Notes on Linear Control Systems: Module VIII

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***Abstract*—Realizations. Interconnections. Zero-pole cancellation.**

I. Realizations

A system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \quad (1)$$

has an I/O transfer function $\mathbf{W}(s) = C(sI - A)^{-1}B + D$. In this module, we want to study the inverse problem:

Definition *1.1: (State Space Realization problem). Given a proper rational function $\mathbf{W}(s)$, find matrices $A(n \times n)$, $B(n \times 1)$, $C(1 \times n)$ and $D(1 \times 1)$ such that*

$$\mathbf{W}(s) = C(sI - A)^{-1}B + D \quad (2)$$

We also say that (A, B, C, D) is a (state space) realization of $\mathbf{W}(s)$. If $\mathbf{W}(s)$ is proper but not strictly proper, it is always possible to find $D(1 \times 1)$ such that

$$\mathbf{W}(s) = \mathbf{W}_0(s) + D \quad (3)$$

where $\mathbf{W}_0(s)$ is strictly proper. If

$$\mathbf{W}(s) := \frac{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1} + b_n s^n}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n} \quad (4)$$

then it is possible to find reals D and $b'_0, \ldots, b'_{n-1}$ such that

$$\begin{aligned} \mathbf{W}(s) &= \frac{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1} + b_n s^n}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n} \\ &= \frac{b'_0 + b'_1 s + \cdots + b'_{n-1}s^{n-1}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n} + D \\ &= \mathbf{W}_0(s) + D. \end{aligned} \quad (5)$$

Indeed, by multiplying both parts of the above equality by $a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n$

$$\begin{aligned} &b_0 - Da_0 + (b_1 - Da_1)s + \cdots + (b_{n-1} - Da_{n-1})s^{n-1} \\ &+(b_n - D)s^n = b'_0 + b'_1 s + \cdots + b'_{n-1}s^{n-1} \end{aligned} \quad (6)$$

and equating the coefficients of the similar monomials, we obtain a bunch of linear equations in the unknowns D and $b'_0, \ldots, b'_{n-1}$:

$$\begin{aligned} &D = b_n,\ b'_0 = b_0 - Da_0 =,\ b'_1 = b_1 - Da_0,\ \ldots \\ &b'_{n-1} = b_{n-1} - Da_{n-1}. \end{aligned} \quad (7)$$

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It can be directly checked that a set of matrices $A(n \times n)$, $B(n \times 1)$, $C(1 \times n)$ such that

$$\mathbf{W}_0(s) = C(sI - A)^{-1}B \quad (8)$$

is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}$$
$$B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} b'_0 & b'_1 & \cdots & b'_{n-3} & b'_{n-2} & b'_{n-1} \end{pmatrix} \quad (9)$$

Therefore, (A, B, C, D), with (A, B, C) in (9) and D in (7), is a (state space) realization of $\mathbf{W}(s)$.

Proposition *1.1: The state space realization (A, B, C, D), with (A, B, C) in (9) and D in (7), is a controllable and observable system.*

It can be directly checked that

$$\begin{aligned} R &= \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & * \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & * & \cdots & * & * \\ 1 & * & * & \cdots & * & * \end{pmatrix} \end{aligned}$$

where the asterisks denote some unspecified quantities. R is nonsingular and, therefore, the realization (9) is a controllable system. Moreover, assume by absurd that the system described by the realization (9) is not observable. As we have seen, in this case there exists a coordinate transformation $z = Sx$ such that $A(n \times n)$, $B(n \times 1)$, $C(1 \times n)$ and $D(1 \times 1)$ in the new coordinates are

$$\begin{aligned} &\widetilde{A} = SAS^{-1} = \begin{pmatrix} \widetilde{A}_{11} & 0 \\ \widetilde{A}_{12} & \widetilde{A}_{22} \end{pmatrix},\ \widetilde{B} = SB = \begin{pmatrix} \widetilde{B}_1 & \widetilde{B}_2 \end{pmatrix} \\ &\widetilde{C} = CS^{-1} = \begin{pmatrix} \widetilde{C}_1 & 0 \end{pmatrix},\ \widetilde{D} = D \end{aligned} \quad (10)$$

with $\widetilde{A}_{11}((n-s)\times(n-s))$, $\widetilde{B}_1((n-s)\times 1)$, $\widetilde{C}_1(1\times(n-s))$ and $n > s := \text{rank}\{O\}$. But

