Stochastic Stability of the Discrete-Time Extended Kalman Filter

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***Abstract*—In this paper, the authors analyze the error behavior for the discrete-time extended Kalman filter for general nonlinear systems in a stochastic framework. In particular, it is shown that the estimation error remains bounded if the system satisfies the nonlinear observability rank condition and the initial estimation error as well as the disturbing noise terms are small enough. This result is verified by numerical simulations for an example system.**

***Index Terms*—Discrete-time systems, Kalman filtering, nonlinear systems, observability, stability, stochastic systems.**

I. Introduction

EXTENDED Kalman filtering is a widely used method in many areas of signal processing, control, and optimization, e.g., adaptive filtering [20], estimation [19], [30], prediction [11], robust control [42], state observation [31], system identification [25], [28], target tracking [12], training of neural networks [35], and many others.

However, despite its superior practical usefulness the extended Kalman filter has not been analyzed in a rigorous mathematical way for a long time. The convergence and stability properties have been treated for important special cases, e.g., if the state equations are given in a special form [17], [44] or if it is used as a parameter estimator for linear systems [27], [50]. Another well-developed situation is the zero noise case, i.e., deterministic state estimation [5], [9], [14], [16], [36], [38], [45]. Moreover, a rigorous treatment of the discrete-time extended Kalman filter for finite data sets in terms of recursive least squares is given in [6]–[8]. However, in addition to these results a study of a more general nonlinear case in a stochastic framework would also be of some interest.

In this paper, we combine the stability results for the usual Kalman filter (see, e.g., [24, ch. 7]) and stochastic stability analysis for more general nonlinear estimation problems [48], [51] to analyze the error behavior of the extended Kalman filter. The main contribution consists of the proof that, under certain conditions, the estimation error of the extended Kalman filter remains bounded.

The paper is organized as follows. In Section II we recall the state estimation problem for nonlinear stochastic discrete-time systems and some auxiliary results from stochastic stability theory. Then, in Section III the extended Kalman filter is introduced and the error boundedness is proved, if certain conditions are satisfied. The role of nonlinear observability in this context is discussed in Section IV. Section V contains numerical simulation results for an example system to illustrate the theoretical results of Sections III and IV. In Section VI, some conclusions are drawn.

Throughout this article, $||\cdot||$ denotes the Euclidian norm of real vectors or the spectral norm of real matrices and $E\{x\}$ is the expectation value of x, $E\{x\,|\,y\}$, the expectation value of x conditional on y. Moreover, N_0 denotes the set of natural numbers including zero, R^q the real q-dimensional vector space, and C^1 the continuously differentiable functions.

II. State Estimation and Stochastic Boundedness

We consider a nonlinear discrete-time system represented by

$$z_{n+1} = f(z_n, x_n) + G_n w_n \tag{1}$$

$$y_n = h(z_n) + D_n v_n \tag{2}$$

where $n \in N_0$ is the discrete time, $z_n \in R^q$ is the state, $x_n \in R^p$ the input, and $y_n \in R^m$ the output. Moreover, v_n, w_n, are R^k and R^l valued uncorrelated zero-mean white noise processes with identity covariance and D_n, G_n time varying matrices of size $m \times k$ and $q \times l$. For simplicity we consider a constant initial condition z_0 with probability one. The functions f and h are assumed to be C^1-functions. For this system we introduce a state estimator given by

$$\hat{z}_{n+1} = f(\hat{z}_n, x_n) + K_n(y_n - h(\hat{z}_n)) \tag{3}$$

where the observer gain K_n is a matrix-valued stochastic process of size $q \times m$. The estimated states are denoted by $\hat{z}_n$. Because f and h are C^1-functions, they can be expanded via

$$f(z_n, x_n) - f(\hat{z}_n, x_n) = A_n(z_n - \hat{z}_n) + \varphi(z_n, \hat{z}_n, x_n) \tag{4}$$

and

$$h(z_n) - h(\hat{z}_n) = C_n(z_n - \hat{z}_n) + \chi(z_n, \hat{z}_n) \tag{5}$$

with a $q \times q$ matrix-valued stochastic process A_n and a $m \times q$ matrix-valued stochastic process C_n given by

$$A_n = \frac{\partial f}{\partial z}(\hat{z}_n, x_n) \tag{6}$$

and

$$C_n = \frac{\partial h}{\partial z}(\hat{z}_n) \tag{7}$$

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respectively. We define the estimation error by

$$\zeta_n = z_n - \hat{z}_n. \tag{8}$$

Subtracting (3) from (1) and using (2) as well as (4)–(7) yields

$$\zeta_{n+1} = (A_n - K_n C_n)\zeta_n + r_n + s_n \tag{9}$$

where

$$r_n = \varphi(z_n, \hat{z}_n, x_n) - K_n \chi(z_n, \hat{z}_n) \tag{10}$$
$$s_n = G_n w_n - K_n D_n v_n. \tag{11}$$

For the analysis of the error dynamics (9) we make use of the following two concepts for the boundedness of stochastic processes [2], [48].

Definition 2.1: The stochastic process ζ_n is said to be exponentially bounded in mean square, if there are real numbers $\eta, \nu > 0$ and $0 < \vartheta < 1$ such that

$$E\{\|\zeta_n\|^2\} \leq \eta \|\zeta_0\|^2 \vartheta^n + \nu \tag{12}$$

holds for every $n \geq 0$.

Definition 2.2: The stochastic process is said to be bounded with probability one, if

$$\sup_{n \geq 0} \|\zeta_n\| < \infty \tag{13}$$

holds with probability one.

For later use we recall some standard results about the boundedness of stochastic processes.

Lemma 2.1: Assume there is a stochastic process $V_n(\zeta_n)$ as well as real numbers $\underline{v}, \bar{v}, \mu > 0$ and $0 < \alpha \leq 1$ such that

$$\underline{v}\|\zeta_n\|^2 \leq V_n(\zeta_n) \leq \bar{v}\|\zeta_n\|^2 \tag{14}$$

and

$$E\{V_{n+1}(\zeta_{n+1}) \mid \zeta_n\} - V_n(\zeta_n) \leq \mu - \alpha V_n(\zeta_n) \tag{15}$$

are fulfilled for every solution of (9). Then the stochastic process is exponentially bounded in mean square, i.e., we have

$$E\{\|\zeta_n\|^2\} \leq \frac{\bar{v}}{\underline{v}} E\{\|\zeta_0\|^2\}(1-\alpha)^n + \frac{\mu}{\underline{v}} \sum_{i=1}^{n-1} (1-\alpha)^i \tag{16}$$

for every $n \geq 0$. Moreover, the stochastic process is bounded with probability one.

Proof: This lemma contains a combination of [2, Sec. 4.1, Th. 1] and [48, Th. 2] (see also [20, Appendix D, Corollary D.5.1, p. 501] as well as [32] and [33]). □

Remark: Using

$$\sum_{i=1}^{n-1} (1-\alpha)^i \leq \sum_{i=1}^{\infty} (1-\alpha)^i = \frac{1}{\alpha} \tag{17}$$

inequality (16) can be rewritten in the form [48]

$$E\{\|\zeta_n\|^2\} \leq \frac{\bar{v}}{\underline{v}} E\{\|\zeta_0\|^2\}(1-\alpha)^n + \frac{\mu}{\underline{v}\alpha}. \tag{18}$$

III. Error Bounds for the Extended Kalman Filter

There are two common formulations of the discrete-time extended Kalman filter, and both are widely used in engineering literature: a two-step recursion consisting of time-update and measurement-update with a relinearization between these two steps (see, e.g., [1] and [13]) or a one-step formulation in terms of the *a priori* variables (see, e.g., [20] and [27]). As remarked in [27], these two formulations may have a different performance and transient behavior, but the convergence properties are the same.

Definition 3.1: A discrete-time extended Kalman filter is given by the following coupled difference equations:

- *Difference equation for state estimate*:

$$\hat{z}_{n+1} = f(\hat{z}_n, x_n) + K_n(y_n - h(\hat{z}_n)). \tag{19}$$

- *Riccati difference equation*:

$$P_{n+1} = A_n P_n A_n^T + Q_n - K_n\big(C_n P_n C_n^T + R_n\big)K_n^T. \tag{20}$$

- *Linearization*:

$$A_n = \frac{\partial f}{\partial z}(\hat{z}_n, x_n) \tag{21}$$
$$C_n = \frac{\partial h}{\partial z}(\hat{z}_n). \tag{22}$$

- *Kalman gain*:

$$K_n = A_n P_n C_n^T \big(C_n P_n C_n^T + R_n\big)^{-1}. \tag{23}$$

The system output of the system to be observed is given by (1) and (2), Q_n is a time-varying symmetric positive definite $q \times q$ matrix and R_n a time-varying positive definite $m \times m$ matrix.

