Notes on Linear Control Systems: Module XI

Stefano Battilotti

***Abstract*—Steady-state performances. Tracking. Disturbance compensation.**

I. Tracking and disturbance compensation in frequency domain

A control system is the interconnection of a certain number components or systems in such a way to guarantee some desired performances, in particular the capability of the output to reproduce some given behaviours. Feedback interconnections are flexible control schemes which allow to meet multiple requirements. An error signal is obtained from comparing at each time the output response with its desired value and then a control signal is generated to improve the performances of the process and to force it to reproduce desired behaviours. The capability of the output to follow (within some tolerances) a given input (*tracking*) must be guaranteed also in the presence of disturbances or noise introduced in the control-loop by actuators and sensors. Therefore, an important property of a control system is the complete or partial elimination of the effects of disturbances. Many systems are not able to follow given behaviours due to disturbances which deteriorate their performances.

II. Steady-state performances

An important requirement for a control system is, besides stability, the capability of its output to follow or track (within a given error) a given reference input (*tracking*). For example, an electrical drive is required to work in steady-state regime with a prescribed angular velocity (possibly within a given error tolerance). If the output $\mathbf{y}(t)$ of system is required to asymptotically track a given input $\mathbf{v}(t)$, the *tracking error* $\mathbf{e}(t) := \mathbf{v}(t) - \mathbf{y}(t)$ must tend to zero as $t \to \infty$. We want to give some necessary and sufficient conditions for asymptotically driving to zero the tracking error of a system $\mathbf{W}(s)$ (i.e. its steady-state error is zero). The situation for which the steady-state tracking error response is constant and non-zero is associated to the notion of *type*.

Definition *2.1: A system $\mathbf{W}(s)$ is said to be of type k if its steady-state error $\mathbf{e}_{ss}(t)$ is constant and non-zero for an input $\mathbf{v}(t) := \frac{t^k}{k!}$.*

The (tracking) *error* transfer function is defined as

$$\mathbf{W}_e(s) := \frac{\mathfrak{L}[\mathbf{e}(t)](s)}{\mathfrak{L}[\mathbf{v}(t)](s)} \tag{1}$$

where $\mathbf{e}(t)$ denotes the forced (tracking) error response with input $\mathbf{v}(t)$. We have

$$\begin{aligned}\mathbf{W}_e(s) &:= \frac{\mathfrak{L}[\mathbf{e}(t)](s)}{\mathfrak{L}[\mathbf{v}(t)](s)} = \frac{\mathfrak{L}[\mathbf{v}(t)](s) - \mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{v}(t)](s)} \\ &= 1 - \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{v}(t)](s)} = 1 - \mathbf{W}(s)\end{aligned} \tag{2}$$

As we have seen in module V, the steady-state tracking error response $\mathbf{e}_{ss}^{(k)}(t)$ to an input $\mathbf{v}(t) := \frac{t^k}{k!}$ is

$$\mathbf{e}_{ss}^{(k)}(t) = \sum_{j=0}^{k} \frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} \frac{t^{k-j}}{(k-j)!j!} \tag{3}$$

Notice that $\mathbf{e}_{ss}^{(k)}(t)$ is constant and non-zero if and only if

$$\frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} = 0,\ j = 0, 1, \dots, k-1 \tag{4}$$

$$\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} \neq 0 \tag{5}$$

If we consider the Laurent expansion of $\mathbf{W}_e(s)$ around $s = 0$ (we assume that $\mathbf{W}(s)$ and, therefore, $\mathbf{W}_e(s)$ has all poles in $\mathbb{C}^-$)

$$\mathbf{W}_e(s) = \sum_{i=0}^{+\infty} \frac{d^i}{ds^i}\mathbf{W}_e\Big|_{s=0} \frac{s^i}{i!}, \tag{6}$$

on account of (5) a system is of type k if and only if

$$\begin{aligned}\mathbf{W}_e(s) &= \sum_{i=0}^{+\infty} \frac{d^i}{ds^i}\mathbf{W}_e\Big|_{s=0} \frac{s^i}{i!} = \sum_{i=k}^{+\infty} \frac{d^i}{ds^i}\mathbf{W}_e\Big|_{s=0} \frac{s^i}{i!} \\ &= s^k\Big(\frac{1}{k!}\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} + \sum_{i=k+1}^{+\infty} \frac{d^i}{ds^{i-k}}\mathbf{W}_e\Big|_{s=0} \frac{s^{i-k}}{i!}\Big) \\ &= s^k \widetilde{\mathbf{W}}_e(s)\end{aligned} \tag{7}$$

with $\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} \neq 0$, hence $\widetilde{\mathbf{W}}_e(0) \neq 0$, on account (4). Therefore, $\mathbf{W}_e$ has k zeroes in $s = 0$ or, equivalently, a zero $s = 0$ with multiplicity k.

Proposition *2.1: A system is of type k if and only if the error transfer function $\mathbf{W}_e$ has a zero $s = 0$ with multiplicity k.*

As established above, if $\mathbf{W}(s)$ is of type k then the steady-state error is constant and non-zero. Denote by $e_{ss}^{(k)}$ this constant and non-zero number. From (3)-(5) we get

$$\mathbf{e}_{ss}^{(k)}(t) = e_{ss}^{(k)} = \frac{1}{k!}\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} \tag{8}$$

Also, since (8) is exactly the coefficient of s^k in the Laurent expansion (7) ,

$$\mathbf{e}_{ss}^{(k)}(t) = e_{ss}^{(k)} = \frac{1}{s^k}\mathbf{W}_e(s)\Big|_{s=0} \tag{9}$$

which is an equivalent formula for computing the steady-state error $\mathbf{e}_{ss}^{(k)}(t)$ for a type-k system.

S. Battilotti is with Department of Computer, Control, and Management Engineering "Antonio Ruberti", Sapienza University of Rome, Via Ariosto 25, Italy.

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What is the value of the steady-state error $\mathbf{e}_{ss}^{(h)}(t)$, $h \geqslant 0$, with an input $\mathbf{v}(t) := \frac{t^h}{h!}$ if $\mathbf{W}(s)$ is of type $0 \leqslant k < h$ (or $k > h$)? If $\mathbf{W}(s)$ is of type $0 \leqslant k < h$ then (see (4)-(5))

$$\begin{aligned} &\frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} = 0,\ j = 0, 1, \ldots, k-1 \\ &\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} \neq 0 \end{aligned} \tag{10}$$

while the steady-state error $\mathbf{e}_{ss}^{(h)}(t)$ is (see (3))

$$\begin{aligned} \mathbf{e}_{(ss,h)}(t) &= \sum_{j=0}^{h} \frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} \frac{t^{h-j}}{(h-j)!j!} \\ &= \sum_{j=k}^{h} \frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} \frac{t^{h-j}}{(h-j)!j!} \end{aligned} \tag{11}$$

which goes (in norm) to infinity as $t \to +\infty$ since $\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} \neq 0$. On the other hand, if $\mathbf{W}(s)$ is of type $k > h$

$$\begin{aligned} &\frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} = 0,\ j = 0, 1, \ldots, k-1 \\ &\frac{d^k}{ds^k}\mathbf{W}_e\Big|_{s=0} \neq 0 \end{aligned} \tag{12}$$

and the steady-state error $\mathbf{e}_{ss}^{(h)}(t)$ is

$$\mathbf{e}_{ss}^{(h)}(t) = \sum_{j=0}^{h} \frac{d^j}{ds^j}\mathbf{W}_e\Big|_{s=0} \frac{t^{h-j}}{(h-j)!j!} = 0 \tag{13}$$

Proposition *2.2: The system $\mathbf{W}(s)$ is of type k if and only $\mathbf{W}_e(s)$ has a zero $s = 0$ with multiplicity k. Moreover, the steady-state error $\mathbf{e}_{ss}^{(h)}(t)$ of $\mathbf{W}(s)$ to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ with $h > k$ tends to infinity as $t \to +\infty$ and the steady-state error to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ is $\mathbf{e}_{ss}^{(h)}(t) = 0$ if $h < k$.*