$$\begin{aligned}&\mathbf{W}(s) = C(sI-A)^{-1}B + D\\&= CS^{-1}S(sI-A)^{-1}S^{-1}TB + D\\&= CS^{-1}(sI-SAS^{-1})^{-1}SB + D = \widetilde{C}(sI-\widetilde{A})^{-1}\widetilde{B} + \widetilde{D}\\&= \begin{pmatrix}\widetilde{C}_1 & 0\end{pmatrix}(sI - \begin{pmatrix}\widetilde{A}_{11} & 0\\ \widetilde{A}_{12} & \widetilde{A}_{22}\end{pmatrix})^{-1}\begin{pmatrix}\widetilde{B}_1 & \widetilde{B}_2\end{pmatrix} + \widetilde{D}\\&= \widetilde{C}_1(sI-\widetilde{A}_{11})^{-1}\widetilde{B}_1 + \widetilde{D} := \widetilde{\mathbf{W}}(s)\end{aligned} \tag{11}$$

But this gives a contradiction since $\mathbf{W}(s)$, which has a denominator polynomial with degree n, is equal to some $\widetilde{\mathbf{W}}(s)$, which has a denominator polynomial with degree $n-s<n$. This is not possible if zero-pole cancellations are ruled out. Therefore, the system described by the realization (9) is also observable.

Notice that if $A(n\times n)$, $B(n\times 1)$, $C(1\times n)$ and $D(1\times 1)$ is a state space realization of $\mathbf{W}(s)$ any other set of matrices $\widetilde{A}(n\times n)$, $\widetilde{B}(n\times 1)$, $\widetilde{C}(1\times n)$ and $\widetilde{D}(1\times 1)$ such that for some nonsingular $T(n\times n)$

$$\widetilde{A} = TAT^{-1},\ \widetilde{B} = TB,\ \widetilde{C} = CT^{-1},\ \widetilde{D} = D$$

is a state space realization of $\mathbf{W}(s)$. Indeed, if $A(n\times n)$, $B(n\times 1)$, $C(1\times n)$ and $D(1\times 1)$ is is a state space realization of $\mathbf{W}(s)$

$$\begin{aligned}&\mathbf{W}(s) = C(sI-A)^{-1}B + D\\&= CT^{-1}T(sI-A)^{-1}T^{-1}TB + D\\&= CT^{-1}(sI-TAT^{-1})^{-1}TB + D\\&= \widetilde{C}(sI-\widetilde{A})^{-1}\widetilde{B} + \widetilde{D}\end{aligned}$$

Notice also that if $A(n\times n)$, $B(n\times 1)$, $C(1\times n)$ and $D(1\times 1)$ is a state space realization of $\mathbf{W}(s)$ then $A^T(n\times n)$, $C^T(n\times 1)$, $B^T(1\times n)$ and $D(1\times 1)$ is another state space realization of $\mathbf{W}(s)$. Indeed, if $A(n\times n)$, $B(n\times 1)$, $C(1\times n)$ and $D(1\times 1)$ is a state space realization of $\mathbf{W}(s)$

$$\begin{aligned}&B^T(sI-A^T)^{-1}C^T + D = (B^T(sI-A^T)^{-1}C^T + D)^T\\&= C((sI-A^T)^{-1})^TB + D = C((sI-A^T)^T)^{-1}B + D\\&= C(sI-A)^{-1}B + D = \mathbf{W}(s)\end{aligned}$$

II. Interconnections

In this section we will study how the interconnection of two or more systems change the input/output transfer function and the state space equations. Moreover, the interconnection may alter the controllability and observability properties of each system. We discuss in details the following three interconnections: series, parallel and feedback interconnection.

A. Series interconnection

The series interconnection of two systems described by state-space models

$$\Sigma_1 : \begin{cases}\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + B_1\mathbf{u}_1(t)\\ \mathbf{y}_1(t) = C_1\mathbf{x}_1(t)\end{cases}$$

$$\Sigma_2 : \begin{cases}\dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{u}_2(t)\\ \mathbf{y}_2(t) = C_2\mathbf{x}_2(t)\end{cases}$$

is defined through the following input/output constraints:

$$\mathbf{u} = \mathbf{u}_1,\ \mathbf{y} = \mathbf{y}_2,\ \mathbf{u}_2 = \mathbf{y}_1 \tag{12}$$

where $\mathbf{u}$ and $\mathbf{y}$ denote the input and, respectively, the output of the interconnection (see Figure 1). The series interconnection is a system with state $\mathbf{x} = \begin{pmatrix}\mathbf{x}_1^\top & \mathbf{x}_2^\top\end{pmatrix}^\top$. In view of (12) the state space interconnection of Σ_1 and Σ_2 is given by

$$\Sigma : \begin{cases}\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + B_1\mathbf{u}(t)\\ \dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{y}_1(t)\\ \mathbf{y}(t) = \mathbf{y}_2(t)\end{cases}$$