Remark: A usual choice for the matrices Q_n and R_n are the covariances for the corrupting noise terms in (1) and (2), i.e.,

$$Q_n = G_n G_n^T \tag{24}$$
$$R_n = D_n D_n^T. \tag{25}$$

However, this is not the only possibility. Especially for deterministic estimation problems, i.e., for

$$G_n G_n^T = 0 \tag{26}$$
$$D_n D_n^T = 0 \tag{27}$$

or for systems with severe nonlinearities supplementary alternatives are of particular interest [37]–[40].

With this prerequisite we are able to state a main result of this paper.

Theorem 3.1: Consider a nonlinear stochastic system given by (1), (2) and an extended Kalman filter as stated in Definition 3.1. Let the following assumptions hold.

1) There are positive real numbers $\bar{a}, \bar{c}, \underline{p}, \bar{p} > 0$ such that the following bounds on various matrices are fulfilled for every $n \geq 0$:

$$\|A_n\| \leq \bar{a} \tag{28}$$
$$\|C_n\| \leq \bar{c} \tag{29}$$
$$\underline{p}I \leq P_n \leq \bar{p}I \tag{30}$$
$$\underline{q}I \leq Q_n \tag{31}$$
$$\underline{r}I \leq R_n. \tag{32}$$

2) A_n is nonsingular for every $n \geq 0$.
3) There are positive real numbers $\epsilon_\varphi, \epsilon_\chi, \kappa_\varphi, \kappa_\chi > 0$ such that the nonlinear functions φ, χ in (10) are bounded via

$$\|\varphi(z, \hat{z}, x)\| \leq \kappa_\varphi \|z - \hat{z}\|^2 \tag{33}$$
$$\|\chi(z, \hat{z})\| \leq \kappa_\chi \|z - \hat{z}\|^2 \tag{34}$$

for $z, \hat{z} \in R^p$ with $\|z - \hat{z}\| \leq \epsilon_\varphi$ and $\|z - \hat{z}\| \leq \epsilon_\chi$, respectively.

Then the estimation error ζ_n given by (8) is exponentially bounded in mean square and bounded with probability one, provided that the initial estimation error satisfies

$$\|\zeta_0\| \leq \epsilon \tag{35}$$

and the covariance matrices of the noise terms are bounded via

$$G_n G_n^T \leq \delta I \tag{36}$$
$$D_n D_n^T \leq \delta I \tag{37}$$

for some $\delta, \epsilon > 0$.

Remarks:

1) The inequalities (28)–(30) should be understood in the sense that they can be verified during the state estimation process. If the numerically calculated values for A_n, C_n, P_n satisfy the required bounds, then the state estimates are reliable in the sense of Theorem 3.1.
2) For linear systems with $Q_n = G_n G_n^T$ and $R_n = D_n D_n^T$ the matrix P_n is the covariance matrix for the estimation error and therefore this theorem is trivially true in this case. However, this theorem treats only the case of nonsingular A_n.
3) Inequality (30) is closely related to observability and detectability properties of the linearized system [10], [22], [43]. The relations to nonlinear observability are shown in Section IV.
4) To obtain the error bounds, the matrices Q_n and R_n need not to be the covariances of the noise terms. Any other positive definite matrices can be chosen as well.
5) The proof of the theorem following below can be modified easily for the case:

$$\|\varphi(z, \hat{z}, x)\| \leq \kappa_\varphi \|z - \hat{z}\|^{1+\beta}$$
$$\|\chi(z, \hat{z})\| \leq \kappa_\chi \|z - \hat{z}\|^{1+\beta}$$

with some $0 < \beta \leq 1$ instead of (33), (34) treating a larger class of functions φ and χ.
6) The proof for this theorem contains explicit estimation formulas for ϵ and δ. These estimates, however, may be very conservative; see also the end of Section V, where these estimates are compared to values from numerical simulations.

The proof of this theorem is divided into several lemmas.

Lemma 3.1: Under the conditions of Theorem 3.1, there is a real number $0 < \alpha < 1$ such that $\Pi_n = P_n^{-1}$ satisfies the inequality

$$(A_n - K_n C_n)^T \Pi_{n+1} (A_n - K_n C_n) \leq (1 - \alpha) \Pi_n \tag{38}$$

for $n \geq 0$ with K_n given by (23).

Proof: From (20) and (23) we have

$$P_{n+1} = A_n P_n A_n^T + Q_n - A_n P_n C_n^T K_n^T \tag{39}$$

and rearranging the terms yields

$$P_{n+1} = (A_n - K_n C_n) P_n (A_n - K_n C_n)^T + Q_n + K_n C_n P_n (A_n - K_n C_n)^T. \tag{40}$$

As the next step we take care of the term $K_n C_n P_n (A_n - K_n C_n)^T$ on the right-hand side of (40). With (23) it can be verified that

$$A_n^{-1}(A_n - K_n C_n) P_n = P_n - P_n C_n^T \left(C_n P_n C_n^T + R_n\right)^{-1} C_n P_n \tag{41}$$

is a symmetric matrix, and applying the matrix inversion lemma (see, e.g., [26, Appendix A2, p. 347]) we obtain

$$A_n^{-1}(A_n - K_n C_n) P_n = \left(P_n^{-1} + C_n^T R_n^{-1} C_n\right)^{-1} > 0 \tag{42}$$

because $P_n^{-1} > 0$. Moreover, we have from (23) using $P_n > 0$ and $R_n > 0$

$$A_n^{-1} K_n C_n = P_n C_n^T \left(C_n P_n C_n^T + R_n\right)^{-1} C_n \geq 0. \tag{43}$$

Combining (42) and (43) and using $P_n = P_n^T$ we establish that

$$K_n C_n P_n (A_n - K_n C_n)^T = A_n \left[A_n^{-1} K_n C_n\right] \left[A_n^{-1}(A_n - K_n C_n) P_n\right]^T A_n^T \geq 0 \tag{44}$$

holds, and inserting into (40) leads to

$$P_{n+1} \geq (A_n - K_n C_n) P_n (A_n - K_n C_n)^T + Q_n.$$

Inequality (42) implies that $(A_n - K_n C_n)^{-1}$ exists and therefore we may write

$$P_{n+1} \geq (A_n - K_n C_n) \times [P_n + (A_n - K_n C_n)^{-1} Q_n (A_n - K_n C_n)^{-T}] \times (A_n - K_n C_n)^T. \tag{45}$$

From (23), (28)–(32) and $C_n P_n C_n^T \geq 0$ we have

$$\|K_n\| \leq \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}} \tag{46}$$

and with (28)–(32) we obtain

$$P_{n+1} \geq (A_n - K_n C_n)\left[P_n + \frac{\underline{q}}{(\bar{a} + \bar{a}\bar{p}\bar{c}^2/\underline{r})^2} I\right] \times (A_n - K_n C_n)^T. \tag{47}$$

Taking the inverse of both sides (this is allowed since $P_n \geq \underline{p} I$ and $A_n - K_n C_n$ are nonsingular), multiplying from left and right with $(A_n - K_n C_n)^T$ and $A_n - K_n C_n$, and using (30) we get finally with $\Pi_n = P_n^{-1}$

$$(A_n - K_n C_n)^T \Pi_{n+1} (A_n - K_n C_n) \leq \left[1 + \frac{\underline{q}}{\bar{p}(\bar{a} + \bar{a}\bar{p}\bar{c}^2/\underline{r})^2}\right]^{-1} \Pi_n \tag{48}$$

i.e., inequality (38) with

$$1 - \alpha = 1 \Big/ \left(1 + \frac{\underline{q}}{\bar{p}(\bar{a} + \bar{a}\bar{p}\bar{c}^2/\underline{r})^2}\right). \tag{49}$$

□

Lemma 3.2: Let the conditions of Theorem 3.1 be fulfilled, let $\Pi_n = P_n^{-1}$ and K_n, r_n be given by (23), (10). Then there are positive real numbers $\epsilon', \kappa_{\text{nonl}} > 0$ such that

$$r_n^T \Pi_n [2(A_n - K_n C_n)(z_n - \hat{z}_n) + r_n] \leq \kappa_{\text{nonl}} \|z_n - \hat{z}_n\|^3 \tag{50}$$

holds for $\|z_n - \hat{z}_n\| \leq \epsilon'$.