If $\mathbf{W}$ (s) is the result of the feedback interconnection of $\mathbf{G}(s)\mathbf{P}(s)$ with unitary feedback, where $\mathbf{P}(s)$ is the controlled process and $\mathbf{G}(s)$ is the controller, we have

$$\mathbf{W}(s) = \frac{\mathbf{P}(s)\mathbf{G}(s)}{1 + \mathbf{P}(s)\mathbf{G}(s)}.$$

and

$$\mathbf{W}_e(s) = 1 - \mathbf{W}(s) = \frac{1}{1 + \mathbf{P}(s)\mathbf{G}(s)}.$$

Hence, $\mathbf{W}_e$ has a zero $s = 0$ with multiplicity k if and only if $\mathbf{P}(s)\mathbf{G}(s)$ has a pole $s = 0$ with multiplicity k. From Proposition 2.1 and (11) and (13) we conclude that

Proposition *2.3: The feedback interconnection $\mathbf{W}(s)$ of $\mathbf{P}(s)\mathbf{G}(s)$ with unitary feedback is of type k if and only $\mathbf{P}(s)\mathbf{G}(s)$ has a pole $s = 0$ with multiplicity k. Moreover, the steady-state error $\mathbf{e}_{ss}^{(h)}(t)$ of $\mathbf{W}(s)$ to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ with $h > k$ tends to infinity as $t \to +\infty$ and the steady-state error to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ is $\mathbf{e}_{ss}^{(h)}(t) = 0$ if $h < k$.*

Therefore, if the controlled process $\mathbf{P}(s)$ has a number of poles $h < k$ in $s = 0$ and we aim at a feedback system $\mathbf{W}(s)$ of type k (i.e. a constant and non-zero steady-state error with an input $\mathbf{v}(t) := \frac{t^k}{k!}$), it is necessary to introduce the additional $k - h$ poles in $s = 0$ through the controller $\mathbf{G}(s)$, i.e.

$$\mathbf{G}(s) = \frac{1}{s^{k-h}} \tag{14}$$

so that $\mathbf{P}(s)\mathbf{G}(s)$ has the required number of poles in $s = 0$. Since $\frac{1}{s}$ represents an *integral* control action ($\int y$) and since the transfer function of a series interconnection of $\mathbf{S}_1$ and $\mathbf{S}_2$ is the product of the transfer functions of $\mathbf{S}_1$ and $\mathbf{S}_2$, this is equivalent to apply a series of $k - h$ integral control actions directly to $\mathbf{P}(s)$. On the other hand, if the controlled process $\mathbf{P}(s)$ has a number of poles $h < k$ in $s = 0$ and we aim at a feedback system $\mathbf{W}(s)$ of type $> k$ (i.e. a zero steady-state error with an input $\mathbf{v}(t) := \frac{t^k}{k!}$), it is necessary to introduce *at least* $k - h + 1$ poles in $s = 0$ through the controller $\mathbf{G}(s)$, i.e.

$$\mathbf{G}(s) = \frac{1}{s^{k-h+l}},\ l \geqslant 1. \tag{15}$$

We can relate the situation enlighted by proposition 2.3 with the number of poles at $s = 0$ of $\mathbf{G}(s)\mathbf{P}(s)$.

Proposition *2.4: The feedback interconnection of $\mathbf{G}(s)\mathbf{P}(s)$ with unitary feedback i) has a constant and non-zero steady-state error $\mathbf{e}_{ss}^{(h)}(t)$ to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ if and only $\mathbf{G}(s)\mathbf{P}(s)$ has a pole in $s = 0$ with multiplicity h, ii) has steady-state error $\mathbf{e}_{ss}^{(h)}(t) = 0$ to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ if and only if $\mathbf{G}(s)\mathbf{P}(s)$ has a pole in $s = 0$ with multiplicity $k > h$ and iii) the steady-state error $\mathbf{e}_{ss}^{(h)}(t)$ to an input $\mathbf{v}(t) := \frac{t^h}{h!}$ tends to infinity as $t \to +\infty$ if and only if $\mathbf{G}(s)\mathbf{P}(s)$ has a pole $s = 0$ with multiplicity $k < h$.*

Notice that the Laplace transform of $\mathbf{v}(t) := \frac{t^k}{k!}$ is $\frac{1}{s^{k+1}}$. In order to have $\mathbf{e}_{ss}^{(k)}(t) = 0$, from proposition 2.3 it follows that $\mathbf{P}(s)$ should have a pole $s = 0$ with multiplicity $k + 1$ or, equivalently, $\mathbf{G}(s)\mathbf{P}(s)$ must contain a factor $\frac{1}{s^{k+1}}$ which is exactly the Laplace transform of $\mathbf{v}(t) := \frac{t^k}{k!}$. It should be also noticed that increasing the number of poles in $s = 0$ of $\mathbf{G}(s)\mathbf{P}(s)$ the closed-loop system $\mathbf{W}(s)$ become unstable, as it may be seen from the Nyquist plot, since for each pole at $s = 0$ in $\mathbf{G}(s)$ we have a clockwise half rotation of $\mathbf{G}(j\omega)\mathbf{P}(j\omega)$ around $-1 + 0j$ with ω varying from $-\infty$ to $+\infty$.

A. Steady-state error attenuation

Under this regard, instead of having $\mathbf{e}_{ss}^{(k)}(t) = 0$ with $k + 1$ integral actions in $\mathbf{G}(s)\mathbf{P}(s)$, it is however possible to require only k integral actions in $\mathbf{G}(s)\mathbf{P}(s)$ and reduce the non-zero steady-state error $e_{ss}^{(h)}$ within a given tolerance $M \in (0, 1)$ simply by increasing the value of $(s^k\mathbf{PG})|_{s=0}$, i.e. the *generalized gain* of $\mathbf{G}(s)\mathbf{P}(s)$, using a proportional action in $\mathbf{G}(s)$. To illustrate this point, assume that $\mathbf{P}(s)\mathbf{G}(s)$ has no zeroes at $s = 0$ (otherwise, we have a zero-pole cancellation at $s = 0$ with the integral actions provided in $\mathbf{P}(s)\mathbf{G}(s)$ for guaranteeing the tracking performances, resulting in internal instability). If $k = 0$ we have using (9)

$$\begin{aligned} \mathbf{e}_{ss}^{(0)}(t) = e_{ss}^{(0)} &= \mathbf{W}_e(s)|_{s=0} \\ &= \frac{1}{1 + \mathbf{G}(s)\mathbf{P}(s)}\Big|_{s=0} = \frac{1}{1 + (\mathbf{GP})(0)} \end{aligned} \tag{16}$$

Therefore, since $|\mathbf{P}(0)| < +\infty$ (when $k = 0$, $\mathbf{P}(s)\mathbf{G}(s)$ has no poles in $s = 0$), for each given tolerance $M \in (0, 1)$

$$\begin{aligned} |e_{ss}^{(0)}| \leqslant M \Leftrightarrow \Big|\frac{1}{1 + (\mathbf{GP})(0)}\Big| \leqslant M \\ \Leftrightarrow \Big\{\mathbf{G}(0) \geqslant -\frac{1}{\mathbf{P}(0)} + \frac{1}{|\mathbf{P}(0)|M}\Big\} \\ \bigcup \Big\{\mathbf{G}(0) \leqslant -\frac{1}{\mathbf{P}(0)} - \frac{1}{|\mathbf{P}(0)|M}\Big\} \end{aligned} \tag{17}$$

which means that $|\mathbf{G}(0)|$ must be selected sufficiently large. More simply, taking into account the inequality $|a + b| \geqslant |a| - |b|$ for all $a, b \in \mathbb{R}$, it is possible to characterize the inequality $|e_{ss}^{(0)}| \leqslant M$ with a smaller but simpler set of values of $\mathbf{G}(0)$ as

$$|e_{ss}^{(0)}| \leqslant M \Leftarrow |\mathbf{G}(0)| \geqslant \frac{1}{|\mathbf{P}(0)|}\Big(1 + \frac{1}{M}\Big). \tag{18}$$

Notice that, unlike (17) in which the arrows are bidirectional, the values of $|\mathbf{G}(0)| \geqslant \frac{1}{|\mathbf{P}(0)|}\Big(1 + \frac{1}{M}\Big)$ in (18) imply $|e_{ss}^{(0)}|$ but the converse is false in general. However, for simplicity, we consider the set of values of $\mathbf{G}(0)$ in (18) to guarantee the attenuation condition $|e_{ss}^{(0)}| \leqslant M$.