Therefore,

$$\Sigma : \begin{cases}\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) = C\mathbf{x}(t)\end{cases}$$

with

$$A := \begin{pmatrix}A_1 & 0\\ B_2C_1 & A_2\end{pmatrix},\ B := \begin{pmatrix}B_1\\ 0\end{pmatrix}, C := \begin{pmatrix}0 & C_2\end{pmatrix}$$

Note that $\sigma(A) = \sigma(A_1)\cup\sigma(A_2)$. Moreover, if we assume that $A_j(n_j\times n_j)$, $B_j(n_j\times 1)$ and $C_j(1\times n_j)$, $j = 1,2$, the controllability matrix of Σ is with $n := n_1 + n_2$

$$\begin{aligned}&R := \begin{pmatrix}B & AB & \cdots & A^{n-1}B\end{pmatrix}\\&= \begin{pmatrix}B_1 & A_1B_1 & \cdots & A_1^{n-1}B_1\\ 0 & B_2C_1B_1 & & \sum_{j=1}^{n-1}A_2^{j-1}B_2C_1A_1^{n-1-j}B_1\end{pmatrix}\end{aligned}$$

while its observability matrix is

$$\begin{aligned}O &:= \begin{pmatrix}C\\ CA\\ \cdots\\ CA^{n_1+n_2-1}\end{pmatrix}\\&= \begin{pmatrix}0 & C_2\\ C_2B_2C_1 & C_2A_2\\ \vdots & \vdots\\ \sum_{j=1}^{n-1}C_1A_1^{j-1}C_2A_2^{n-1-j}B_2 & C_2A_2^{n-1}\end{pmatrix}\end{aligned}$$

The transfer functions of Σ_1 and Σ_2 are

$$\frac{\mathfrak{L}[\mathbf{y}_j(t)](s)}{\mathfrak{L}[\mathbf{u}_j(t)](s)} = \mathbf{P}_j(s) = C_j(sI-A_j)^{-1}B_j,\ j = 1,2. \tag{13}$$

($\mathbf{y}_j(t)$ denotes the forced output response of Σ_j to the input $\mathbf{u}_j(t)$). In view of (12) the transfer function of the series interconnection of Σ_1 and Σ_2 is

$$\begin{aligned}\mathbf{W}(s) &= \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{u}(t)](s)} = \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}\\&= \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}\frac{\mathfrak{L}[\mathbf{u}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_2(t)](s)}\\&= \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_2(t)](s)}\frac{\mathfrak{L}[\mathbf{u}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}\\&= \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_2(t)](s)}\frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)} = \mathbf{P}_2(s)\mathbf{P}_1(s)\end{aligned} \tag{14}$$

($\mathbf{y}(t)$ denotes the forced output response of the interconnection of Σ_1 and Σ_2 to the input $\mathbf{u}(t)$).

Proposition *2.1: The transfer functions of the series interconnection of two systems Σ_j, $j = 1,2$, with transfer function $\mathbf{P}_j(s)$ is $\mathbf{W}(s) = \mathbf{P}_1(s)\mathbf{P}_2(s)$.*

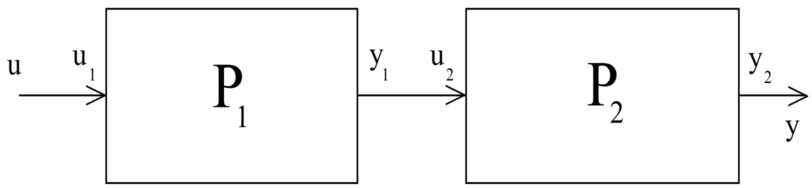

Figure 1. Series interconnection.

Since $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ the stability of a series interconnection is guaranteed by the stability of the two systems Σ_1 and Σ_2.

The series interconnection of two systems may lead to a system which is either not controllable or not observable. This happens when a zero of $\mathbf{P}_1(s)$ (resp. $\mathbf{P}_2(s)$) cancels a pole of $\mathbf{P}_2(s)$ (resp. $\mathbf{P}_1(s)$).