Proof: From (23), (28)–(32), and $C_n P_n C_n^T > 0$ we have

$$\|K_n\| \leq \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}} \tag{51}$$

and inserting into (10) yields

$$\|r_n\| \leq \|\varphi(z_n, \hat{z}_n, x_n)\| + \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}}\|\chi(z_n, \hat{z}_n)\|. \tag{52}$$

Choosing $\epsilon' = \min(\epsilon_\varphi, \epsilon_\chi)$ and using (33), (34) we obtain

$$\|r_n\| \leq \kappa_\varphi \|z_n - \hat{z}_n\|^2 + \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}}\kappa_\chi \|z_n - \hat{z}_n\|^2 \tag{53}$$

for $\|z_n - \hat{z}_n\| \leq \epsilon'$, i.e.,

$$\|r_n\| \leq \kappa' \|z_n - \hat{z}_n\|^2 \tag{54}$$

with

$$\kappa' = \kappa_\varphi + \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}}\kappa_\chi. \tag{55}$$

From (53) and (28)–(32) we get with $\Pi_n = P_n^{-1}$ for $\|z_n - \hat{z}_n\| \leq \epsilon'$

$$\begin{aligned} & r_n^T \Pi_n [2(A_n - K_n C_n)(z_n - \hat{z}_n) + r_n] \\ & \quad \leq \kappa' \|z_n - \hat{z}_n\|^2 \frac{1}{\underline{p}} \left(2\left(\bar{a} + \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}}\bar{c} \right) \|z_n - \hat{z}_n\| \right. \\ & \qquad \left. + \kappa' \epsilon' \|z_n - \hat{z}_n\| \right) \end{aligned} \tag{56}$$

i.e., (50) with

$$\kappa_{\text{nonl}} = \kappa' \frac{1}{\underline{p}} \left(2\left(\bar{a} + \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}}\bar{c} \right) + \kappa' \epsilon' \right). \tag{57}$$

□

Lemma 3.3: Let the conditions of Theorem 3.1 hold, let $\Pi_n = P_n^{-1}$, and K_n, s_n be given by (23), (11). Then there is a positive real number $\kappa_{\text{noise}} > 0$ independent of δ, such that

$$E\{s_n^T \Pi_{n+1} s_n\} \leq \kappa_{\text{noise}} \delta \tag{58}$$

holds.

Proof: Since v_n and w_n are uncorrelated the expectation value of the crossterms containing both v_n and w_n will vanish and we focus on the following terms:

$$s_n^T \Pi_{n+1} s_n = w_n^T G_n^T \Pi_{n+1} G_n w_n + v_n^T D_n^T K_n^T \Pi_{n+1} K_n v_n. \tag{59}$$

From (23), (28)–(32) and $C_n P_n C_n^T > 0$ we have

$$\|K_n\| \leq \bar{a}\bar{p}\bar{c}\frac{1}{\underline{r}}. \tag{60}$$

Inserting into (59) and using (30) we get with $\Pi_n = P_n^{-1}$

$$s_n^T \Pi_{n+1} s_n \leq \frac{1}{\underline{p}} w_n^T G_n^T G_n w_n + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2}{\underline{p}\underline{r}^2} v_n^T D_n^T D_n v_n. \tag{61}$$

Because both sides of (61) are scalars, we may take the trace on the right-hand side of (61) without changing its value

$$s_n^T \Pi_{n+1} s_n \leq \frac{1}{\underline{p}} \text{tr}\big(w_n^T G_n^T G_n w_n\big) + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2}{\underline{p}\underline{r}^2} \text{tr}\big(v_n^T D_n^T D_n v_n\big). \tag{62}$$

Using the well-known matrix identity

$$\text{tr}(\Gamma\Delta) = \text{tr}(\Delta\Gamma) \tag{63}$$

where Γ, Δ are such matrices that the above matrix multiplication and the trace operations make sense (see, e.g., [52]), we get

$$s_n^T \Pi_{n+1} s_n^T \leq \frac{1}{\underline{p}} \text{tr}\big(G_n w_n w_n^T G_n^T\big) + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2}{\underline{p}\underline{r}^2} \text{tr}\big(D_n v_n v_n^T D_n^T\big) \tag{64}$$

and taking the mean value yields

$$\begin{aligned} E\{s_n^T \Pi_{n+1} s_n\} \leq & \frac{1}{\underline{p}} \text{tr}\big(G_n E\{w_n w_n^T\} G_n^T\big) \\ & + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2}{\underline{p}\underline{r}^2} \text{tr}\big(D_n E\{v_n v_n^T\} D_n^T\big) \end{aligned} \tag{65}$$

where we have used that D_n and G_n are deterministic matrices. Because v_n and w_n are standard vector-valued white noise processes, the conditions

$$E\{v_n v_n^T\} = I \tag{66}$$

$$E\{w_n w_n^T\} = I \tag{67}$$

hold and therefore we have

$$E\{s_n^T \Pi_{n+1} s_n\} \leq \frac{1}{\underline{p}} \text{tr}\big(G_n G_n^T\big) + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2}{\underline{p}\underline{r}^2} \text{tr}\big(D_n D_n^T\big). \tag{68}$$

Using (36) and (37) we get

$$\text{tr}\big(G_n G_n^T\big) \leq \delta \text{tr}(I) = q\delta \tag{69}$$

$$\text{tr}\big(D_n D_n^T\big) \leq \delta \text{tr}(I) = m\delta \tag{70}$$

where q and m are the number of the rows for G_n and D_n, respectively. Setting

$$\kappa_{\text{noise}} = \frac{q}{\underline{p}} + \frac{\bar{a}^2 \bar{c}^2 \bar{p}^2 m}{\underline{p}\underline{r}^2} \tag{71}$$

it follows with (68) and (71) that

$$E\{s_n^T \Pi_{n+1} s_n\} \leq \kappa_{\text{noise}} \delta \tag{72}$$

holds, yielding the desired inequality (58). □

Proof of Theorem 3.1: We choose

$$V_n(\zeta_n) = \zeta_n^T \Pi_n \zeta_n \tag{73}$$

with $\Pi_n = P_n^{-1}$, which exists since P_n is positive definite [cf. (30)]. From (30) we have

$$\frac{1}{\bar{p}}\|\zeta_n\|^2 \le V_n(\zeta_n) \le \frac{1}{\underline{p}}\|\zeta_n\|^2 \tag{74}$$

i.e., (14) with $\underline{v} = 1/\bar{p}$ and $\bar{v} = 1/\underline{p}$. To satisfy the requirements for an application of Lemma 2.1, we need an upper bound on $E\{V_{n+1}(\zeta_{n+1}) \mid \zeta_n\}$ as in (15). From (9) we have

$$\begin{aligned} V_{n+1}(\zeta_{n+1}) = &\left[\zeta_n^T(A_n - K_nC_n)^T + r_n^T + s_n^T\right]\Pi_{n+1} \\ &\times \left[(A_n - K_nC_n)\zeta_n + r_n + s_n\right] \end{aligned} \tag{75}$$

and applying Lemma 3.1 we obtain with (73)

$$\begin{aligned} &V_{n+1}(\zeta_{n+1}) \\ &\quad \le (1-\alpha)V_n(\zeta_n) + r_n^T\Pi_{n+1}[2(A_n - K_nC_n)\zeta_n + r_n] \\ &\qquad + 2s_n^T\Pi_{n+1}[(A_n - K_nC_n)\zeta_n + r_n] + s_n^T\Pi_{n+1}s_n. \end{aligned} \tag{76}$$

Taking the conditional expectation $E\{V_{n+1}(\zeta_{n+1}) \mid \zeta_n\}$ and considering the white noise property it can be seen that the term $E\{s_n^T\Pi_{n+1}((A_n - K_nC_n)\zeta_n + r_n) \mid \zeta_n\}$ vanishes since neither $\Pi_{n+1} = P_{n+1}^{-1}$ nor $A_n, C_n, K_n, r_n, \zeta_n$ depend on v_n or w_n. The remaining terms are estimated via Lemmas 3.2 and 3.3 yielding

$$\begin{aligned} &E\{V_{n+1}(\zeta_{n+1}) \mid \zeta_n\} - V_n(\zeta_n) \\ &\quad \le -\alpha V_n(\zeta_n) + \kappa_{\text{nonl}}\|\zeta_n\|^3 + \kappa_{\text{noise}}\delta \end{aligned} \tag{77}$$

for $\|\zeta_n\| \le \epsilon'$. Defining

$$\epsilon = \min\left(\epsilon', \frac{\alpha}{2\bar{p}\kappa_{\text{nonl}}}\right) \tag{78}$$

we obtain with (73), (74) for $\|\zeta_n\| \le \epsilon$

$$\kappa_{\text{nonl}}\|\zeta_n\|\|\zeta_n\|^2 \le \frac{\alpha}{2\bar{p}}\|\zeta_n\|^2 \le \frac{\alpha}{2}V_n(\zeta_n).$$