On the other hand, using (9), for $k > 0$

$$\begin{aligned} \mathbf{e}_{ss}^{(k)}(t) = e_{ss}^{(k)} = \frac{1}{s^k}\mathbf{W}_e(s)\Big|_{s=0} = \frac{1}{s^k}\frac{1}{1 + \mathbf{G}(s)\mathbf{P}(s)}\Big|_{s=0} \\ = \frac{1}{s^k + s^k\mathbf{G}(s)\mathbf{P}(s)}\Big|_{s=0} = \frac{1}{(s^k\mathbf{GP})|_{s=0}} \end{aligned} \tag{19}$$

Therefore, for each given tolerance $M \in (0, 1)$

$$\begin{aligned} |e_{ss}^{(k)}| \leqslant M \Leftrightarrow \Big|\frac{1}{(s^k\mathbf{GP})|_{s=0}}\Big| \leqslant M \\ \Leftrightarrow |(s^k\mathbf{GP})|_{s=0}| \geqslant \frac{1}{M} \end{aligned} \tag{20}$$

which means that $(s^k\mathbf{GP})|_{s=0}$ must be selected sufficiently large (through the proportional action $\mathbf{G}(0) = K$). Notice that in this case $\mathbf{P}(s)$ may contain some poles at $s = 0$ so that $\mathbf{P}(0)$ may be not finite.

Since the control system may assume different forms from a standard feedback interconnection with unitary feedback, for the design of the controller $\mathbf{G}(s)$ it is better to use the results of Proposition 2.2 which are independent of the internal structure of the control system.

B. The internal model principle

It is possible to consider inputs to be tracked different from polynomials $\frac{t^k}{k!}$, as for instance sinusoidal or exponential inputs. Let $\mathbf{W}(s)$ be a given (asymptotically stable) system and $\mathbf{W}_e(s) = 1 - \mathbf{W}(s)$ its error transfer function. The following result is a consequence of the formula in Laplace domain for the forced error response $\mathbf{e}(s)$ to an input $\mathbf{v}(s)$, taking into account that the poles of $\mathbf{W}(s)$ and, therefore, of $\mathbf{W}_e(s)$ are in $\mathbb{C}^-$.

Theorem *2.1: (Internal model principle) A system $\mathbf{W}(s)$ is such that $\mathbf{e}_{ss}(t) = 0$ with input $\mathbf{v}(t)$ if $\mathbf{W}(s)$ is asymptotically stable and $\mathbf{W}_e(s)$ has among its zeroes all the poles of $\mathfrak{L}[\mathbf{v}(t)](s)$.*

Notice that when $\mathbf{v}(t) = \frac{t^k}{k!}$ we recover the results of the previous section. In particular, $\mathbf{W}_e(s)$ must contain (at least) a factor s^{k+1} or, equivalently, a zero $s = 0$ with multiplicity (at least) $k + 1$ or, equivalently, $\mathbf{W}(s)$ must be of type (at least) $k + 1$.

Let us consider a feedback system $\mathbf{W}(s)$ with unitary feedback and $\mathbf{G}(s)\mathbf{P}(s)$ on the direct path, $\mathbf{P}(s)$ the controlled process and $\mathbf{G}(s)$ the controller, i.e. $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ and $\mathbf{W}_e(s) = \frac{1}{1+\mathbf{G}(s)\mathbf{P}(s)}$.

Theorem *2.2: (Internal model principle for feedback systems). The feedback system $\mathbf{W}(s)$ is such that $\mathbf{e}_{ss}(t) = 0$ with input $\mathbf{v}(t)$ if $\mathbf{W}(s)$ is asymptotically stable and $\mathbf{G}(s)\mathbf{P}(s)$ has among its poles all the poles of $\mathfrak{L}[\mathbf{v}(t)](s)$.*

Notice that when $\mathbf{v}(t) = \frac{t^k}{k!}$ we recover the results of the previous section. In particular, $\mathbf{G}(s)\mathbf{P}(s)$ must contain (at least a factor) $\frac{1}{s^{k+1}}$ or, equivalently, a pole $s = 0$ with multiplicity (at least) $k + 1$ (compare with Proposition 2.4).

We will show the principle for feedback systems in the particular case

$$\mathbf{v}(t) = \alpha\cos(\omega t) + \beta\sin(\omega t).$$

We have

$$\mathfrak{L}[\mathbf{v}(t)](s) = \frac{\alpha s + \beta\omega}{s^2 + \omega}$$

The feedback system has error transfer function

$$\mathbf{W}_e(s) = 1 - \mathbf{W}(s) = \frac{1}{1 + \mathbf{G}(s)\mathbf{P}(s)}$$

Let assume for $\mathbf{G}(s)\mathbf{P}(s)$ a form

$$\mathbf{G}(s)\mathbf{P}(s) = \mathbf{P}'(s)\frac{1}{s^2 + \omega}$$

i.e. $\mathbf{G}(s)\mathbf{P}(s)$ has among its poles all the poles of $\mathfrak{L}[\mathbf{v}(t)](s)$, and for $\mathbf{P}'(s)$ a form

$$\mathbf{P}'(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)}$$

We also assume that $\mathbf{W}(s)$ is asymptotically stable. Since

$$\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1 + \mathbf{G}(s)\mathbf{P}(s)} = \frac{\mathbf{n}(s)}{\mathbf{n}(s) + \mathbf{d}(s)(s^2 + \omega)},$$

the polynomial $\mathbf{n}(s) + \mathbf{d}(s)(s^2 + \omega)$ is Hurwitz. It follows that

$$\mathbf{W}_e(s) = \frac{1}{1 + \mathbf{G}(s)\mathbf{P}(s)} = \frac{\mathbf{d}(s)(s^2 + \omega)}{\mathbf{n}(s) + \mathbf{d}(s)(s^2 + \omega)}$$

Therefore,

$$\begin{aligned} \mathfrak{L}[\mathbf{e}(t)](s) = \mathbf{W}_e(s)\mathfrak{L}[\mathbf{v}(t)](s) \\ = \frac{\mathbf{d}(s)(\alpha s + \beta\omega)}{\mathbf{n}(s) + \mathbf{d}(s)(s^2 + \omega)} \end{aligned} \tag{21}$$

Since the polynomial $\mathbf{n}(s) + \mathbf{d}(s)(s^2 + \omega)$ is Hurwitz and using the residuals theorem (module III)

$$\mathbf{e}(t) = \mathfrak{L}^{-1}\Big[\frac{\mathbf{d}(s)(\alpha s + \beta\omega)}{\mathbf{n}(s) + \mathbf{d}(s)(s^2 + \omega)}\Big](t) \to 0 \tag{22}$$

as $t \to +\infty$, i.e. $\mathbf{e}_{ss}(t) = 0$.

III. Disturbance rejection and attenuation

The capability of the output to follow or track (with a certain steady-state error) a given input (*tracking*) must be retained also in the presence of disturbances or noise introduced in the control-loop by actuators and sensors. If we consider the disturbance d as an input for the system we can inherit all the conclusions of the previous section as long as we replace e with y and v with d. Let $\mathbf{W}(s)$ be a given system and $\mathbf{W}_d$ be the *disturbance-to-output transfer function*

$$\mathbf{W}_d(s) := \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} \tag{23}$$

Here, $\mathbf{y}(t)$ is the forced output response to a disturbance $\mathbf{d}(t)$ (i.e. zeroing all the other inputs as well as $\mathbf{v}(t)$). By following the conclusions of proposition 2.2 we have the following result.