Exercize *2.1:* Consider the series interconnection of the two systems described by state-space models

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1(t) = -\mathbf{x}_1(t) + \mathbf{u}_1(t) \\ \mathbf{y}_1(t) = \mathbf{x}_1(t) \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2(t) = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}_2(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{u}_2(t) \\ \mathbf{y}_2(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x}_2(t) \end{cases}$$

The series interconnection in the state space is

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

with

$$A := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix},\ B := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C := \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

Note that Σ is controllable since

$$R = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

but not observable since

$$O = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -2 & 0 & 2 \end{pmatrix}$$

which has rank 2. Since Σ is not observable we can apply the PBH observability test (see module VI) to determine the invariant spectrum $\mathfrak{F}_O$ of $A - KC$: the spectrum of A is $\{-1, -2, 0\}$ and

$$\mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} (-1)I - A \\ C \end{pmatrix} \right\} = \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\} = 2 \Rightarrow \{-1\} \in \mathfrak{F}_O$$

$$\mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} (-2)I - A \\ C \end{pmatrix} \right\} = \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\} = 3 \Rightarrow \{-1\} \notin \mathfrak{F}_O$$

$$\mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} (0)I - A \\ C \end{pmatrix} \right\} = \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \right\} = 3 \Rightarrow \{0\} \notin \mathfrak{F}_O$$

This can be seen also by defining a coordinate transformation $z = Sx$ with

$$S = \begin{pmatrix} C \\ CA \\ w_1^T \end{pmatrix}$$

and

$$w_1 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

By direct calculations

$$S^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.333 & 0.666 & 0.333 \\ 0.666 & 0.333 & -0.333 \\ 0.333 & -0.333 & 0.333 \end{pmatrix}$$

The matrices A, B and C in the new coordinates are

$$\widetilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -1 \end{pmatrix},\ \widetilde{B} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \widetilde{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (15)$$

Since $n - s = \mathrm{rank}_{\mathbb{R}}\{O\} = 2$ with $n = 3$ (hence, $s = 1$),

$$\widetilde{A}_{11} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix},\ \widetilde{A}_{12} = \begin{pmatrix} -1 & 1 \end{pmatrix},\ \widetilde{A}_{22} = -1$$

$$\widetilde{B}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \widetilde{B}_2 = 1,\ \widetilde{C}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (16)$$

and the invariant spectrum $\mathfrak{F}_O$ of $A-KC$ is $\sigma(\widetilde{A}_{22}) = \{-1\}$. Note that the observable system of (15) is:

$$\widetilde{A}_{11} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix},\ \widetilde{B}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \widetilde{C}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{17}$$

The transfer functions of the two systems Σ_1 and Σ_2 are

$$\begin{aligned} \mathbf{P}_1(s) &= \frac{1}{s+1} \\ \mathbf{P}_2(s) &= \frac{s+1}{s(s+2)} \end{aligned}$$

Therefore, the transfer function of the series interconnection is

$$\mathbf{W}(s) = \mathbf{P}_1(s)\mathbf{P}_2(s) = \frac{1}{s(s+2)}$$

The invariant spectrum $\mathfrak{F}_O = \{-1\}$ of $A-KC$ is exactly the pole $s=-1$ of $\mathbf{P}_1(s)$ which has been canceled by the zero $s=-1$ of $\mathbf{P}_2(s)$. Note also that $\mathbf{W}(s)$ has a realization

$$\Sigma_O : \begin{cases} \dot{\mathbf{x}}(t) = A_O\mathbf{x}(t) + B_O\mathbf{u}(t) \\ \mathbf{y}(t) = C_O\mathbf{x}(t) \end{cases} \tag{18}$$

where

$$A_O := \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix},\ B_O := \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ C_O := \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The system (18) is equal to (17): therefore, Σ_O is the observable system of (15). $\lhd$

B. Parallel interconnection

The parallel interconnection of two systems described by state-space models

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + B_1\mathbf{u}_1(t) \\ \mathbf{y}_1(t) = C_1\mathbf{x}_1(t) \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{u}_2(t) \\ \mathbf{y}_2(t) = C_2\mathbf{x}_2(t) \end{cases}$$

is defined through the following input/output constraints:

$$\mathbf{u} = \mathbf{u}_1 = \mathbf{u}_2,\ \mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \tag{19}$$

where $\mathbf{u}$ and $\mathbf{y}$ denote the input and, respectively, the output of the interconnection (see Figure 2). The parallel interconnection is a system with state $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{pmatrix}^\top$. In view of (19) the state space interconnection of Σ_1 and Σ_2 is given by

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + B_1\mathbf{u}(t), \\ \dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{y}_1(t) + \mathbf{y}_2(t) \end{cases}$$

Therefore,

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

with

$$A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\ B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C := \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