Inserting into (77) yields

$$E\{V_{n+1}(\zeta_{n+1}) \mid \zeta_n\} - V_n(\zeta_n) \le -\frac{\alpha}{2}V_n(\zeta_n) + \kappa_{\text{noise}}\delta \tag{79}$$

for $\|\zeta_n\| \le \epsilon$. Therefore we are able to apply Lemma 2.1 with $\|\zeta_0\| \le \epsilon, \underline{v} = 1/\bar{p}$, $\bar{v} = 1/\underline{p}$, and $\mu = \kappa_{\text{noise}}\delta$. However, we have to take care that for $\tilde{\epsilon} \le \|\zeta_n\| \le \epsilon$ with some $\tilde{\epsilon} < \epsilon$ the supermartingale inequality

$$E\{V_{n+1}(\zeta_{n+1}) \mid \zeta_n\} - V_n(\zeta_n) \le -\frac{\alpha}{2}V_n(\zeta_n) + \kappa_{\text{noise}}\delta \le 0 \tag{80}$$

is fulfilled to guarantee the boundedness of the estimation error (cf. [18, Sec. 5.1, Th. 5.2, p. 129]). Choosing

$$\delta = \frac{\alpha\tilde{\epsilon}^2}{2\bar{p}\kappa_{\text{noise}}} \tag{81}$$

with some $\tilde{\epsilon} < \epsilon$ we have for $\|\zeta_n\| \ge \tilde{\epsilon}$

$$\kappa_{\text{noise}}\delta \le \frac{\alpha}{2\bar{p}}\|\zeta_n\|^2 \le \frac{\alpha}{2}V_n(\zeta_n) \tag{82}$$

i.e., (80) holds. Therefore, we conclude that the estimation error remains bounded if the initial error and the noise terms are bounded by (35)–(37). □

Remark: The proof of Theorem 3.1 contains a constructive way to quantify the error bound according to (18), (35), and $1 - \alpha < 1$, $\underline{v} = 1/\bar{p}$, $\bar{v} = 1/\underline{p}$, $\mu = \kappa_{\text{noise}}\delta$ with (71) and (81). Other interesting possibilities to quantify the error are discussed in [29] and [47].

In this section, we have proved that the estimation error of the extended Kalman filter remains bounded, if certain conditions are satisfied. This is an important result, since the stability and convergence of the extended Kalman filter has been analyzed in the literature only for special cases [17], [27], [45], which are different from the case treated in this paper.

IV. The Significance of Nonlinear Observability for the Extended Kalman Filter

For simplicity and clarity we restrict in this section to nonlinear autonomous systems with measurement noise only, i.e., to systems in the form

$$z_{n+1} = f(z_n) \tag{83}$$

$$y_n = h(z_n) + D_nv_n. \tag{84}$$

A generalization to systems with process noise will be briefly sketched after the proof of Theorem 4.1 given below. For the nonlinear autonomous system given by (83), (84) we recall the following observability rank condition (cf., [15], [34], and [46]).

Definition 4.1: The nonlinear system given by (83), (84) satisfies the nonlinear observability rank condition at $z_n \in R^q$, if the nonlinear observability matrix

$$U(z_n) = \begin{bmatrix} \frac{\partial h}{\partial z}(z_n) \\ \frac{\partial h}{\partial z}(z_{n+1})\frac{\partial f}{\partial z}(z_n) \\ \vdots \\ \frac{\partial h}{\partial z}(z_{n+q-1})\frac{\partial f}{\partial z}(z_{n+q-2})\cdots\frac{\partial f}{\partial z}(z_n) \end{bmatrix} \tag{85}$$

has full rank q at z_n. Now we are able to state the following result.

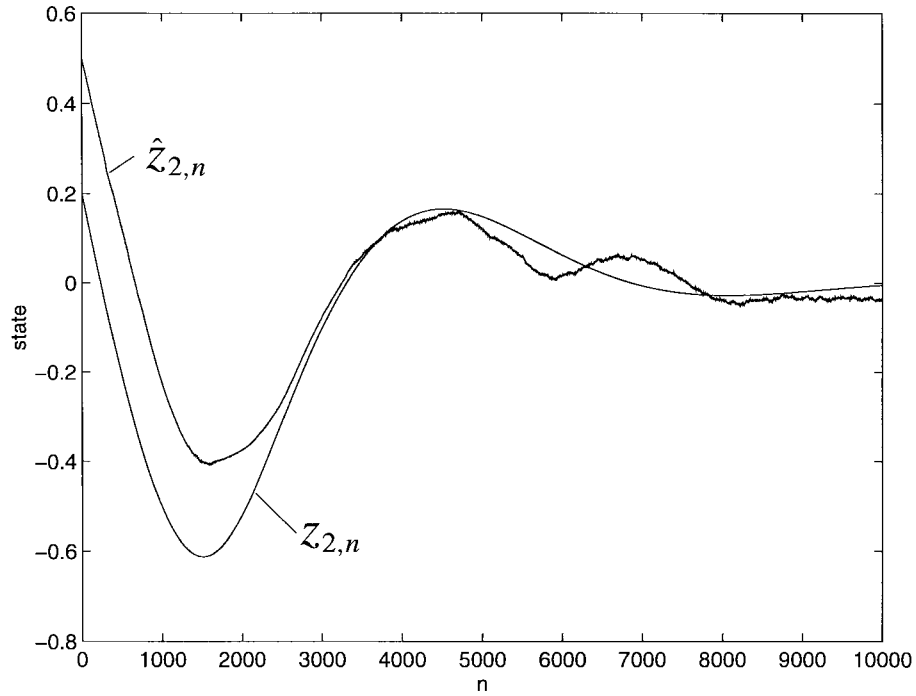
Theorem 4.1: Consider a nonlinear autonomous system given by (83), (84) and an extended Kalman filter as in Definition 3.1. Assume there are real numbers $\underline{r}, \underline{q} > 0$ with

$$\underline{q}I \le Q_n \tag{86}$$

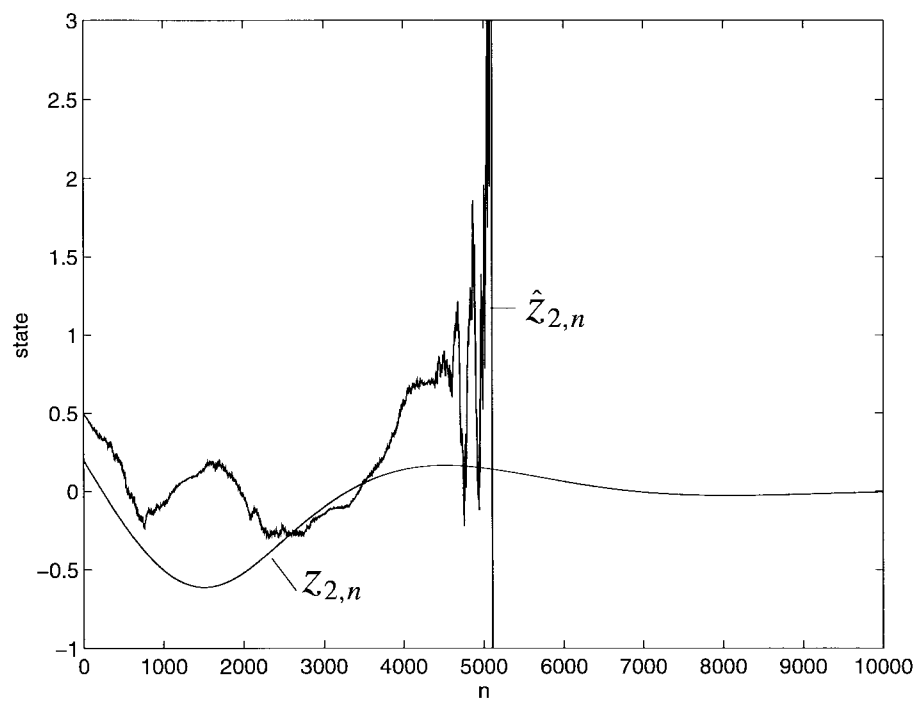
$$\underline{r}I \le R_n \tag{87}$$

for $n \ge 0$ and a compact subset $\mathcal{K}$ of R^q, such that the following conditions hold.

1) The nonlinear system given by (83), (84) satisfies the observability rank condition for every $z_n \in \mathcal{K}$.
2) The nonlinear functions f, h are twice continuously differentiable and $(\partial f/\partial z)(z) \ne 0$ holds for every $z \in \mathcal{K}$.



(a)



(b)

Fig. 1. Numerical simulations for the considered example system with zero process noise. The diagrams show the state component $z_{2,n}$ and its estimate $\hat{z}_{2,n}$ as a function of the discrete time n for several cases: (a) small initial error and small noise; (b) large measurement noise.

3) The sample paths of z_n are bounded with probability one, and $\mathcal{K}$ contains these sample paths as well as all points with distance smaller than $\epsilon_{\mathcal{K}}$ from these sample paths, where $\epsilon_{\mathcal{K}} > 0$ is a real number independent of n.