Proposition *3.1: A system $\mathbf{W}(s)$ has a constant and non-zero steady-state output response $\mathbf{y}_{ss}^{(k)}$ to a disturbance $\mathbf{d}(t) := \frac{t^k}{k!}$ if and only if $\mathbf{W}_d(s)$ has a zero in $s = 0$ with multiplicity k. The value of this constant non-zero steady-state output response is*

$$y_{ss}^{(k)} := \frac{1}{s^k}\mathbf{W}_d(s)\Big|_{s=0} = \frac{1}{k!}\frac{d^k}{d^k s}\mathbf{W}_d\Big|_{s=0}. \tag{24}$$

Moreover, if $\mathbf{W}_d(s)$ has a zero in $s = 0$ with multiplicity k the steady-state output response $\mathbf{y}_{ss}^{(h)}(t)$ to a disturbance $\mathbf{d}(t) := \frac{t^h}{h!}$ tends to infinity as $t \to +\infty$ if $h > k$ and $\mathbf{y}_{ss}^{(h)}(t) = 0$ if $h < k$.

Note that the conditions for zeroing the steady-state output response to a disturbance $\mathbf{d}(t) := \frac{t^k}{k!}$ and the steady-state error response to an input $\mathbf{v}(t) := \frac{t^k}{k!}$ are the same. Since the output response of the closed–loop system with disturbance d and input v is the superposition of the input-to-output response contributed by v (with $d = 0$) and the disturbance-to-output response contributed by d (with $v = 0$), it follows that if the steady-state error response to an input $\mathbf{v}(t) := \frac{t^k}{k!}$ is zero in the absence of disturbances it remains zero even in the presence of a disturbance $\mathbf{d}(t) := \frac{t^k}{k!}$.

In general, the disturbance may affect the system in different ways, depending on its physical source. We consider only the cases for which the disturbance adds up to the output of $\mathbf{P}(s)$ (*additive output disturbances*) or to the input of $\mathbf{P}(s)$ (*additive input disturbances*). The additive output disturbances correspond to disturbances introduced in the control loop by measurement devices (sensors), while additive input disturbances correspond to disturbances introduced in the control loop by actuating devices (actuators). Moreover, we will consider a feedback system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$.

A. Output disturbances

For additive output disturbances we assume $\mathbf{y} = \mathbf{y}' + \mathbf{d}$ (see Fig 1 where $\mathbf{G}(s) = 1$) where $\mathbf{y}$ is the output of the feedback system $\mathbf{W}(s)$ and $\mathbf{y}'$ is the output of $\mathbf{G}(s)\mathbf{P}(s)$. We compute

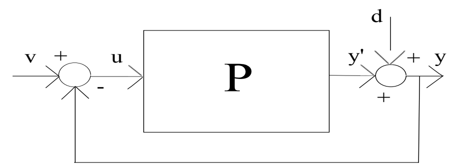

Figure 1. Output disturbances.

$\mathbf{W}_d(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)}$ by setting $\mathbf{v} = 0$. Since in the feedback interconnection $\mathbf{u} = -\mathbf{y}$ when $\mathbf{v} = 0$, we have

$$\begin{aligned}
&\mathbf{W}_d(s) := \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} \\
&= \frac{\mathfrak{L}[\mathbf{y}'(t)](s) + \mathfrak{L}[\mathbf{d}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} \\
&= \frac{\mathbf{G}(s)\mathbf{P}(s)\mathfrak{L}[\mathbf{u}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} + 1 = -\frac{\mathbf{G}(s)\mathbf{P}(s)\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} + 1 \\
&= -\mathbf{G}(s)\mathbf{P}(s)\mathbf{W}_d(s) + 1
\end{aligned}$$

Therefore,

$$\mathbf{W}_d(s) = \frac{1}{1 + \mathbf{P}(s)} \tag{25}$$

Since in this case $\mathbf{W}_d(s) = \mathbf{W}_e(s)$, from proposition 2.4

Proposition *3.2: The feedback interconnection of $\mathbf{P}(s)\mathbf{G}(s)$ with unitary feedback and additive output disturbance $\mathbf{d}(t)$ i) has a constant and non-zero steady-state output response $\mathbf{y}_{ss}^{(h)}(t)$ to a disturbance $\mathbf{d}(t) := \frac{t^h}{h!}$ if and only $\mathbf{P}(s)\mathbf{G}(s)$ has a pole in $s = 0$ with multiplicity h, ii) has zero steady-state output response $\mathbf{y}_{ss}^{(h)}(t)$ to a disturbance $\mathbf{d}(t) := \frac{t^h}{h!}$ if and only if $\mathbf{P}(s)\mathbf{G}(s)$ has a pole in $s = 0$ with multiplicity $k > h$ and iii) the steady-state output response $\mathbf{y}_{ss}^{(h)}(t)$ to a disturbance $\mathbf{d}(t) := \frac{t^h}{h!}$ tends to infinity as $t \to +\infty$ if and only if $\mathbf{P}(s)\mathbf{G}(s)$ has a pole $s = 0$ with multiplicity $k < h$.*

Therefore, if the open-loop process $\mathbf{P}(s)$ has a number of poles $h < k$ in $s = 0$ and we aim at a constant and non-zero steady-state output response $\mathbf{y}_{ss}^{(k)}(t)$ it is necessary to introduce the additional $k - h$ poles in $s = 0$ in $\mathbf{G}(s)$, i.e.

$$\mathbf{G}(s) = \frac{1}{s^{k-h}} \tag{26}$$

On the other hand, if the open-loop process $\mathbf{P}(s)$ has a number of poles $h < k$ in $s = 0$ and we aim at a steady-state output response $\mathbf{y}_{ss}^{(k)}(t) = 0$ it is necessary to introduce (at least) additional $k - h + 1$ poles at $s = 0$ in $\mathbf{G}(s)$, i.e.

$$\mathbf{G}(s) = \frac{1}{s^{k-h+l}},\ l \geqslant 1. \tag{27}$$

B. Output disturbance attenuation

Let $\mathbf{y}_{ss}^{(k)}(t)$ be the steady-state output response to a disturbance $\mathbf{d}(t) = \frac{t^k}{k!}$ and $y_{ss}^{(k)}$ denote its constant and non-zero value. We see how to reduce $y_{ss}^{(k)}$ within a given tolerance $M \in (0,1)$ by means of suitable control actions. We assume that $\mathbf{GP}$ does not have zeroes at $s = 0$ (see section II-A). Since in this case $\mathbf{W}_d(s) = \mathbf{W}_e(s)$ and taking into account (24), for each given tolerance $M \in (0,1)$ and if $k = 0$

$$\begin{aligned} &|\mathbf{y}_{ss}^{(0)}(t)| = |y_{ss}^{(0)}| \leqslant M \Leftrightarrow |\frac{1}{1+\mathbf{P}(0)\mathbf{G}(0)}| \leqslant \\ &= M \Leftrightarrow \Big\{\mathbf{G}(0) \geqslant -\frac{1}{\mathbf{P}(0)} + \frac{1}{|\mathbf{P}(0)|M}\Big\} \\ &\bigcup \Big\{\mathbf{G}(0) \leqslant -\frac{1}{\mathbf{P}(0)} - \frac{1}{|\mathbf{P}(0)|M}\Big\} \end{aligned} \tag{28}$$

which once again means that $|\mathbf{G}(0)|$ must be selected sufficiently large. Next, taking into account (24), for each given tolerance $M \in (0,1)$ and for $k > 0$

$$\begin{aligned} &|\mathbf{y}_{ss}^{(k)}(t)| = |y_{ss}^{(k)}| \leqslant M \Leftrightarrow |\frac{1}{(s^k\mathbf{GP})|_{s=0}}| \leqslant M \\ &\Leftrightarrow |(s^k\mathbf{GP})|_{s=0}| \geqslant \frac{1}{M} \end{aligned} \tag{29}$$

which means that the generalized gain of $\mathbf{GP}$ must be selected sufficiently large in absolute value (through the proportional action in $\mathbf{G}(\mathbf{s})$).