Note that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$. Moreover, if we assume that $A_j(n_j \times n_j)$, $B_j(n_j \times 1)$ and $C_j(1 \times n_j)$, $j = 1, 2$, the controllability matrix of Σ is

$$\begin{aligned} R &:= \begin{pmatrix} B & AB & \cdots & A^{n_1+n_2-1}B \end{pmatrix} \\ &= \begin{pmatrix} B_1 & A_1B_1 & \cdots & A_1^{n_1+n_2-1}B_1 \\ B_2 & A_2B_2 & \cdots & A_2^{n_1+n_2-1}B_2 \end{pmatrix} \end{aligned}$$

while its observability matrix is

$$O := \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n_1+n_2-1} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_1A_1 & C_2A_2 \\ \vdots & \vdots \\ C_1A_1^{n_1+n_2-1} & C_2A_2^{n_1+n_2-1} \end{pmatrix}$$

The transfer functions of Σ_1 and Σ_2 are

$$\frac{\mathfrak{L}[\mathbf{y}_j(t)](s)}{\mathfrak{L}[\mathbf{u}_j(t)](s)} = \mathbf{P}_j(s) = C_j(sI - A_j)^{-1}B_j,\ j = 1, 2. \tag{20}$$

In view of (19) the transfer function of the parallel interconnection of Σ_1 and Σ_2 is

$$\begin{aligned} \mathbf{W}(s) &= \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{u}(t)](s)} \\ &= \frac{\mathfrak{L}[\mathbf{y}_1(t)](s) + \mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}(t)](s)} \\ &= \frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}(t)](s)} + \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}(t)](s)} \\ &= \frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)} + \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_2(t)](s)} = \mathbf{P}_2(s) + \mathbf{P}_1(s) \end{aligned}$$

Proposition *2.2: The transfer functions of the parallel interconnection of two systems Σ_j, $j = 1, 2$, with transfer functions $\mathbf{P}_j(s)$ is $\mathbf{W}(s) = \mathbf{P}_1(s) + \mathbf{P}_2(s)$.*

Since $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ the stability of a parallel interconnection is guaranteed by the stability of the two systems Σ_1 and Σ_2.

The parallel interconnection of two systems may lead to a system which is not controllable and not observable. This happens when a pole of $\mathbf{P}_1(s)$ (resp. $\mathbf{P}_2(s)$) is a also a pole of $\mathbf{P}_2(s)$ (resp. $\mathbf{P}_1(s)$).

Exercize *2.2:* Consider the series interconnection of the two systems described by state-space models

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{u}_1(t) \\ \mathbf{y}_1(t) = \mathbf{x}_1(t) \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2(t) = \mathbf{u}_2(t) \\ \mathbf{y}_2(t) = \mathbf{x}_2(t) \end{cases}$$

The parallel interconnection in the state space is

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

with

$$A := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ B := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C := \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Note that Σ is not controllable since

$$R = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

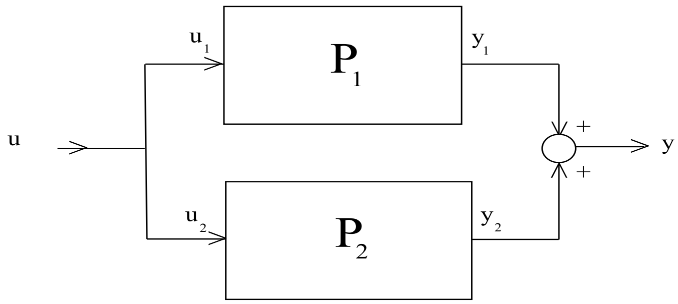

Figure 2. Parallel interconnection.

and not observable since

$$O = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Since Σ is not observable we can apply the PBH observability test to determine the invariant spectrum $\mathfrak{F}_O$ of $A-KC$: the spectrum of A is $\{0,0\}$ and

$$\begin{aligned} &\operatorname{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} (0)I - A \\ C \end{pmatrix} \right\} \\ &= \operatorname{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} = 1 \Rightarrow \{0\} \in \mathfrak{F}_O \end{aligned}$$

Since Σ is also not controllable we can apply the PBH controllability test to determine the invariant spectrum $\mathfrak{F}_R$ of $A+BF$:

$$\begin{aligned} &\operatorname{rank}_{\mathbb{R}}\{\begin{pmatrix}(0)I - A & B\end{pmatrix}\} \\ &= \operatorname{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\} = 1 \Rightarrow \{0\} \in \mathfrak{F}_R \end{aligned}$$

This can be seen also by defining a coordinate transformation $z = Sx$ with

$$S := \begin{pmatrix} C \\ w_1^T \end{pmatrix}$$

where

$$w_1 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

By direct calculations

$$S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The matrices A, B and C in the new coordinates are

$$\widetilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \widetilde{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \widetilde{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{21}$$