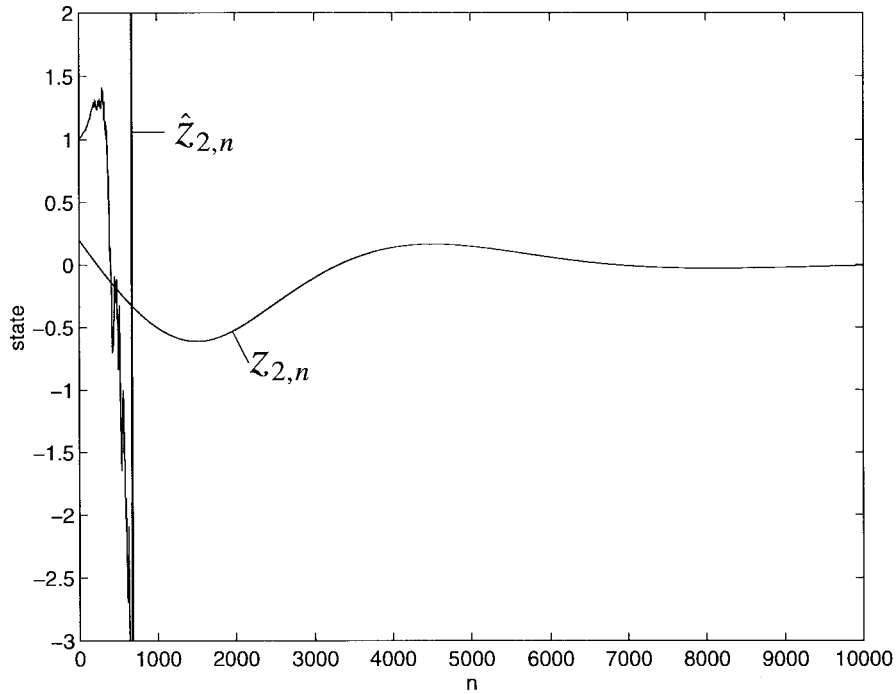
Then the estimation error ζ_n given by (8) is exponentially bounded in mean square and bounded with probability one, provided that the initial estimation error satisfies

$$||\zeta_0|| \leq \epsilon \tag{88}$$

and the covariance matrix of the noise term is bounded via

$$D_n D_n^T \leq \delta I \tag{89}$$

for some $\delta, \epsilon > 0$.



(c)

Fig. 1. (*Continued.*) Numerical simulations for the considered example system with zero process noise. The diagrams show the state component $z_{2,n}$ and its estimate $\hat{z}_{2,n}$ as a function of the discrete time n for several cases: (c) large initial error.

Remark: See also Remark 6 after Theorem 3.1.

For the proof of this theorem we make use of some auxiliary results. First we recall the uniform observability of linear time-varying systems (cf., [3] or [24, Section 7.5, p. 231]).

Definition 4.2: Consider time-varying matrices A_n, C_n, $n \geq 0$ and let the observability gramian be given by

$$M_{n+k,n} = \sum_{i=n}^{n+k} \Phi_{i,n}^T C_i^T C_i \Phi_{i,n} \tag{90}$$

for some integer $k > 0$ with $\Phi_{n,n} = I$ and

$$\Phi_{i,n} = A_{i-1} \cdots A_n \tag{91}$$

for $i > n$. The matrices $A_n, C_n, n \geq 0$ are said to satisfy the uniform observability condition, if there are real numbers $\underline{m}, \bar{m} > 0$ and an integer $k > 0$, such that the following inequality holds:

$$\underline{m} I \leq M_{n+k,n} \leq \bar{m} I. \tag{92}$$

Lemma 4.1: Consider a solution P_n for $n \geq 0$ of the Riccati difference equation (20) and let the following conditions hold.

1) There are real numbers $\underline{q}, \bar{q}, \underline{r}, \bar{r} > 0$ such that the matrices Q_n, R_n in (20) are bounded by

$$\underline{q} I \leq Q_n \leq \bar{q} I \tag{93}$$

$$\underline{r} I \leq R_n \leq \bar{r} I. \tag{94}$$

2) The matrices A_n, C_n satisfy the uniform observability condition.
3) The initial condition P_0 of the Riccati difference equation (20) is positive definite.

Then there are real numbers $\underline{p}, \bar{p} > 0$ such that the solution of the Riccati difference equation (20) is bounded via

$$\underline{p} I \leq P_n \leq \bar{p} I \tag{95}$$

for every $n \geq 0$.

Proof: This lemma follows directly from [3, Corollary 5.2]. □

Remark: The observability gramian $M_{n+k,n}$ given by (90) can be rewritten in the form (cf., [21, Sec. 2.5, p. 44])

$$M_{n+k,n} = \left[\Phi_{n+k,n}^T C_{n+k}^T \ \cdots \ \Phi_{n+1,n}^T C_{n+1}^T \quad C_n^T\right] \begin{bmatrix} C_n \\ C_{n+1}\Phi_{n+1,n} \\ \vdots \\ C_{n+k}\Phi_{n+k,n} \end{bmatrix}. \tag{96}$$

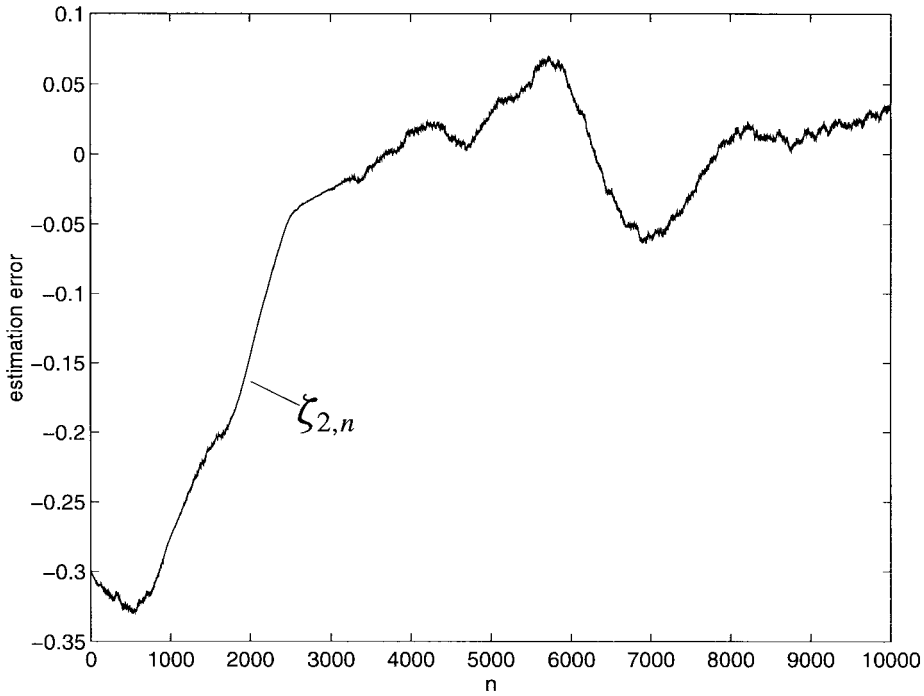
Moreover, for $\hat{z}_n = z_n \in R^q$ we have $A_n = (\partial f/\partial z)(\hat{z}_n)$ and $C_n = (\partial h/\partial z_n)(\hat{z}_n)$, and using (85) and (91) we get from (96) with $k = q - 1$

$$M_{n+q-1,n} = U^T(z_n) U(z_n). \tag{97}$$

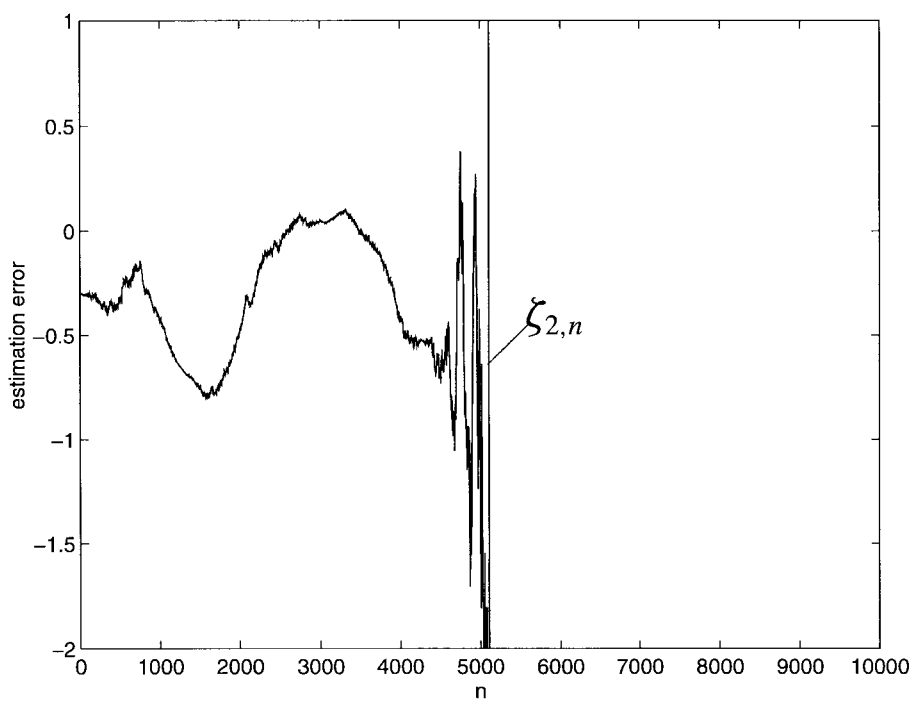
Lemma 4.2: Consider a compact subset $\mathcal{K} \subset R^q$ and assume that the nonlinear system given by (83) and (84) satisfies the nonlinear observability condition for every $z_n \in \mathcal{K}$. Then there is a real number $\epsilon_{\text{obs}} > 0$ such that the matrices

$$A_n = \frac{\partial f}{\partial z}(\hat{z}_n) \tag{98}$$

$$C_n = \frac{\partial h}{\partial z}(\hat{z}_n) \tag{99}$$



(a)



