sum_{j=1}^{m} \int_0^t \sum_{i=1}^{r} e^{\lambda_i (t-\tau)} c_{i,j} z_i [\mathbf{u}(\tau)]_j d\tau + \\
&+ \sum_{j=1}^{m} \int_0^t \sum_{i=1}^{\frac{n-r}{2}} m_{i,j} e^{\alpha_i t}(v_i \sin(\omega_i (t-\tau) + \phi_{i,j}) \\
&+ w_i \cos(\omega_i (t-\tau) + \phi_{i,j}))[\mathbf{u}(\tau)]_j d\tau \\
&= \sum_{i=1}^{r} z_i (\sum_{j=1}^{m} c_{i,j} \int_0^t e^{\lambda_i (t-\tau)} [\mathbf{u}(\tau)]_j d\tau) + \\
&+ \sum_{i=1}^{\frac{n-r}{2}} \{v_i (\sum_{j=1}^{m} m_{i,j} \int_0^t e^{\alpha_i (t-\tau)} \sin(\omega_i (t-\tau) + \phi_{i,j}) [\mathbf{u}(\tau)]_j d\tau) \\
&\quad + w_i (\sum_{j=1}^{m} m_{i,j} \int_0^t e^{\alpha_i (t-\tau)} \sin(\omega_i (t-\tau) + \phi_{i,j}) [\mathbf{u}(\tau)]_j d\tau)\}
\end{aligned} \tag{100}$$

With an impulsive input

$$\mathbf{u}(t) = \boldsymbol{\delta}^{(0)}(t) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{(j-th component)} \tag{101}$$

from (100) we have

$$\mathbf{x}^{(u)}(t,\mathbf{u}) = e^{At}[B]_j = [\mathbf{H}(t)]_j = \sum_{i=1}^{r} c_{i,j} e^{\lambda_i t} z_i + \sum_{i=1}^{\frac{n-r}{2}} m_{i,j} e^{\alpha_i t}(v_i \sin(\omega_i t + \phi_{i,j}) + w_i \cos(\omega_i t + \phi_{i,j})) \tag{102}$$

Definition *5.1: A mode which can be isolated from the forced state response with impulsive inputs is said to be excitable with impulsive inputs.*

By inspection of the modal decomposition of the forced state response we obtain the following necessary and sufficient condition for excitability of a mode.

Proposition *5.1: The i-th aperiodic mode, $i = 1, \ldots, r$, is excitable with impulsive inputs if and only if*

$$c_{i,j} \neq 0$$

for at least one j and the i-th aperiodic mode, $i = 1, \ldots, (n-r)/2$, is excitable with impulsive inputs if and only if

$$\begin{pmatrix} g_{i,j} & h_{i,j} \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$$

for at least one j.

As the unforced output response, we can decompose the forced output response as

$$\begin{aligned}
\mathbf{y}^{(u)}(t,\mathbf{u}) &= \int_0^t (Ce^{At}B + D\boldsymbol{\delta}^{(0)}(t-\tau))\mathbf{u}(\tau) d\tau \\
&= \sum_{j=1}^{m} \int_0^t (Ce^{A(t-\tau)}[B]_j + \boldsymbol{\delta}^{(0)}(t-\tau)[D]_j)[\mathbf{u}(\tau)]_j d\tau \\
&= \sum_{i=1}^{r} Cz_i \sum_{j=1}^{m} c_{i,j} \int_0^t e^{\lambda_i (t-\tau)} [\mathbf{u}(\tau)]_j d\tau + \\
&+ \sum_{i=1}^{\frac{n-r}{2}} Cv_i \sum_{j=1}^{m} m_{i,j} \int_0^t e^{\alpha_i (t-\tau)} \sin(\omega_i (t-\tau) + \phi_{i,j}) [\mathbf{u}(\tau)]_j d\tau \\
&+ Cw_i \sum_{j=1}^{m} m_{i,j} \int_0^t e^{\alpha_i (t-\tau)} \cos(\omega_i (t-\tau) + \phi_{i,j}) [\mathbf{u}(\tau)]_j d\tau \\
&+ \sum_{j=1}^{m} [D]_j [\mathbf{u}(t)]_j
\end{aligned} \tag{103}$$