Since $n - s = \dim(\operatorname{Ker}\{O\}) = 1$ with $n = 2$ (hence, $s = 1$),

$$\begin{aligned} &\widetilde{A}_{11} = \widetilde{A}_{12} = \widetilde{A}_{22} = 0 \\ &\widetilde{B}_1 = 2,\ \widetilde{B}_2 = 0,\ \widetilde{C}_1 = 1 \end{aligned} \tag{22}$$

and the invariant spectrum $\mathfrak{F}_O$ of $A - KC$ is $\sigma(\widetilde{A}_{22}) = \{0\}$. Note that the observable system of (21) is:

$$\widetilde{A}_{11} = 0,\ \widetilde{B}_1 = 2,\ \widetilde{C}_1 = 1 \tag{23}$$

The transfer functions of the two systems Σ_j, $j = 1, 2$, are

$$\mathbf{P}_1(s) = \frac{1}{s}$$
$$\mathbf{P}_2(s) = \frac{1}{s}$$

Therefore, the transfer function of the parallel interconnection is

$$\mathbf{W}(s) = \mathbf{P}_1(s) + \mathbf{P}_2(s) = \frac{2}{s}$$

The invariant spectrum $\mathfrak{F}_O = \{-1\}$ of $A - KC$ is exactly to the common pole $s = 0$ of $\mathbf{P}_1(s)$ and $\mathbf{P}_2(s)$. Note also that $\mathbf{W}(s)$ has a realization

$$\Sigma_O : \begin{cases} \dot{\mathbf{x}}(t) = A_O\mathbf{x}(t) + B_O\mathbf{u}(t) \\ \mathbf{y}(t) = C_O\mathbf{x}(t) \end{cases} \tag{24}$$

where

$$A_O := 0,\ B_O := 1,\ C_O := 2$$

The system (24) is equivalent under coordinate transformation to (23): i.e.

$$\widetilde{A}_{11} = QA_OQ^{-1},\ \widetilde{B}_1 = QB_O\ \widetilde{C}_1 = C_OQ^{-1}$$

with $Q := 2$. ◁

C. Feedback interconnection

The feedback interconnection of two systems described by state-space models

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + B_1\mathbf{u}_1(t) \\ \mathbf{y}_1(t) = C_1\mathbf{x}_1(t) \end{cases}$$
$$\Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{u}_2(t) \\ \mathbf{y}_2(t) = C_2\mathbf{x}_2(t) \end{cases}$$

is defined through the following input/output constraints:

$$\mathbf{u}_1 = \mathbf{u} - \mathbf{y}_2,\ \mathbf{y} = \mathbf{y}_1,\ \mathbf{u}_2 = \mathbf{y}_1 \tag{25}$$

where $\mathbf{u}$ and $\mathbf{y}$ denote the input and, respectively, the output of the interconnection (see Figure 3). The feedback interconnection is a system with state $\mathbf{x} = \begin{pmatrix}\mathbf{x}_1^\top & \mathbf{x}_2^\top\end{pmatrix}^\top$. In view of (25) the state space interconnection of Σ_1 and Σ_2 is given by

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) - B_1\mathbf{y}_2(t) + B_1\mathbf{u}(t), \\ \dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{y}_1(t) \\ \mathbf{y}(t) = \mathbf{y}_1(t) \end{cases}$$

Therefore,

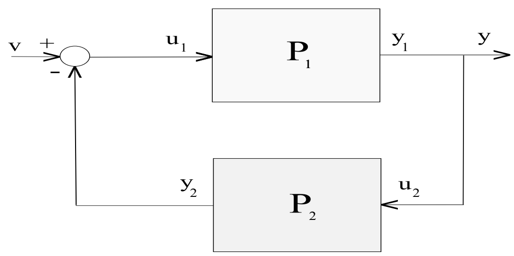

Figure 3. Feedback interconnection.

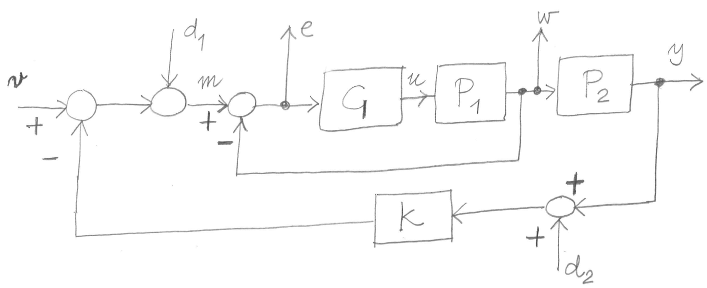

Figure 4. Feedback control system.