(b)

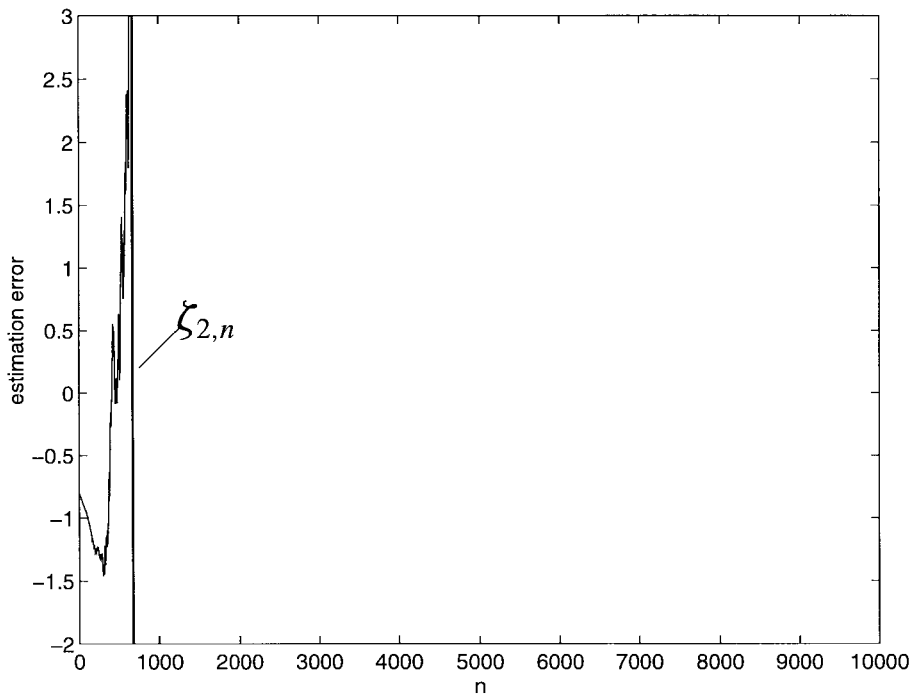
Fig. 2. Numerical simulations for the considered example system with zero process noise. The diagrams show the estimation error component $\zeta_{2,n}$ as a function of the discrete time n for several cases: (a) small initial error and small noise; (b) large measurement noise.

satisfy the uniform observability condition, provided that $||z_n - \hat{z}_n|| \leq \epsilon_{\text{obs}}$.

Proof: See [45, Proposition 4.1]. □

Proof of Theorem 4.1: For the proof of Theorem 4.1 we make use of Theorem 3.1. We now show that the inequalities (86), (87) and Conditions 1–3 in Theorem 4.1 together with the observability results in Lemmas 4.1 and 4.2 imply the Conditions 1–3 in Theorem 3.1. It can be seen immediately that (31), (32) and (86), (87) coincide, i.e., (31), (32) are satisfied if (86), (87) are.

The bounds (33), (34) for $z, \hat{z} \in \mathcal{K}$ can be obtained by a standard estimation via an integral formula (see, e.g., [23, ch. XX, Sec. 1.68] or [49, Sec. 8.1.3]). Let f_i, h_i be the components of f and h, respectively. Since f, h are twice



(c)

Fig. 2. (*Continued.*) Numerical simulations for the considered example system with zero process noise. The diagrams show the estimation error component $\zeta_{2,n}$ as a function of the discrete time n for several cases: (c) large initial error.

differentiable for every $z \in \mathcal{K}$ according to Assumption 2 and $\mathcal{K}$ is compact, it follows that the Hessian matrices of f_i and h_i are bounded with respect to the spectral norm of matrices. The constants $\kappa_\varphi, \kappa_\chi$ are therefore given by

$$\kappa_\varphi = \max_{1 \le i \le q} \sup_{z \in \mathcal{K}} \|\text{Hess } f_i(z)\| \tag{100}$$

$$\kappa_\chi = \max_{1 \le i \le m} \sup_{z \in \mathcal{K}} \|\text{Hess } h_i(z)\|. \tag{101}$$

Concerning the remaining conditions of Theorem 3.1 we use the fact that according to [45] it is sufficient to ensure these conditions one time-step in advance. However, we have to take care that the constants $\bar{a}, \bar{c}, \underline{p}, \bar{p}$ in (28)–(30) can be chosen *independently* of the time n. This means, we have to show that the boundedness with probability one of ζ_n and z_n implies the desired bounds on A_n, C_n and P_n. Then we obtain from Theorem 3.1 the boundedness of ζ_{n+1}, i.e., for the next time step. Repeating this procedure we get bounds on $A_{n+1}, C_{n+1}, P_{n+1}$ and therefore on ζ_{n+2}. Continuing this strategy yields the desired result. To establish the bounds on A_n, C_n, P_n we divide the cases $0 \le n < q$ and $n \ge q$ and treat them separately. This is due to the fact that we need $q-1$ steps in order to set up the uniform observability condition.

First we consider the setup cycle $0 \le n < q$. From the Lemma 3.1, inequality (47), and considering the boundedness with probability one of z_n, ζ_n and therefore of $\hat{z}_n$ and A_n, C_n it follows that $P_{n+1} > 0$ if $P_n > 0$ (this fact is obvious since the Riccati difference equation determines the evolution for the error covariance, which is positive definite if Q_n is). Taking the minimum and maximum eigenvalue of P_n and the maximum singular value of A_n, C_n for $0 \le n < q$ we obtain the bounds (28)–(30) for $0 \le n < q$.

Secondly, we treat the case $n \ge q$. We have to take care that neither any eigenvalue of P_n converges to zero nor any of the matrices A_n, C_n, P_n diverges. The bound

$$\underline{p} I \le P_n \le \bar{p} I \tag{102}$$

follows according to Lemmas 4.1 and 4.2 utilizing the boundedness with probability one of ζ_i for $q \le i \le n$ in the region $\|\zeta_i\| \le \epsilon_{\text{obs}}$. Moreover, the boundedness for A_n, C_n follows from the continuity of $\partial f / \partial z$ and $\partial h / \partial z$, the compactness of $\mathcal{K}$ and the fact that $\hat{z}_n \in \mathcal{K}$ with probability one according to Assumption 3, using $\|\zeta_n\| = \|z_n - \hat{z}_n\| \le \epsilon_{\mathcal{K}}$. By these arguments, Theorem 3.1 can be applied with the change that we have to set

$$\epsilon = \min\left(\epsilon_{\mathcal{K}}, \epsilon_{\text{obs}}, \frac{\alpha}{2\bar{p}\kappa_{\text{nonl}}}\right) \tag{103}$$

instead of (78). □

Remark: The proof can be generalized to nonlinear systems with process noise, i.e.,