C. Input disturbances

For additive input disturbances we assume $\mathbf{u}'(s) = \mathbf{G}(s)(\mathbf{v}(s) - \mathbf{y}(s)) + \mathbf{d}(s)$ (see Fig. 2) where $\mathbf{u}'$ is the input of $\mathbf{P}(s)$ and $\mathbf{y}$ is the output of $\mathbf{P}(s)$. We compute $\mathbf{W}_d(s) = \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)}$ by setting $\mathbf{v} = 0$. Since in the feedback interconnection $\mathbf{u}'(s) = -\mathbf{G}(s)\mathbf{y}(s) + \mathbf{d}(s)$ when $\mathbf{v} = 0$, we have

$$\begin{aligned} \mathbf{W}_d(s) &:= \frac{\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} = \frac{\mathbf{P}(s)\mathbf{u}'(s)}{\mathfrak{L}[\mathbf{d}(t)](s)} \\ &= \frac{\mathbf{P}(s)(-\mathbf{G}(s)\mathfrak{L}[\mathbf{y}(t)](s) + \mathfrak{L}[\mathbf{d}(t)](s))}{\mathfrak{L}[\mathbf{d}(t)](s)} \\ &= -\frac{\mathbf{P}(s)\mathbf{G}(s)\mathfrak{L}[\mathbf{y}(t)](s)}{\mathfrak{L}[\mathbf{d}(t)](s)} + \mathbf{P}(s) \\ &= -\mathbf{P}(s)\mathbf{G}(s)\mathbf{W}_d(s) + \mathbf{P}(s) \end{aligned}$$

Therefore,

$$\mathbf{W}_d(s) = \frac{\mathbf{P}(s)}{1 + \mathbf{G}(s)\mathbf{P}(s)} \tag{30}$$

Notice that, if $\mathbf{P}(s)$ has no zeroes at $s = 0$, $\mathbf{W}_d(s)$ has a zero at $s = 0$ with multiplicity k if and only if $\mathbf{G}(s)$ has k poles at $s = 0$, independently of the fact that $\mathbf{P}(s)$ may have or not some poles at $s = 0$.

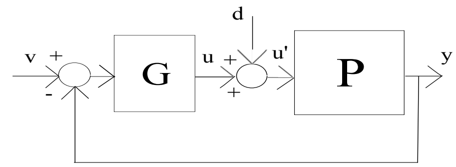

Figure 2. Control action at the entrance point of the disturbance.

Proposition *3.3: Assume that $\mathbf{P}(s)$ has no zeroes in $s = 0$. The feedback interconnection of $\mathbf{G}(s)\mathbf{P}(s)$ with unitary feedback and additive input disturbance $\mathbf{d}(t)$ i) has a constant and non-zero steady-state output response $\mathbf{y}_{ss}^{(k)}(t)$ to a disturbance $\mathbf{d}(t) := \frac{t^k}{k!}$ if and only $\mathbf{G}(s)$ has a pole $s = 0$ with multiplicity k, ii) has a steady-state output response $\mathbf{y}_{ss}^{(h)}(t) = 0$ to a disturbance $\mathbf{d}(t) := \frac{t^h}{h!}$ if and only if $\mathbf{G}(s)$ has a pole $s = 0$ with multiplicity $k > h$ and iii) has a steady-state output response $\mathbf{y}_{ss}^{(h)}(t) = 0$ to a disturbance $\mathbf{d}(t) := \frac{t^h}{h!}$ which tends to infinity as $t \to +\infty$ if and only if $\mathbf{G}(s)$ has a pole $s = 0$ with multiplicity $k < h$.*

We can also use the internal model principle to state the following result for disturbances different from polynomial disturbances $\mathbf{d}(t) = \frac{t^k}{k!}$. Let us consider a system $\mathbf{W}(s)$ with output $\mathbf{y}(t)$, disturbance $\mathbf{d}(t)$ and disturbance-to-output transfer function $\mathbf{W}_d(s)$.

Theorem *3.1: The system $\mathbf{W}(s)$ is such that $\mathbf{y}_{ss}(t) = 0$ with input $\mathbf{d}(t)$ if $\mathbf{W}(s)$ is asymptotically stable and $\mathbf{W}_d(s)$ has among its zeroes all the poles of $\mathfrak{L}[\mathbf{d}(t)](s)$.*

In using this result, we have to design the controller $\mathbf{G}(s)$ inside the control loop in such a way that $\mathbf{W}_d(s)$ has the claimed properties, i.e. all the poles of $\mathfrak{L}[\mathbf{v}(t)](s)$ among its zeroes. Moreover, if $\mathbf{d}(t) = \frac{t^k}{k!}$ and for feedback systems $\mathbf{W}(s)$ we recover the results of propositions 3.3.

D. Input disturbance attenuation

Let see how to reduce a non-zero constant steady-state output response $\mathbf{y}_{ss}^{(k)}(t)$ to a disturbance $\mathbf{d}(t) = \frac{t^k}{k!}$. We still assume that $\mathbf{GP}$ has no zeroes at $s = 0$. We relate its constant and non-zero value $y_{ss}^{(k)}$ with $(s^k\mathbf{GP})|_{s=0}$, the generalized gain of $\mathbf{G}(s)\mathbf{P}(s)$. For each given tolerance $M \in (0,1)$ and if $k = 0$

$$|\mathbf{y}_{ss}^{(0)}(t)| = |y_{ss}^{(0)}| \leqslant M \Leftrightarrow \Big|\frac{\mathbf{P}}{1+\mathbf{GP}}\Big|_{s=0} \leqslant M$$
$$\Leftrightarrow \Big\{\mathbf{G}(0) \geqslant -\frac{1}{\mathbf{P}(0)} + \frac{1}{M}\Big\}$$
$$\bigcup \Big\{\mathbf{G}(0) \leqslant -\frac{1}{\mathbf{P}(0)} - \frac{1}{M}\Big\} \tag{31}$$

which means that $|\mathbf{G}(0)|$ must be selected sufficiently large. Next, for each given tolerance $M \in (0,1)$ and $k > 0$

$$|\mathbf{y}_{ss}^{(k)}(t)| = |y_{ss}^{(k)}| \leqslant M \Leftrightarrow \Big|\frac{1}{s^k}\frac{\mathbf{P}}{1+\mathbf{GP}}\Big|_{s=0} \leqslant M.$$
$$\Leftrightarrow \Big|\frac{1}{s^k}\frac{\mathbf{P}}{\mathbf{GP}}\Big|_{s=0} \leqslant M \Leftrightarrow |(s^k\mathbf{G})|_{s=0}| \geqslant \frac{1}{M} \tag{32}$$

which means that $|(s^k\mathbf{G})|_{s=0}|$ must be selected sufficiently large (through a proportional action K in $\mathbf{G}(s)$).

Since the way disturbances may affect the feedback systems may be various inside the loop and not always be recognizable as the cases III-A and III-C and the control system may assume different forms from a standard feedback form, for the design of the controller $\mathbf{G}(s)$ it is better to use the results of Proposition 3.1 which are independent of the way disturbances affect the feedback system and from the internal structure of the control system.

Exercize *3.1: With reference to the feedback system in Figure 3-a with*

$$\mathbf{P}(s) = \frac{s-2}{s^3-2s^2-2}$$

design the controller $\mathbf{G}(s)$ *in such a way that the (i) the steady-state error response to sinusoidal inputs* $\mathbf{v}(t) = \sin(t)$ *is zero (ii) the steady-state output response to constant disturbances* $\mathbf{d}_2(t)$ *is zero and (iii) the steady-state output response to disturbances* $\mathbf{d}_1(t) = t$ *is in absolute value less than* 0.1.