From (103) with an impulsive input (101) we have

$$\begin{aligned}
\mathbf{y}^{(u)}(t,\mathbf{u}) &= Ce^{At}[B]_j + [D]_j \boldsymbol{\delta}^{(0)}(t) = [\mathbf{W}(t)]_j \\
&= \sum_{i=1}^{r} (Cz_i) c_{i,j} e^{\lambda_i t} + \sum_{i=1}^{\frac{n-r}{2}} m_{i,j} e^{\alpha_i t}((Cv_i)\sin(\omega_i t + \phi_{i,j}) \\
&+ (Cw_i)\cos(\omega_i t + \phi_{i,j})) + [D]_j \boldsymbol{\delta}^{(0)}(t)
\end{aligned} \tag{104}$$

Definition *5.2: A mode which can be isolated both from the forced state response with impulsive inputs and from the unforced output response is said to be excitable with impulsive inputs and observable from the outputs.*

By inspection of the modal decomposition of the forced output response we obtain the following necessary and sufficient condition for excitability and observability of a mode.

Proposition *5.2: The i-th aperiodic mode, $i = 1, \dots, r$, is excitable with impulsive inputs and observable from the outputs if and only if*

$$c_{i,j} \neq 0$$

for at least one j and

$$Cz_i \neq 0$$

and the i-th pseudoperiodic mode, $i = 1, \dots, (n-r)/2$, is excitable with impulsive inputs and observable from the outputs if and only if

$$m_{i,j} \neq 0$$

for at least one j and

$$C \begin{pmatrix} v_i & w_i \end{pmatrix} \neq 0.$$

The condition $c_{i,j} \neq 0$ for at least one j means that at least one column $[B]_j$ of B has a nonzero component in the direction of z_i.

Exercize *5.1: Consider the matrix A in (66) together with*

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{105}$$

Discuss the excitability of the modes with impulsive inputs.

The forced state response to an impulsive input $\mathbf{u}(t) = \boldsymbol{\delta}^{(0)}(t)$ is

$$\mathbf{x}^{(u)}(t, \mathbf{u}) = e^{At}B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} c_1 e^t + \\ + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} m_1 e^{-t} \sin(t + \phi_1) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} m_1 e^{-t} \cos(t + \phi_1)$$

with

$$\begin{aligned} m_1 &:= \sqrt{g_1^2 + h_1^2} \\ \sin\phi_1 &:= \frac{g_1}{m_1}, \ \cos\phi_1 := \frac{h_1}{m_1} \end{aligned} \tag{106}$$

and c_1, g_1, h_1 such that

$$B = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + g_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + h_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{107}$$

By direct calculations $c_1 = 1$, $g_1 = h_1 = 0$ and $\phi_1 = 0$ so that

$$\mathbf{x}^{(u)}(t, \mathbf{u}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t \tag{108}$$

Notice that B has a nonzero component only along the direction of the eigenvector associated to the real eigenvalue of A (i.e. $c_1 \neq 0$, $g_1 = h_1 = 0$). Therefore, only the aperiodic mode is excitable with impulsive inputs, as it is also clear from (108). $\triangleleft$

Exercize *5.2: Consider the matrix A in (66) together with*

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \ D = 0 \tag{109}$$

Discuss the excitability and observability of the modes.

As we have already seen the forced state response input $\mathbf{u}(t) = \boldsymbol{\delta}^{(0)}(t)$ is

$$\mathbf{x}^{(u)}(t, \mathbf{u}) = \mathbf{H}(t) = e^{At}B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$$

Therefore, the forced output response to an impulsive input input $\mathbf{u}(t) = \boldsymbol{\delta}^{(0)}(t)$ is

$$\mathbf{y}^{(u)}(t, \mathbf{u}) = \mathbf{W}(t) = Ce^{At}B = 0 \tag{110}$$

Notice that B has a nonzero component only along the direction of the eigenvector associated to the real eigenvalue of A and $Cz_1 = 0$ and $Cv_1 = 0$ but $Cw_1 \neq 0$. Therefore, the aperiodic mode is excitable with impulsive inputs and the pseudoperiodic mode is observable from the output but none of the modes is both excitable and observable, as it is also clear from (110). $\triangleleft$