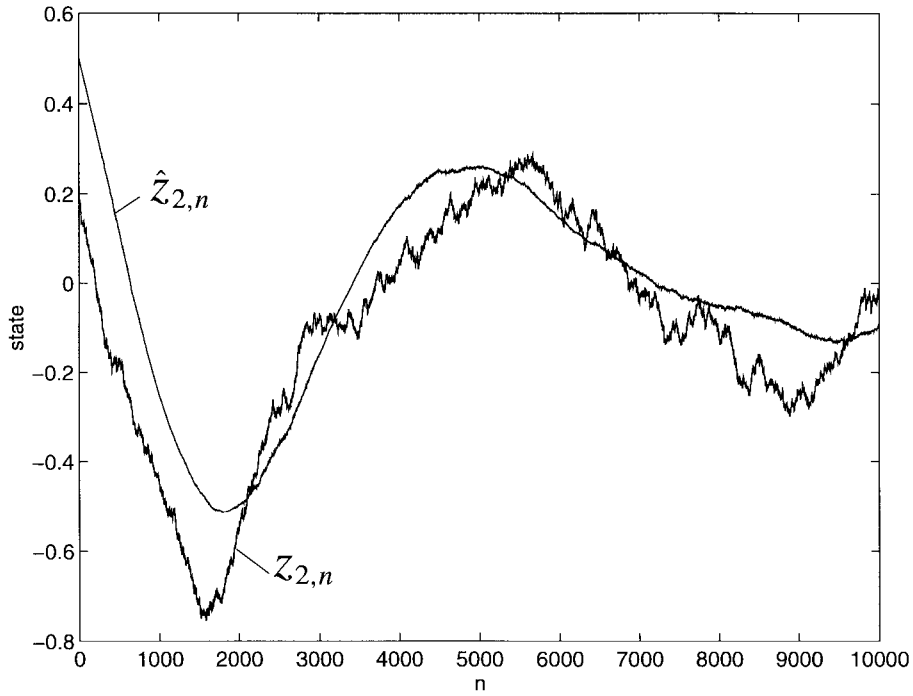
$$\tilde{z}_{n+1} = f(\tilde{z}_n) + w_n \tag{104}$$

$$\tilde{y}_n = h(\tilde{z}_n) + v_n \tag{105}$$

if we assume that the solution $\tilde{z}_n$ for $n \ge 0$ is bounded with probability one sufficiently close to the nominal solution $z_n, n \ge 0$ of (83), i.e., $\|\tilde{z}_n - z_n\| \le \epsilon_w$ for some $\epsilon_w > 0$. The main modification is to apply Lemma 4.2 in this case, where

$$\|z_n - \hat{z}_n\| \le \|z_n - \tilde{z}_n\| + \|\tilde{z}_n - \hat{z}_n\|. \tag{106}$$

is used.



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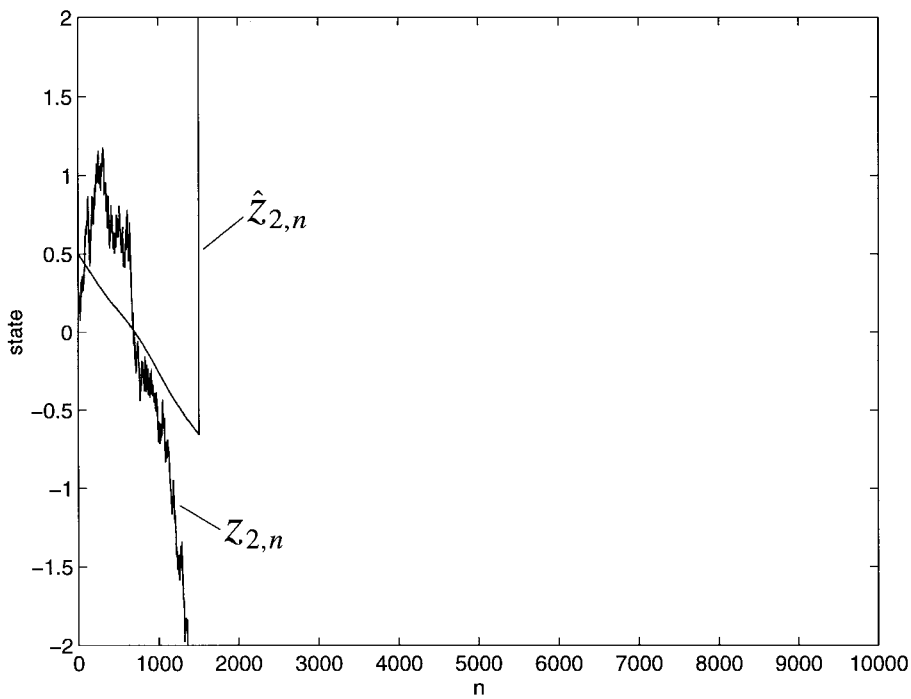


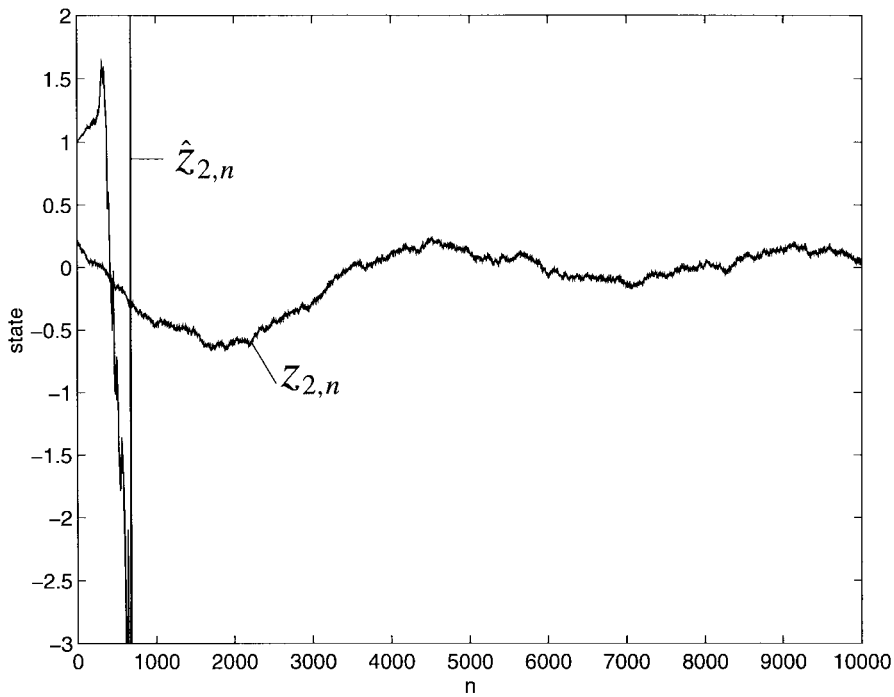
(b)

Fig. 3. Numerical simulations for the considered example system with nonzero process noise. The diagrams show the state component $z_{2,n}$ and its estimate $\hat{z}_{2,n}$ as a function of the discrete time n for several cases: (a) small initial error and small noise; (b) large process noise.

V. Numerical Simulations

The results in the preceding two sections show that the estimation error of the discrete-time extended Kalman filter remains bounded, if the nonlinear system to be observed satisfies appropriate conditions. These conditions include the requirement of a sufficiently small initial estimation error and sufficiently small noise. To illustrate the significance of these conditions, in this section we apply the extended Kalman filter to an example system and verify the error behavior by numerical simulations. We consider a nonlinear stochastic example system given by (1), (2) with

$$f(z_n, x_n) = \begin{bmatrix} z_{1,n} + \tau z_{2,n} \\ z_{2,n} + \tau\big(-z_{1,n} + (z_{1,n}^2 + z_{2,n}^2 - 1)z_{2,n}\big) \end{bmatrix} \tag{107}$$



(c)

Fig. 3. (*Continued.*) Numerical simulations for the considered example system with nonzero process noise. The diagrams show the state component $z_{2,n}$ and its estimate $\hat{z}_{2,n}$ as a function of the discrete time n for several cases: (c) large initial error.

and

$$h(z_n) = z_{1,n}. \tag{108}$$

This system can be obtained from the corresponding system in continuous-time

$$\dot{z}(t) = \tilde{f}(z(t), x(t)) + \tilde{G}(t)w(t) \tag{109}$$

and

$$y(t) = h(z(t)) + \tilde{D}(t)v(t) \tag{110}$$

by approximation employing Euler's method with sampling time $\tau > 0$ (see, e.g., [26, Part I, Sec. 2.4, pp. 80–83]). Furthermore, we remark that using white noise processes $w(t), v(t)$ for the continuous-time system formulations are not misleading in this case, because the Ito and Stratonovich forms coincide (cf., [4, Sec. 10.2, p. 178] or [18, Sec. 3.4, p. 95]). Especially for constant weighting matrices $\tilde{G}(t), \tilde{D}(t)$ it follows $G_n = \tilde{G}(t)\sqrt{\tau}$, $D_n = \tilde{D}(t)/\sqrt{\tau}$. From (107), (108) we compute

$$\begin{aligned} A_n &= \frac{\partial f}{\partial z}(\hat{z}_n, x_n) \\ &= \begin{bmatrix} 1 & \tau \\ (-1 + 2\hat{z}_{1,n}\hat{z}_{2,n})\tau & 1 + \left(\hat{z}_{1,n}^2 + 3\hat{z}_{2,n}^2 - 1\right)\tau \end{bmatrix} \end{aligned} \tag{111}$$

and

$$C_n = \frac{\partial h}{\partial z}(\hat{z}_n) = [1 \quad 0]. \tag{112}$$

This section is divided in two parts. First we verify the theoretical results from Section IV for systems with measurement noise, later the theoretical results from Section III for systems with process noise. Generally we choose

$$Q_n = I\tau \tag{113}$$

$$R_n = 1/\tau \tag{114}$$

$$P_0 = I \tag{115}$$

$$z_0 = [0.8 \quad 0.2]^T \tag{116}$$

and the sampling time $\tau = 10^{-3}$, executing $n = 10^4$ steps. The remaining matrices G_n and D_n as well as the initial value $\hat{z}_0$ are chosen particularly for each case and are shown in Table I.