Notice that the disturbances $\mathbf{d}_1(t)$ and $\mathbf{d}_2(t)$ enter the loop according to a paradigm which is not covered by the analogue cases studied in Figures 1 and 2. In general, it is convenient to write for the closed-loop system the Laplace transform of the output as a function of the Laplace transforms of the inputs and disturbances so as to identify each closed-loop transfer function (input to output and disturbance to output) and apply the results we have discussed above to each single transfer function. First we write the Laplace transform of the output $\mathbf{y}(t)$ as a function of the Laplace transforms of the input $\mathbf{m}(t)$ and disturbances $\mathbf{d}_1(t)$ and $\mathbf{d}_2(t)$ (see Figure 3-b). To do this, we determine:

- the disturbance $\mathbf{d}_2$ to output $\mathbf{y}$ (open loop) transfer function $\mathbf{W}_{d_2,y}(s)$ (setting $m = 0$ and $\mathbf{d}_2 = 0$):

$$\mathbf{W}_{d_2,y}(s) = \frac{1}{1+\mathbf{P}(s)} \tag{33}$$

- the disturbance $\mathbf{d}_1$ to output $\mathbf{y}$ (open loop) transfer function $\mathbf{W}_{d_1,y}(s)$ is (setting $m = 0$ and $\mathbf{d}_1 = 0$):

$$\mathbf{W}_{d_1,y}(s) = \frac{\mathbf{P}(s)}{1+\mathbf{P}(s)} \tag{34}$$

- the input $\mathbf{m}$ to output $\mathbf{y}$ (open loop) transfer function $\mathbf{W}_{m,y}(s)$ is (setting $\mathbf{d}_1 = 0$ and $\mathbf{d}_2 = 0$):

$$\mathbf{W}_{m,y}(s) = \frac{\mathbf{P}(s)}{1+\mathbf{P}(s)} \tag{35}$$

Therefore, as in Figure 3-b we obtain the equivalent diagram block from the inputs $,\mathbf{d}_1,\mathbf{d}_2$ to the output $\mathbf{y}$

$$\mathbf{y}(s) = \mathbf{W}_{m,y}(s)\mathbf{m}(s) + \mathbf{W}_{d_1,y}(s)\mathbf{d}_1(s) + \mathbf{W}_{d_2,y}(s)\mathbf{d}_2(s) \tag{36}$$

Next, in the feedback system from $\mathbf{m}$ to $\mathbf{y}$ we replace the relation (36) so as to obtain Figure 3-c.

Now, we can easily determine

- the input $\mathbf{v}$ to output $\mathbf{y}$ closed-loop transfer function $\mathbf{W}(s)$ (setting $\mathbf{d}_1 = 0$ and $\mathbf{d}_2 = 0$):

$$\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{W}_{m,y}(s)}{1+\mathbf{G}(s)\mathbf{W}_{m,y}(s)} = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{P}(s)+\mathbf{G}(s)\mathbf{P}(s)} \tag{37}$$

- the disturbance $\mathbf{d}_2$ to output $\mathbf{y}$ closed-loop transfer function $\mathbf{W}_{d_2}(s)$ (setting $\mathbf{v} = 0$ and $\mathbf{d}_1 = 0$):

$$\mathbf{W}_{d_2}(s) = \frac{\mathbf{W}_{d_2,y}(s)}{1+\mathbf{G}(s)\mathbf{W}_{m,y}(s)} = \frac{1}{1+\mathbf{P}(s)+\mathbf{G}(s)\mathbf{P}(s)} \tag{38}$$

- the disturbance $\mathbf{d}_1$ to output $\mathbf{y}$ closed-loop transfer function $\mathbf{W}_{d_1}(s)$ is (setting $v = 0$ and $\mathbf{d}_2 = 0$)

$$\mathbf{W}_{d_1}(s) = \frac{\mathbf{W}_{\mathbf{d}_1,y}(s)}{1+\mathbf{G}(s)\mathbf{W}_{m,y}(s)} = \frac{\mathbf{P}(s)}{1+\mathbf{P}(s)+\mathbf{G}(s)\mathbf{P}(s)} \tag{39}$$

Therefore, the output of the closed-loop system is

$$\mathbf{y}(s) = \mathbf{W}(s)\mathbf{v}(s) + \mathbf{W}_{d_1}(s)\mathbf{d}_1(s) + \mathbf{W}_{\mathbf{d}_2}(s)\mathbf{d}_2(s) \tag{40}$$

Next, to meet the requirements (i)-(iii) we analyse each transfer function $\mathbf{W}(s)$, $\mathbf{W}_{d_2}(s)$ and $\mathbf{W}_{\mathbf{d}_1}(s)$.

Let us begin with (i). In order to have that a steady-state error response to sinusoidal inputs $\mathbf{v}(t) = \sin(t)$ equal to the error transfer function $\mathbf{W}_e(s) = 1 - \mathbf{W}(s)$ must have among its zeroes all the poles of the Laplace transform of $\mathbf{v}(t) = \sin(t)$ (by the internal model principle, restated version). Since

$$\mathfrak{L}[\sin(t)](s) = \frac{1}{s^2+1}$$

and

$$\mathbf{W}_e(s) = \frac{1+\mathbf{P}(s)}{1+\mathbf{P}(s)+\mathbf{G}(s)\mathbf{P}(s)} \tag{41}$$

we assume for $\mathbf{G}(s)$ a structure

$$\mathbf{G}(s) = \mathbf{G}_1(s)\frac{1}{s^2+1} \tag{42}$$

As to point (ii), in order to have a steady-state output response to constant disturbances $\mathbf{d}_2(t)$ equal to zero the transfer function $\mathbf{W}_{d_2}(s)$ must have a zero at $s = 0$. Since

$$\mathbf{W}_{d_2}(s) = \frac{1}{1+\mathbf{P}(s)+\mathbf{G}(s)\mathbf{P}(s)} \tag{43}$$