A. An application: modal decomposition for the longitudinal and lateral motion of the aircraft (OPTIONAL)

Consider the equations of the longitudinal motion of the aircraft linearized around a given equilibrium flight condition(trim condition). By choosing v_X (velocity along X-axis in the body-axis system), v_Z (velocity along Z-axis in the body-axis system), ω_Y (angular velocity along Y-axis in the body-axis system) and θ (pitch attitude) as state variables and η (elevator angle) and τ (thrust) as control inputs we obtain the following state space representation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{111}$$

where

$$\mathbf{x} = \begin{pmatrix} v_X \\ v_Z \\ \omega_Y \\ \theta \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} \eta \\ \tau \end{pmatrix},$$

$$A = \begin{pmatrix} \alpha_{v_X} & \alpha_{v_Z} & \alpha_{\omega_Y} & \alpha_\theta \\ \beta_{v_X} & \beta_{v_Z} & \beta_{\omega_Y} & \beta_\theta \\ \gamma_{v_X} & \gamma_{v_Z} & \gamma_{\omega_Y} & \gamma_\theta \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} \alpha_\eta & \alpha(\tau) \\ \beta_\eta & \beta(\tau) \\ \gamma_\eta & \gamma(\tau) \\ 0 & 0 \end{pmatrix} \tag{112}$$

The coefficients of A and B are the aerodynamic stability derivatives and, respectively, control derivatives, calculated at the trim condition and referred to the body-axis system of the aircraft.

The eigenvalues are the roots of the characteristic polynomial of A

$$\begin{aligned} (\lambda I - A) &= (\lambda^2 + 2\zeta_p\omega_{n,p}\lambda + {\omega_{n,p}}^2) \times \\ &\quad \times (\lambda^2 + 2\zeta_s\omega_{n,s}\lambda + {\omega_{n,s}}^2) \end{aligned}$$

Usually, $0 < \zeta_i < 1$ and $\omega_{n,(i)} > 0$ for $i = s, p$. Therefore, the eigenvalues of A are two distinct pairs of complex conjugate numbers with negative real part and we have two convergent pseudoperiodic modes, known as *short-period* and, respectively, *phugoid*, mode. In particular each pair is given by

$$\lambda_\pm^i = (-\zeta_i \pm j\sqrt{1 - \zeta_i^2})\omega_{n,i}, \ i = s, p. \tag{113}$$

The short-period mode is a damped oscillation in pitch θ about the Y-axis in the body-axis system. Whenever the aircraft is disturbed from its pitch equilibrium state (trim condition) the short-period mode is excited and shows itself as a classical second-order oscillation. The natural frequency of the mode is usually in the range 1 rad/sec to 10 rad/sec and the oscillation tends to vanish (stable damping), although lower than desired. Since the period of the mode is short, v_X remains approximately zero during a short-term perturbation. Since the short-term behaviour is dominated by the short-period mode, it is convenient to reduce the linearized equations of the longitudinal motion of the aircraft by suppressing the phugoid mode thereby providing a deeper insight into the physical behaviour of the aircraft. The model reduction is performed by assuming $v_X \approx 0$, initial steady level flight and referring the equations of motion to the wind axis ($v_{Xe} = V_0$, the trim velocity along the X-axis coincident with the wind axis, and $\theta_e = \alpha_e = 0$, the trim pitch and angle of attack). Since under this conditions $\beta_\theta \approx 0$ and $\gamma_\theta \approx 0$, from the equations of motion we obtain

$$\dot{\mathbf{x}}_s = A_s \mathbf{x}_s(t) + B_s \mathbf{u}_s(t), \tag{114}$$

where

$$\begin{aligned} x_s &= \begin{pmatrix} v_Z \\ \omega_Y \end{pmatrix},\ u_s = \begin{pmatrix} \eta \\ \tau \end{pmatrix}, \\ A_s &= \begin{pmatrix} \beta_{v_Z} & \beta_{\omega_Y} \\ \gamma_{v_Z} & \gamma_{\omega_Y} \end{pmatrix},\ B_s = \begin{pmatrix} \beta_\eta & \beta(\tau) \\ \gamma_\eta & \gamma(\tau) \end{pmatrix} \end{aligned} \tag{115}$$