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

with

$$A := \begin{pmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix}, \ B := \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, C := \begin{pmatrix} C_1 & 0 \end{pmatrix}$$

The transfer functions of Σ_1 and Σ_2 are

$$\frac{\mathfrak{L}[\mathbf{y}_j(t)](s)}{\mathfrak{L}[\mathbf{u}_j(t)](s)} = \mathbf{P}_j(s) = C_j(sI - A_j)^{-1}B_j, \ j = 1, 2. \quad (26)$$

In view of (25) the transfer function of the feedback interconnection of Σ_1 and Σ_2 is

$$\begin{aligned} \mathbf{W}(s) &= \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{u}(t)](s)} \\ &= \frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s) + \mathfrak{L}[\mathbf{y}_2(t)](s)} = \frac{\frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}}{1 + \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}} \\ &= \frac{\frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}}{1 + \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_2(t)](s)} \frac{\mathfrak{L}[\mathbf{u}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}} \\ &= \frac{\frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}\{\mathbf{u}_1(t)\}(s)}}{1 + \frac{\mathfrak{L}[\mathbf{y}_2(t)](s)}{\mathfrak{L}[\mathbf{u}_2(t)](s)} \frac{\mathfrak{L}[\mathbf{y}_1(t)](s)}{\mathfrak{L}[\mathbf{u}_1(t)](s)}} = \frac{\mathbf{P}_1(s)}{1 + \mathbf{P}_2(s)\mathbf{P}_1(s)} \end{aligned}$$

Proposition *2.3: The transfer functions of the feedback interconnection of two systems Σ_j, $j = 1, 2$, with transfer functions $\mathbf{P}_j(s)$ is $\mathbf{W}(s) = \frac{\mathbf{P}_1(s)}{1+\mathbf{P}_2(s)\mathbf{P}_1(s)}$.*

The feedback interconnection of two systems may lead to a system which is not controllable and not observable. Since the feedback interconnection is a series interconnection of Σ_1 and Σ_2 closed on itself, this happens when a pole of $\mathbf{P}_1(s)$ (resp. $\mathbf{P}_2(s)$) is a also a pole of $\mathbf{P}_2(s)$ (resp. $\mathbf{P}_1(s)$).

The stability of a feedback interconnection when the process $\mathbf{P}_2(s)$ on the feedback path is constant (i.e. $\mathbf{P}_2(s) = K$) can be analyzed (parametrically with respect to K) by means of the Routh criterion applied to the numerator polynomial of the transfer function of the interconnection

$$\mathbf{W}(s) = \frac{\mathbf{P}_1(s)}{1 + K\mathbf{P}_1(s)}$$

or in other words, if $\mathbf{P}_1(s) := \frac{\mathbf{n}_1(s)}{\mathbf{d}_1(s)}$ for suitable polynomials $\mathbf{n}_1(s)$ and $\mathbf{d}_1(s)$,

$$\mathbf{d}_1(s) + K\mathbf{n}_1(s)$$

Clearly, the Routh criterion can be applied if we have the exact knowledge of the polynomials $\mathbf{n}_1(s)$ and $\mathbf{d}_1(s)$ or, in other words, the transfer function $\mathbf{P}_1(s)$. A useful criterion for the stability of the feedback interconnection with $\mathbf{P}_2(s) = K$ will be discussed in the next chapter and it requires the knowledge of the (approximate) Bode diagrams of $\mathbf{P}_1(s)$ and the number of the poles of $\mathbf{P}_1(s)$ with positive real part.

D. General interconnections

The combination of series, parallel and feedback interconnections gives rise to general interconnections. It is possible to determine the transfer functions of these general interconnections using the formulas for the I/O transfer functions of series, parallel and feedback interconnections.

Exercize *2.3: Determine the transfer functions*

$$\mathbf{W}_{v,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{v}(t)](s)},$$
$$\mathbf{W}_{d_1,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}_1(t)](s)}$$
$$\mathbf{W}_{d_2,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}_2(t)](s)} \tag{27}$$

in the control scheme of Figure 4.

Notice that the control scheme has 3 input variables ($\mathbf{v}$, $\mathbf{d}_1$ and $\mathbf{d}_2$) and one output variable $\mathbf{y}$. The output $\mathbf{y}(s)$ is equal to (by superposition)

$$\mathbf{y}(s) = \mathbf{W}_{v,y}(s)\mathbf{v}(s) + \mathbf{W}_{d_1,y}(s)\mathbf{d}_1(s) + \mathbf{W}_{d_2,y}(s)\mathbf{d}_2(s) \tag{28}$$

Therefore, the $\mathbf{W}_{v,y}(s)$ can be obtained by setting $\mathbf{d}_1 = \mathbf{d}_2 = 0$ in Figure 4, $\mathbf{W}_{d_1,y}(s)$ is obtained by setting $\mathbf{v} = \mathbf{d}_2 = 0$ and $\mathbf{W}_{d_2,y}(s)$ is obtained by setting $\mathbf{v} = \mathbf{d}_1 = 0$.