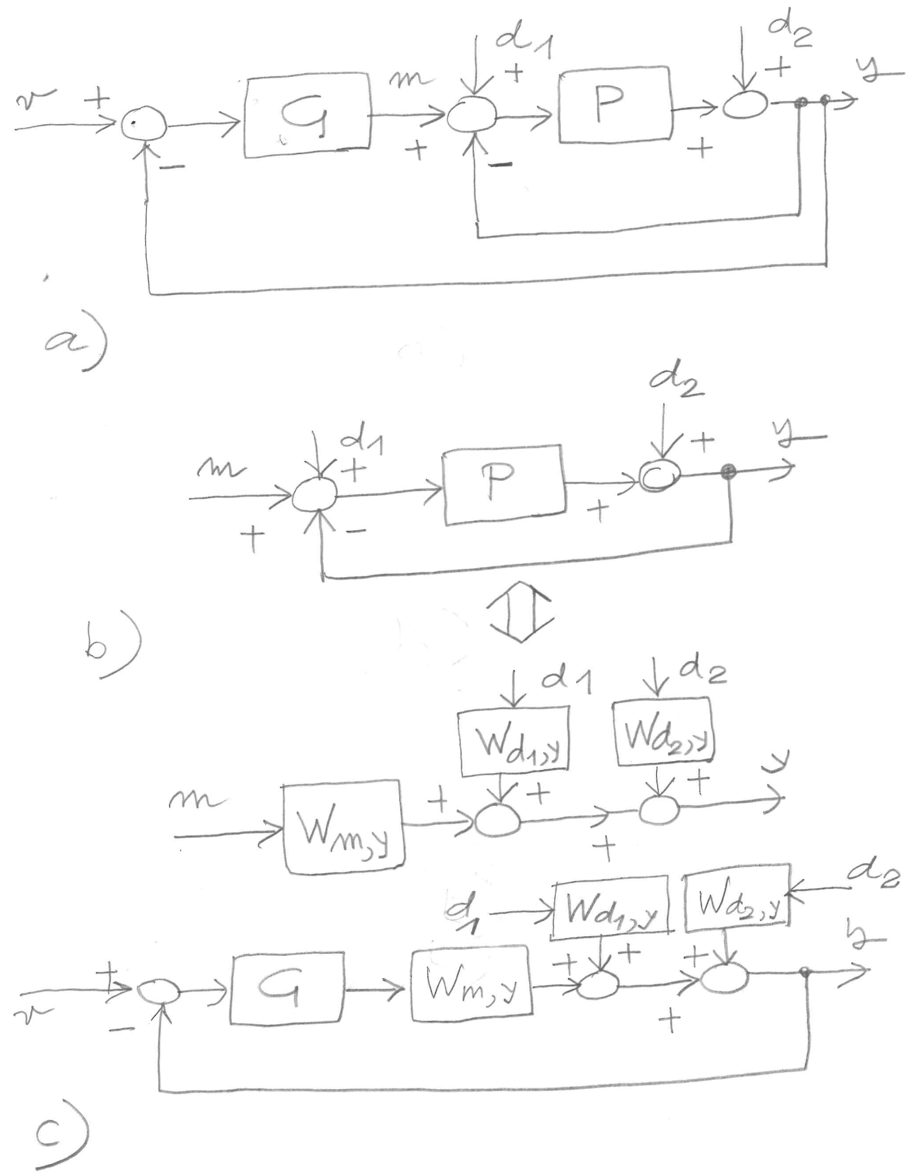

Figure 3. Exercize 3.1.

this clearly implies that $\mathbf{G}_1(s)$ must have a pole at $s = 0$. For this reason we assume for $\mathbf{G}(s)$ a structure

$$\mathbf{G}_1(s) = \mathbf{G}_2(s)\frac{1}{s} \tag{44}$$

We proceed with (iii). In order to have that the steady-state output response to disturbances $\mathbf{d}_1(t) = t$ is in absolute value less than 0.1, the transfer function $\mathbf{W}_{d_1}(s)$ must have a zero at $s = 0$ (this guarantees that the steady-state output response is finite). Moreover, in order to guarantee that the absolute value of the steady-state output response is less than 0.1 we must have

$$|\mathbf{y}_{ss}^{(1)}(t)| = \Big|\frac{\mathbf{W}_{d_1}(s)}{s}\Big|_{s=0} < 0.1 \tag{45}$$

We assume for $\mathbf{G}_2(s)$ a structure

$$\mathbf{G}_2(s) = \mathbf{G}_3(s)K \tag{46}$$

Taking into account (46)

$$\Big|\frac{\mathbf{P}(s)}{s(1+\mathbf{P}(s)+\mathbf{G}(s)\mathbf{P}(s))}\Big|_{s=0} = \Big|\frac{\mathbf{P}(s)}{s\mathbf{P}(s)(1+\mathbf{G}(s))}\Big|_{s=0}$$

$$= \Big|\frac{1}{s\mathbf{G}(s)}\Big|_{s=0} = \Big|\frac{1}{\mathbf{G}_3(0)K}\Big|_{s=0} < 0.1 \tag{47}$$

Therefore, $|\mathbf{G}_3(0)K| > 10$. We can pick $K = 15$ with $|\mathbf{G}_3(0)| \geqslant 1$. The controller

$$\mathbf{G}(s) = \mathbf{G}_3(s)\frac{15}{s(s^2+1)} \tag{48}$$

meets the requirements (i)-(iii) as long as $|\mathbf{G}_3(0)| \geqslant 1$. The remaining part $\mathbf{G}_3(s)$ of the controller $\mathbf{G}(s)$, with $|\mathbf{G}_3(0)| \geqslant 1$, will be designed to stabilize the closed-loop system (for instance, using root locus based techniques: see module X). □

Exercize *3.2: With reference to the feedback system in Figure 4-a where*

$$\mathbf{P}_1(s) = \frac{1}{s-1},\ \mathbf{P}_2(s) = \frac{s+4}{s-2}$$

(i) design the controller $\mathbf{G}_1(s)$ in such a way the forced output response $\mathbf{y}(t)$ to any disturbance $\mathbf{d}(t)$ is 0,
(ii) design the controller $\mathbf{G}_2(s)$ in such a way the feedback system is asymptotically stable and the absolute value of the steady-state error to input $\mathbf{v}(t) = t$ is $\leqslant 1$.

Before going to the solution of our problem, it is convenient to transform the control scheme in Figure 4-a in such a way to easily determine the closed-loop transfer function $\mathbf{W}_d(s)$ from d to y and, respectively, the closed-loop transfer function $\mathbf{W}(s)$ from v to y. To this aim, first consider in Figure 4-a the signal flow from the inputs $\mathbf{d}(t)$ and $\mathbf{m}(t)$ to the output $\mathbf{y}(t)$ and write $\mathbf{y}(s) = \mathfrak{L}[\mathbf{y}(t)](s)$ as function of $\mathbf{m}(s) = \mathfrak{L}[\mathbf{m}(t)](s)$ and $\mathbf{d}(s) = \mathfrak{L}[\mathbf{d}(t)](s)$ as in Figure 4-b:

$$\mathbf{y}(s) = \mathbf{W}_{d,y}(s)d(s) + \mathbf{W}_{m,y}(s)m(s)$$

where $\mathbf{W}_{d,y}(s)$ is the open loop transfer function from $\mathbf{d}$ to $\mathbf{y}$ and, respectively, $\mathbf{W}_{m,y}(s)$ is the open loop transfer function from $\mathbf{m}$ to $\mathbf{y}$. For determining $\mathbf{W}_{d,y}(s)$ we set $\mathbf{m} = 0$ and for determining $\mathbf{W}_{m,y}(s)$ we set $\mathbf{d} = 0$. We have (in particular, notice that $\mathbf{W}_{d,y}(s)$ is the parallel interconnection of 1 and $-\mathbf{G}_1(s)\mathbf{P}_2(s)$)

$$\mathbf{W}_{d,y}(s) = 1 - \mathbf{G}_1(s)\mathbf{P}_2(s),\ \mathbf{W}_{m,y}(s) = \mathbf{P}_2(s).$$

Next, replace the block diagram of Figure 4-b in the feedback loop of Figure 4-a so to obtain the equivalent feedback loop of Figure 4-c. From this we can easily determine the transfer function $\mathbf{W}_d(s)$ from $\mathbf{d}$ to $\mathbf{y}$ (by setting $\mathbf{v} = 0$) and, respectively, the transfer function $\mathbf{W}(s)$ from $\mathbf{v}$ to $\mathbf{y}$ (by setting $d = 0$):

$$\begin{aligned}\mathbf{W}_d(s) &= \mathbf{W}_{d,y}(s)\frac{1}{1+\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{W}_{m,y}(s)}\\ &= \frac{1-\mathbf{G}_1(s)\mathbf{P}_2(s)}{1+\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{P}_2(s)},\\ \mathbf{W}(s) &= \frac{\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{W}_{m,y}(s)}{1+\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{W}_{m,y}(s)}\\ &= \frac{\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}{1+\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}.\end{aligned} \tag{49}$$

From this we also obtain the error transfer function

$$\mathbf{W}_e(s) = 1 - \mathbf{W}(s) = \frac{1}{1+\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}. \tag{50}$$