The eigenvalues are $\mu_1 = \lambda_{+,s}$ and $\mu_2 = \mu_1^* = \lambda_{-,s}$, with

$$\begin{aligned} \omega_{n,s} &= \sqrt{\gamma_{\omega_Y}\beta_{v_Z} - \gamma_{v_Z}\beta_{\omega_Y}}, \\ \zeta_s &= -\frac{\gamma_{\omega_Y} + \beta_{\omega_Y}}{2\omega_{n,s}} \end{aligned} \tag{116}$$

Therefore, we have one pseudoperiodic mode which is exactly the short-period mode. The derivative β_{v_Z} is dependent on the lift curve slope of the wing (i.e. the plot of the lift coefficient versus the angle of attack), γ_{ω_Y} is determined largely by the viscous paddle-damping properties of the tailplane, γ_{v_Z} is a measure of the aerodynamic stiffness in pitch and is also dominated by the aerodynamics of the tail. While β_{v_Z} and γ_{ω_Y} are both negative numbers, the sign of γ_{v_Z} depends on the position of the gravity center, becoming increasingly negative as it moves forward in the airframe. Therefore, the gravity center must be far enough forward in the airframe for the short-period mode to be stable.

Let us write down the modal decomposition of the unforced state response. An eigenvector q_1 associated to $\mu_1 = \alpha_1 + j\omega_1$, with $\alpha_1 = -\zeta_s\omega_{n,s}$ and $\omega_1 = \omega_{n,s}\sqrt{1-\zeta_s}^2$, is

$$q_1 = \begin{pmatrix} -\frac{\gamma_{\omega_Y} - \alpha_1 - j\omega_1}{\gamma_{v_Z}} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\gamma_{\omega_Y} - \alpha_1}{\gamma_{v_Z}} \\ 1 \end{pmatrix} + j\begin{pmatrix} \frac{\omega_1}{\gamma_{v_Z}} \\ 0 \end{pmatrix} \tag{117}$$

Set

$$v_1 = \begin{pmatrix} -\frac{\gamma_{\omega_Y} - \alpha_1}{\gamma_{v_Z}} \\ 1 \end{pmatrix},\ w_1 = \begin{pmatrix} \frac{\omega_1}{\gamma_{v_Z}} \\ 0 \end{pmatrix} \tag{118}$$

The unforced state response is

$$\begin{aligned} &\mathbf{x}_s^{(0)}(t, x_0) \\ &= m_1 e^{-\zeta_s\omega_{n,s}t}\Big(\begin{pmatrix} -\frac{\gamma_{\omega_Y} - \alpha_1}{\gamma_{v_Z}} \\ 1 \end{pmatrix}\sin(\omega_1 t + \phi_1) \\ &+ \begin{pmatrix} \frac{\omega_1}{\gamma_{v_Z}} \\ 0 \end{pmatrix}\cos(\omega_1 t + \phi_1)\Big) \end{aligned} \tag{119}$$

where m_1 and ϕ_1 are defined in the usual way.

The phugoid mode is a lightly damped low-frequency oscillation in speed v_X which couples with pitch attitude θ and height h. The natural frequency of the mode is usually in the range 0.1 rad/sec to 1 rad/sec and the damping ζ_p is typically 0.1 or less. A reduced-order model of the aircraft retaining only the phugoid mode is obtained as follows. During the perturbation the variables v_Z and ω_Y respond in time-scale associated with the short-period mode and therefore it is reasonable that they are approximately constant in the longer time scale associated with the phugoid mode. Moreover, we assume initial steady level flight, α_{ω_Y} insignificantly small and refer the equations of motion to the wind axis. We obtain from the equations of motion the following state space representation for the reduced-order model