First, determine $\mathbf{W}_{v,y}(s)$ and set $\mathbf{d}_1 = \mathbf{d}_2 = 0$ in Figure 4. The transfer function $\mathbf{W}_{e,w}(s)$ from e to w is resulting from the series of $\mathbf{G}$ and $\mathbf{P}_1$:

$$\mathbf{W}_{e,w}(s) = \frac{\mathfrak{L}[\mathbf{w}(t)](s)}{\mathfrak{L}[\mathbf{e}(t)](s)} = \mathbf{G}(s)\mathbf{P}_1(s)$$

The transfer function $\mathbf{W}_{m,w}(s)$ from $\mathbf{m}$ to $\mathbf{w}$ is resulting from the negative feedback interconnection of $\mathbf{W}_{e,w}(s)$ in the direct path and 1 in the feedback path:

$$\mathbf{W}_{m,w}(s) = \frac{\mathfrak{L}[\mathbf{w}(t)](s)}{\mathfrak{L}[\mathbf{m}(t)](s)} = \frac{\mathbf{W}_{e,w}(s)}{1 + \mathbf{W}_{e,w}(s)} = \frac{\mathbf{G}(s)\mathbf{P}_1(s)}{1 + \mathbf{G}(s)\mathbf{P}_1(s)}$$

The transfer function $\mathbf{W}_{m,y}(s)$ from $\mathbf{m}$ to $\mathbf{y}$ is resulting from the series interconnection of $\mathbf{W}_{m,w}$ and $\mathbf{P}_2$:

$$\mathbf{W}_{m,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{m}(t)](s)}$$
$$= \mathbf{W}_{m,w}(s)\mathbf{P}_2(s) = \frac{\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}{1 + \mathbf{G}(s)\mathbf{P}_1(s)}$$

The transfer function $\mathbf{W}_{v,y}(s)$ from $\mathbf{v}$ to $\mathbf{y}$ is resulting from the negative feedback interconnection of $\mathbf{W}_{m,w}$ in the direct path and $\mathbf{K}(s)$ in the feedback path:

$$\mathbf{W}_{v,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{v}(t)](s)}$$
$$= \frac{\mathbf{W}_{m,y}(s)}{1 + \mathbf{W}_{m,y}(s)\mathbf{K}(s)}$$
$$= \frac{\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}{1 + \mathbf{G}(s)\mathbf{P}_1(s) + \mathbf{K}(s)\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}$$

Next, determine $\mathbf{W}_{d_1,y}(s)$ and set $\mathbf{v} = \mathbf{d}_2 = 0$ in Figure 4. The transfer function $\mathbf{W}_{d_1,y}(s)$ from $\mathbf{d}_1$ to $\mathbf{y}$ is resulting from the negative feedback interconnection of $\mathbf{W}_{m,y}(s)$ in the direct path and $\mathbf{K}(s)$ in the feedback path:

$$\mathbf{W}_{d_1,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}_1(t)](s)}$$
$$= \frac{\mathbf{W}_{m,y}(s)}{1 + \mathbf{W}_{m,y}(s)\mathbf{K}(s)}$$
$$= \frac{\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}{1 + \mathbf{G}(s)\mathbf{P}_1(s) + \mathbf{K}(s)\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)} = \mathbf{W}_{v,y}(s)$$

Finally, determine $\mathbf{W}_{d_2,y}(s)$ and set $\mathbf{v} = \mathbf{d}_1 = 0$ in Figure 4. The transfer function $\mathbf{W}_{d_2,y}(s)$ from $\mathbf{d}_1$ to $\mathbf{y}$ is resulting from the negative feedback interconnection of $-\mathbf{K}(s)\mathbf{W}_{m,y}(s)$ in the direct path and -1 in the feedback path:

$$\mathbf{W}_{d_2,y}(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}_2(t)](s)}$$
$$= \frac{\mathbf{W}_{m,y}(s)}{1 + \mathbf{W}_{m,y}(s)\mathbf{K}(s)}$$
$$= \frac{-\mathbf{K}(s)\mathbf{W}_{d_2,y}(s)}{1 + \mathbf{K}(s)\mathbf{W}_{d_2,y}(s)}$$
$$= -\frac{\mathbf{K}(s)\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}{1 + \mathbf{G}(s)\mathbf{P}_1(s) + \mathbf{K}(s)\mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}.$$