(i) For guaranteeing that $\mathbf{y}(t) \equiv 0$ for all disturbances $\mathbf{d}(t)$, one has to define $\mathbf{G}_1(s)$ in such a way that $\mathbf{W}_d(s) \equiv 0$. This is achieved by setting $\mathbf{G}_1(s)\mathbf{P}_2(s) = 1$ that is

$$\mathbf{G}_1(s) = \frac{s-2}{s+4}.$$

(ii) For guaranteeing that the steady-state error $\mathbf{e}_{ss}^{(1)}(t)$ in response to ramp references $\mathbf{v}(t) = t$ satisfies $|\mathbf{e}_{ss}^{(1)}(t)| \leqslant 1$, the transfer function $\mathbf{W}_e(s)$ must have one zero at $s = 0$ and moreover

$$\Big|\frac{\mathbf{W}_e}{s}\Big|_{s=0} \leqslant 1$$

From the formula of $\mathbf{W}_e(s)$ in (50) we see that $\mathbf{G}_2(s)$ should introduce one pole at $s = 0$ (one integral action) for obtaining a closed-loop type-1 system and a proportional action to reduce $|\mathbf{e}_{ss}^{(1)}(t)|$ within the given tolerance. For this reason we assume a structure for $\mathbf{G}_2(s)$ as

$$\mathbf{G}_2(s) = \frac{K_{G_2}}{s}\mathbf{G}_2'(s)$$

Therefore, from condition (51) we get

$$|K_{G_2}\mathbf{G}_2'(0)|_{s=0} \geqslant \frac{1}{|\mathbf{P}_1(0)\mathbf{P}_2(0)|} = \frac{1}{2} \tag{51}$$

We choose $K_{G_2} = \frac{1}{2}$ and keep track in the sequel of the additional condition

$$|\mathbf{G}_2'(0)| \geqslant 1. \tag{52}$$

For stabilizing the closed-loop system (i.e. all poles of $\mathbf{W}(s)$ with negative real part) by choosing the remain part $\mathbf{G}_2'(s)$ of the controller $\mathbf{G}_2$, we notice from Figure 4-c that the transfer function on the direct path (from v to y) of the feedback loop is

$$\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{P}_2(s) = \mathbf{G}_2'(s)\frac{\mathbf{P}_1(s)\mathbf{P}_2(s)}{2s} \tag{53}$$

therefore, $\mathbf{G}_2'(s)$ must stabilize the closed-loop of $\mathbf{P}'(s) = \frac{\mathbf{P}_1(s)\mathbf{P}_2(s)}{2s}$. Since $\mathbf{P}'(s)$ is minimum-phase and its relative degree is 2 we can assume for $\mathbf{G}_2'(s)$ the following structure

$$\mathbf{G}_2'(s) = K_{G_2'}\frac{s-z'}{s-p'} \tag{54}$$

with $z', p' < 0$, i.e. a zero-pole action to move the asymptotes center inside $\mathbb{C}^-$ and a proportional action to move the poles inside $\mathbb{C}^-$ with high gain. Moreover, remind the constraint (52) which reads out here as

$$|K_{G_2'}| \geqslant \frac{|p'|}{|z'|} \tag{55}$$

We will choose $z' < p' < 0$ to move the asymptotes center inside $\mathbb{C}^-$:

$$\frac{p'+3+4-z'}{2} < 0 \implies -p' + z' > 7. \tag{56}$$

Set $z' = -4$ and $p' = -21$. We finally choose $K_{G_2'}$ by applying the Routh criterion on the closed-loop denominator

$$\begin{aligned}&\mathrm{NUM}(1+\mathbf{G}_2(s)\mathbf{P}_1(s)\mathbf{P}_2(s))\\ &= s^4 + 18s^3 + (K_{G_2'} - 61)s^2\\ &+(8K_{G_2'} + 42)s + 16K_{G_2'}\end{aligned} \tag{57}$$

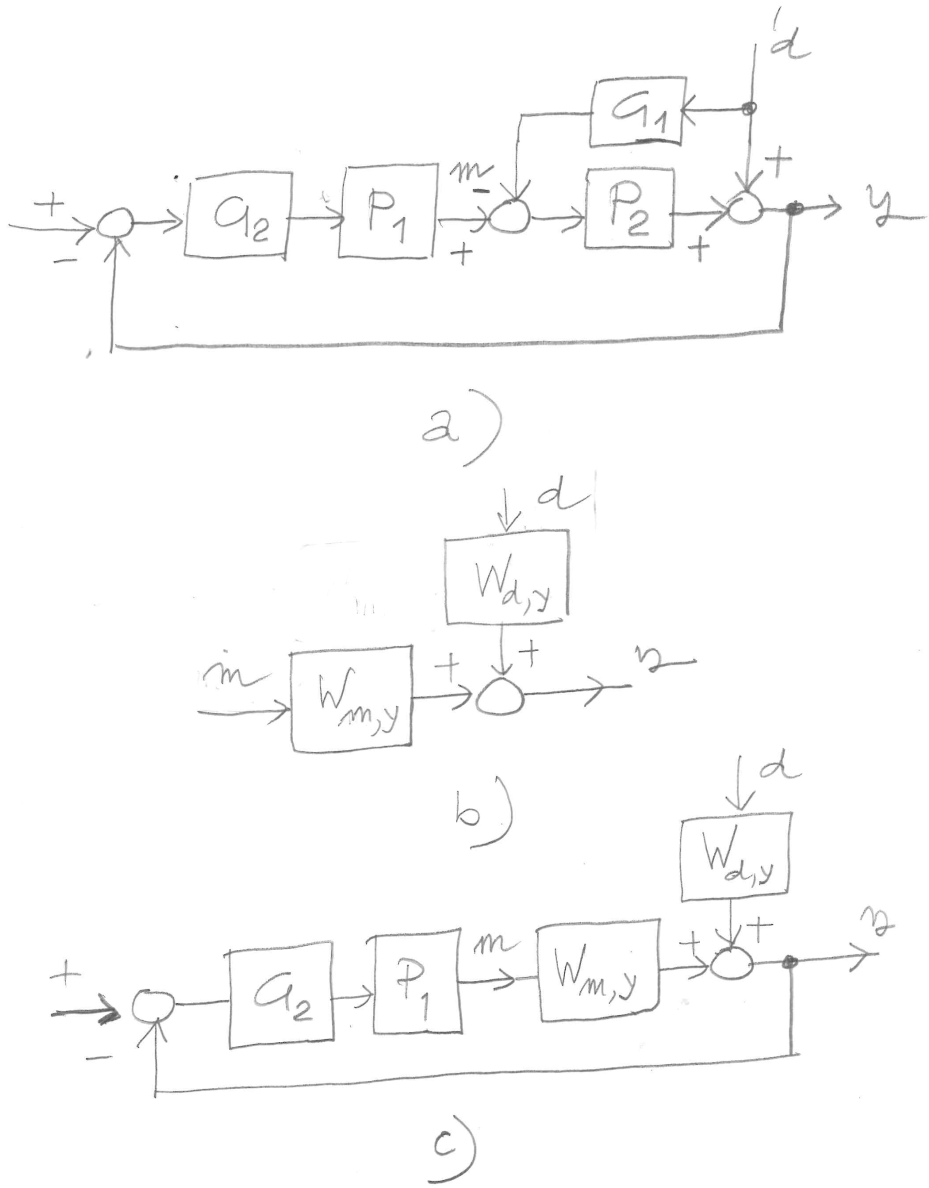

Figure 4. Exercize 3.2.

and at the same time satisfying the constraint (55). The Routh table of the above polynomial is

$$\begin{array}{c|ccc} r^{(4)} & 1 & K_{G_2'}-61 & 16K_{G_2'} \\ r^{(3)} & 9 & 4K_{G_2'}+21 & \\ r^{(2)} & \frac{5K_{G_2'}-570}{9} & 16K_{G_2'} & \\ r^{(1)} & \frac{20K_{G_2'}^2-2319K_{G_2'}-11970}{5K_{G_2'}-570} & & \\ r^{(0)} & K_{G_2'} & & \end{array}$$

We obtain $K_{G_2'} > 241.8$ for having no sign variations in the first column of the Routh table. Taking into account the additional constraint (55)

$$\begin{aligned} K_{G_2'} &> \max\left\{241.8, \frac{|p'|}{|z'|}\right\} \\ &= \max\left\{241.8, \frac{21}{4}\right\} = 241.8 \end{aligned} \tag{58}$$

Set $K_{G_2'} = 250$. The controller is finally

$$\mathbf{G}_2(s) = 125\frac{s+4}{s(s+21)}. \tag{59}$$