$$\dot{\mathbf{x}}_p(t) = A_p \mathbf{x}(t)_p + B_p \mathbf{u}_p(t), \tag{120}$$

where

$$\begin{aligned} \mathbf{x}_p &= \begin{pmatrix} v_X \\ \theta \end{pmatrix},\ \mathbf{u}_p = \eta, \\ A_p &= \begin{pmatrix} \alpha_{v_X} - \alpha_{v_Z}\frac{\gamma_{v_X}U_e - \gamma_{\omega_Y}\beta_{v_X}}{\gamma_{v_Z}U_e - \gamma_{\omega_Y}\beta_{v_Z}} & -g \\ \frac{\gamma_{v_X}\beta_{v_Z} - \gamma_{v_Z}\beta_{v_X}}{\gamma_{v_Z}U_e - \gamma_{\omega_Y}\beta_{v_Z}} & 0 \end{pmatrix}, \\ B_s &= \begin{pmatrix} \alpha_\eta - \frac{\gamma_\eta U_e - \gamma_{\omega_Y}\beta_\eta}{\gamma_{v_Z}U_e - \gamma_{\omega_Y}\beta_{v_Z}} \\ \frac{\gamma_\eta\beta_{v_Z} - \gamma_{v_Z}\beta_\eta}{\gamma_{v_Z}U_e - \gamma_{\omega_Y}\beta_{v_Z}} \end{pmatrix} \end{aligned}$$

The eigenvalues of A_p are $\lambda_{\pm,p}$ and we have one pseudoperiodic mode, the phugoid mode. We have

$$\begin{aligned} \omega_{n,p} &= \sqrt{g\frac{\gamma_{v_X}\beta_{v_Z} - \gamma_{v_Z}\beta_{v_X}}{\gamma_{v_Z}U_e - \gamma_{\omega_Y}\beta_{v_Z}}}, \\ \zeta_p &= -\frac{\alpha_{v_X} - \alpha_{v_Z}\frac{\gamma_{v_X}U_e - \gamma_{\omega_Y}\beta_{v_X}}{\gamma_{v_Z}U_e - \gamma_{\omega_Y}\beta_{v_Z}}}{2\omega_p} \end{aligned} \tag{121}$$

For conventional aircrafts $\gamma_{v_X} \to 0$, $|\gamma_{v_X}\beta_{v_Z}| << |\gamma_{v_Z}\beta_{v_X}|$ and $|\gamma_{v_Z}U_e| << |\gamma_{\omega_Y}\beta_{v_X}|$ and therefore

$$\omega_{n,p} \approx \sqrt{\frac{-g\beta_{v_X}}{U_e}},\ \zeta_p \approx -\frac{\alpha_{v_X}}{2\omega_p} \tag{122}$$

The damping ζ_p is low since it depends directly on the drag to lift ratio which is usually minimized in aircrafts.

A similar modal decomposition can be done for the lateral motion of the aircraft. Consider the equations of the lateral motion of the aircraft linearized around the condition of steady rectilinear symmetric flight (trim condition). By choosing v_Y (velocity along Y-axis in the body-axis system), ω_X (angular velocity along X-axis in the body-axis system), ω_Z (angular velocity along Z-axis in the body-axis system), ϕ and ψ (roll and yaw attitude) as state variables and ξ (aileron angle) and ζ (rudder angle) as control inputs we obtain the following state space representation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{123}$$

where

$$\mathbf{x} = \begin{pmatrix} v_Y \\ \omega_X \\ \omega_Z \\ \phi \\ \psi \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} \xi \\ \zeta \end{pmatrix},$$

$$A = \begin{pmatrix} y_{v_Y} & y_{\omega_X} & y_{\omega_Z} & y_\phi & y_\psi \\ l_{v_Y} & l_{\omega_X} & l_{\omega_Z} & l_\phi & l_\psi \\ n_{,v_Y} & n_{,\omega_X} & n_{,\omega_Z} & n_\phi & n_\psi \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} y_\xi & y_\zeta \\ l_\xi & l_\zeta \\ n_\xi & n_\zeta \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{124}$$

The coefficients of A and B are the aerodynamic stability derivatives and, respectively, control derivatives, calculated at the trim equilibrium and referred to the body-axis system of the aircraft.

The eigenvalues are the roots of the characteristic polynomial of A

$$\det(\lambda I - A) = \lambda(\lambda + \frac{1}{T_s})(\lambda + \frac{1}{T_r})(\lambda^2 + 2\zeta_d\omega_{n,d}\lambda + {\omega_{n,d}}^2) \tag{125}$$

Usually, $0 < \zeta_d < 1$ and $\omega_{n,d}, T_s, T_r > 0$. Therefore, the eigenvalues of A consists of a pair of complex conjugate numbers with negative real part and three reals. correspondigly, we have one convergent pseudoperiodic mode, known as *dutch-roll mode*, one constant aperiodic mode and two convergent aperiodic modes, known, respectively, as *roll-subsidence* and *spiral mode*.

Appendix

A. The step function and the Dirac impulse

Consider the *Heaviside* function (or *unit step* function) defined as

$$\boldsymbol{\delta}^{(-1)}(t) := \begin{cases} 1 & \text{for } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{126}$$

The step function is used also as a truncation for $t < 0$ of a given function: for any real-valued function $f(t)$

$$\boldsymbol{\delta}^{(-1)}(t)\mathbf{f}(t) := \begin{cases} \mathbf{f}(t) & \text{for } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{127}$$

Consider the *impulse* function with duration $T > 0$

$$\begin{aligned} \mathbf{f}(t) \quad &:= \begin{cases} \frac{1}{T} & \text{for } 0 \leqslant t < T \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{T}(\boldsymbol{\delta}^{(-1)}(t) - \boldsymbol{\delta}^{(-1)}(t-T)) \end{aligned} \tag{128}$$

This impulse is called *normalized* since

$$\int_{-\infty}^{+\infty} \mathbf{f}(t)dt = \int_0^{+\infty} \mathbf{f}(t)dt = 1 \tag{129}$$

If we choose $T = \frac{1}{n}$, $n \in \mathbb{N}\backslash\{0\}$, we obtain the family of functions $\{\mathbf{f}_n(t)\}$, $t \to \mathbf{f}_n(t) := n(\boldsymbol{\delta}^{(-1)}(t) - \boldsymbol{\delta}^{(-1)}(t - \frac{1}{n}))$. Note that

$$\lim_{n\to+\infty} \mathbf{f}_n(t) = \boldsymbol{\delta}(t)^{(0)} := \begin{cases} 0 & \text{for } t \neq 0 \\ +\infty & t = 0 \end{cases} \tag{130}$$

and $\boldsymbol{\delta}^{(0)}$ is called the *Dirac impulse* function. A useful property of the the Dirac impulse is:

$$\int_{-\infty}^{+\infty} \boldsymbol{\delta}^{(0)}(t-\tau)\mathbf{f}(\tau)d\tau = \int_{-\infty}^{+\infty} \boldsymbol{\delta}^{(0)}(\tau)\mathbf{f}(t-\tau)d\tau = \mathbf{f}(t)$$

for any function $\mathbf{f}(t)$ and for each $t \geqslant 0$.

B. Recalls on rank, image and kernel

The *column rank* of a matrix A with elements in $\mathbb{R}$ (resp. $\mathbb{C}$) is defined as the number of its linearly independent columns over $\mathbb{R}$ (resp. $\mathbb{C}$), while the *row rank* of a matrix A is defined as the number of its linearly independent rows over $\mathbb{R}$ (resp. $\mathbb{C}$). Row rank and column rank are equal, therefore we simply use the term *rank* of A over $\mathbb{R}$ (resp. $\mathbb{C}$) and we will denote it $\mathrm{rank}_{\mathbb{R}}\{A\}$ (resp. $\mathrm{rank}_{\mathbb{C}}\{A\}$).

For any matrix $A \in \mathbb{R}^{n\times m}$ define the set of all real combinations of the columns of A:

$$\mathrm{Span}_{\mathbb{R}}\{A\} := \{x \in \mathbb{R}^n : x = Az, z \in \mathbb{R}^n\} \tag{131}$$

The set $\mathrm{Span}_{\mathbb{R}}\{A\}$ is a vector subspace of $\mathbb{R}^n$. Indeed, if $z_a, z_b \in \mathrm{Span}_{\mathbb{R}}\{A\}$ then $az_a + bz_b \in \mathrm{Span}_{\mathbb{R}}\{A\}$ for any $a, b \in \mathbb{R}$ since

$$az_a + bz_b = aAz_a + bRz_b = A(az_a + Abz_b)$$

for some $z_a, z_b \in \mathbb{R}^n$. The dimension of $\mathrm{Span}_{\mathbb{R}}\{A\}$ is equal to $\mathrm{rank}_{\mathbb{R}}\{A\}$. For example,

$$\begin{aligned} &\mathrm{Span}_{\mathbb{R}}\{\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}\} \\ &:= \{x \in \mathbb{R}^3 : x = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} z, \ z \in \mathbb{R}^2\} \\ &= \{x \in \mathbb{R}^n : x = \begin{pmatrix} z_1 + 2z_2 \\ 0 \\ z_1 - z_2 \end{pmatrix}, \ z_1, z_2 \in \mathbb{R}\}. \end{aligned}$$

For any matrix $A \in \mathbb{R}^{n\times m}$ define also the set of all vectors $x \in \mathbb{R}^m$ in the kernel of A

$$\mathrm{Ker}_{\mathbb{R}}\{A\} = \{x \in \mathbb{R}^n : Ax = 0\} \tag{132}$$

The set $\mathrm{Ker}_{\mathbb{R}}\{A\}$ is a vector subspace of $\mathbb{R}^m$. Indeed, if $x_a, x_b \in \mathrm{Ker}_{\mathbb{R}}\{A\}$ then $ax_a + bx_b \in \mathrm{Ker}\{A\}$ for any $a, b \in \mathbb{R}$ since

$$A(ax_a + bx_b) = aAx_a + bAx_b = 0. \tag{133}$$

The dimension of $\mathrm{Ker}_{\mathbb{R}}\{A\}$ is $m - \mathrm{rank}_{\mathbb{R}}\{A\}$. For example,

$$\mathrm{Ker}_{\mathbb{R}}\{\begin{pmatrix}1 & 2\\ 0 & 0\\ 1 & 2\end{pmatrix}\} := \{x \in \mathbb{R}^2 : 0 = \begin{pmatrix}1 & 2\\ 0 & 0\\ 1 & 2\end{pmatrix} x\}$$
$$= \{x \in \mathbb{R}^2 : 0 = \begin{pmatrix}x_1 + 2x_2\\ x_1 + 2x_2\end{pmatrix}\}$$
$$= \{x \in \mathbb{R}^2 : x = \begin{pmatrix}1\\ -1\end{pmatrix} z,\ z \in \mathbb{R}\} = \mathrm{Span}_{\mathbb{R}}\{\begin{pmatrix}1\\ -1\end{pmatrix}\}$$

C. Recalls on eigenvalues and eigenvectors

By $\det(A)$ we denote the determinant of any square matrix $A \in \mathbb{R}^n$.

Definition *A.1: For any $A \in \mathbb{R}^n$, the eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) := \det(\lambda I - A)$.*
The characteristic polynomial of $A \in \mathbb{R}^n$ is a real polynomial with degree n. Its roots may be either real or complex conjugate and the multiplicity of each root is called *algebraic multiplicity*. The set of eigenvalues of A is called the *spectrum* of A and it is denoted by $\sigma(A)$.

Definition *A.2: A nonzero vector $z \in \mathbb{C}^n$ such that*

$$(\lambda I - A)z = 0 \tag{134}$$

where λ is an eigenvalue of A, is called a (first order right) eigenvector associated to λ.
Since $\mathrm{rank}_{\mathbb{C}}\{(\lambda I - A)\} < n$ for each $\lambda \in \sigma(A)$, there always exists at least one nonzero solution $z \in \mathbb{C}^n$ of $(\lambda I - A)z = 0$, i.e. a (first order right) eigenvector associated to λ.

Proposition *A.1: For each eigenvalue there exists at least one (first order right) eigenvector.*
When the eigenvalues have all multiplicity 1, we can obtain a set of independent eigenvectors.

Proposition *A.2: Assume that all the eigenvalues of A have algebraic multiplicity 1. For each eigenvalue there exists only one independent eigenvector and all these eigenvectors are independent over $\mathbb{C}$.*
If the eigenvalues of A have algebraic multiplicity 1 there exists a coordinate transformation for which A becomes block-diagonal.

Proposition *A.3: Assume that all the eigenvalues of A have algebraic multiplicity 1. There exists a nonsingular $T \in \mathbb{R}^n$ such that*

$$TAT^{-1} = \mathrm{blockdiag}\{\lambda_1, \ldots, \lambda_r, \Pi_1, \ldots, \Pi_{\frac{n-r}{2}}\} \tag{135}$$

where λ_i, $i = 1, \ldots, r$, are the real eigenvalues and

$$\Pi_i = \begin{pmatrix}\alpha_i & \omega_i\\ -\omega_i & \alpha_i\end{pmatrix}$$

where $\mu_i := \alpha_i + j\omega_i$, $i = 1, \ldots, \frac{n-r}{2}$, with $\mu_i^* = \alpha_i - j\omega_i$, are the complex conjugate eigenvalues.
The transformation T is defined as follows:

$$T := \begin{pmatrix}z_1 & \ldots & z_r & v_1 & w_1 & \ldots & v_s & w_s\end{pmatrix}^{-1}$$

where z_i is any eigenvector associated to λ_i and $z_i = v_i + jw_i$ is any eigenvector associated to μ_i.

We list some useful properties of the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$.

Proposition *A.4:*
i $\sigma(A) = \sigma(A^\top)$,
(ii) A is invertible if and only if it has no null eigenvalues.

Proposition *A.5: If*

$$A = \begin{pmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix} \tag{136}$$

for some matrices $A_{11} \in \mathbb{R}^{r\times r}$, $A_{12} \in \mathbb{R}^{r\times(n-r)}$ and $A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ or

$$A = \begin{pmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{pmatrix} \tag{137}$$

for some matrices $R_{11} \in \mathbb{R}^{r\times r}$, $A_{12}\mathbb{R}^{(n-r)\times r}$ and $R_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, we have

$$\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22}) \tag{138}$$

D. Recalls on matrix exponential

For any $A \in \mathbb{R}^{n\times n}$ we define the *matrix exponential* e^{At}, $t \geqslant 0$, as the power series

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}. \tag{139}$$

with $A^0 := I$.

Proposition *A.6: The power series (139) converges for each $t \geqslant 0$.*
Since the power series is convergent for each $t \geqslant 0$, we can define the derivative of the matrix exponential e^{At} simply by derivating the power series (139) term by term. In particular, by taking derivatives of each term of (139) we obtain for each $t \geqslant 0$

$$\frac{d}{dt}e^{At} = \sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!} = A\sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!} = A\sum_{h=0}^{\infty} \frac{(At)^h}{h!}$$
$$= Ae^{At} \tag{140}$$

But also for each $t \geqslant 0$

$$\frac{d}{dt}e^{At} = \sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!} = (\sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!})A$$
$$= (\sum_{h=0}^{\infty} \frac{(At)^h}{h!})A = e^{At}A$$

We list below some important properties of the matrix exponential e^{At}.

- (*Neutral element*) $e^0 = I$
- (*Commutativity*) For each $t \geqslant 0$ and $X, Y \in \mathbb{R}^{n\times n}$:
$$e^{(X+Y)t} = e^{Xt}e^{Yt} \Leftrightarrow XY = YX \tag{141}$$
- (*Exponential of diagonal matrices*) For any real $\lambda \in \mathbb{R}$
$$e^{\lambda It} = e^{\lambda t}I \tag{142}$$
- (*Exponential of Jordan matrices*) For any real $\lambda \in \mathbb{R}$
$$e^{tJ} = e^{\lambda t}\begin{pmatrix}1 & t & \frac{t^2}{2!} & \cdots & 0 & \sum_{k=0}^{r-1}\frac{t^k}{k!}\\ 0 & 1 & t & \cdots & 0 & \sum_{k=0}^{r-2}\frac{t^k}{k!}\\ \vdots & \vdots & \ldots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & t\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix} \tag{143}$$

where $J(r \times r)$ is a Jordan matrix. i.e.

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \tag{144}$$

- (*Exponential of equivalent matrices*) If

$$\tilde{A} = TAT^{-1} \tag{145}$$

then

$$e^{\tilde{A}t} = Te^{tA}T^{-1} \tag{146}$$