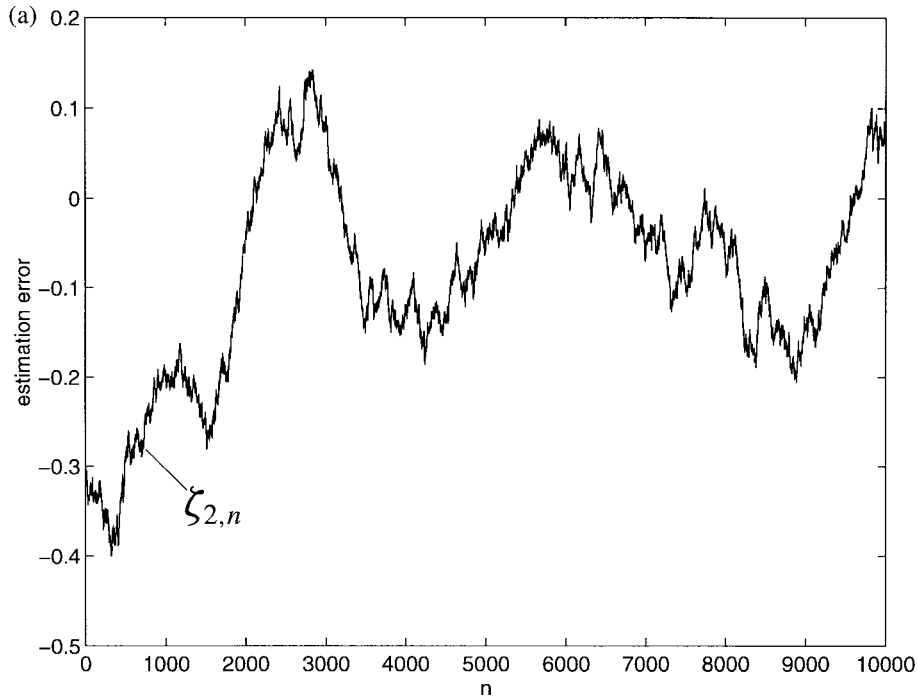
In the first part of this section we present numerical simulations according to the results of Section IV. To fulfill the Assumption 1 in Theorem 4.1 the nonlinear system has to satisfy the observability rank condition for every $z_n \in \mathcal{K}$, where $\mathcal{K}$ denotes a compact subset $\mathcal{K} \subset R^q$. For $\mathcal{K} = B_0(10) = \{b \in R^2 \mid \|b\| \leq 10\}$, using (85) and (111), (112) we get

$$U(z_n) = \begin{bmatrix} 1 & 0 \\ 1 & \tau \end{bmatrix}. \tag{117}$$

With (117) we compute

$$\operatorname{rank} U(z_n) = 2 \tag{118}$$

for $z_n \subset \mathcal{K}$, i.e., Assumption 1 holds. With (107), (108), and (111) it can be easily checked that Assumption 2 is also fulfilled. Assumption 3 is verified by numerical simulations, where the following cases are considered: small initial error and small measurement noise, large measurement noise as well as large initial error. For all cases the process noise is set equal to zero, i.e., $G_n \equiv 0$.



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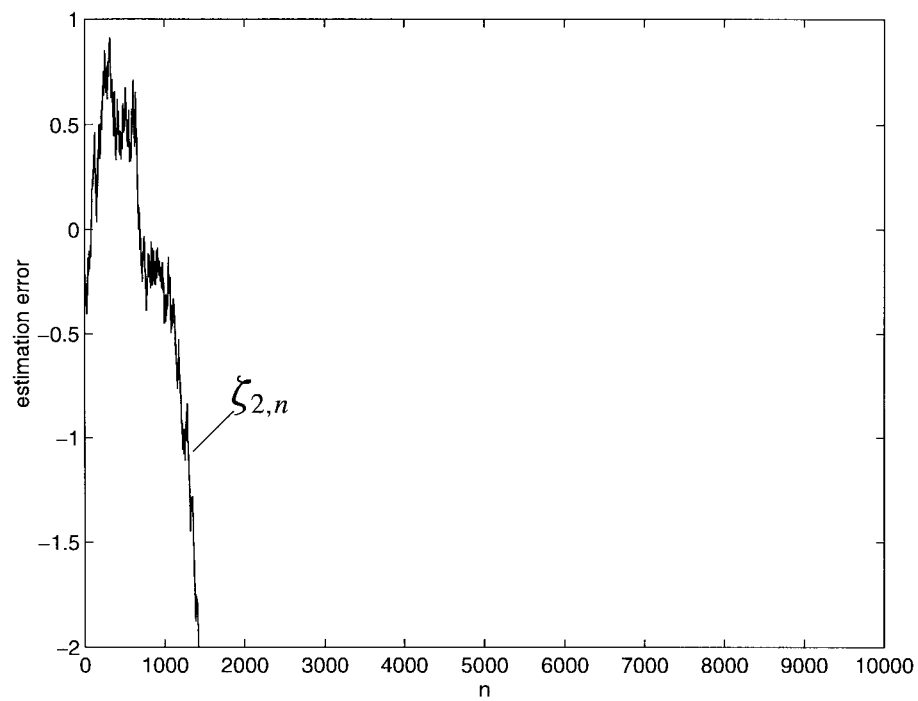

(b)

Fig. 4. Numerical simulations for the considered example system with nonzero process noise. The diagrams show the estimation error component $\zeta_{2,n}$ as a function of the discrete time n for several cases: (a) small initial error and small noise; (b) large process noise.

The simulation results are depicted in Figs. 1 and 2, where sample paths for the unknown state $z_{2,n}$ and the estimated state $\hat{z}_{z,n}$ as well as for the estimation error $\zeta_{2,n}$ are plotted versus n. For small initial error and small measurement noise [cf. (88), (89)] the estimation error remains bounded, as can be verified in Figs. 1(a) and 2(a). In the case of large measurement noise or large initial error (i.e., conditions (88), (89) are violated), the estimation error is divergent [see Figs. 1(b), 1(c) and 2(b), 2(c)], which is due to the high nonlinearities of the example system.

For a treatment of a nonlinear system with process noise we turn to the theoretical results of Section III. Assumption 3 in Theorem 3.1 is satisfied by the same arguments as in Theorem 4.1; moreover, the bounds (33), (34) for $z_n, \hat{z}_n \in \mathcal{K}$ are

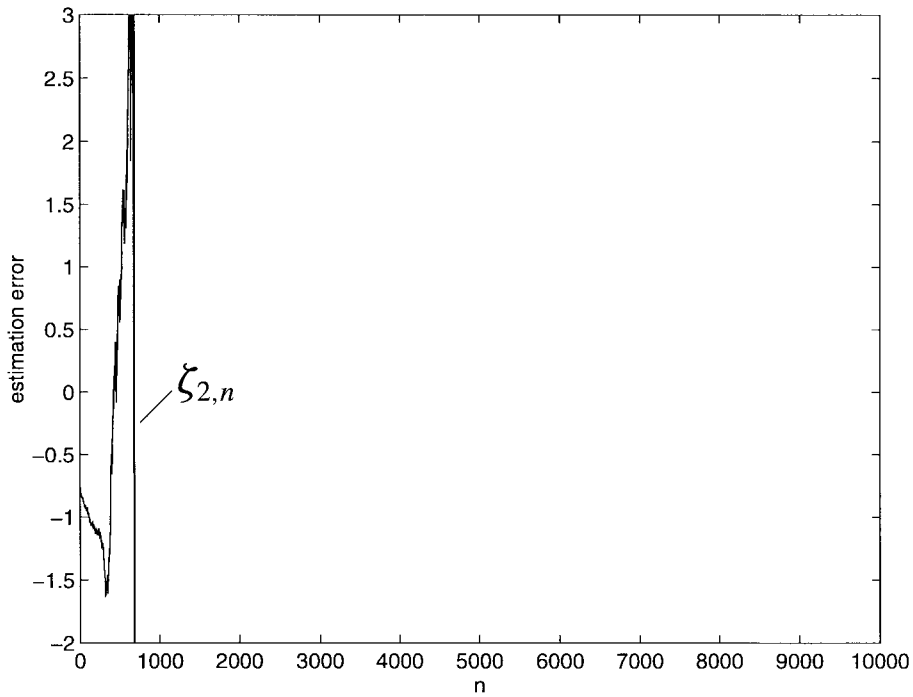

Fig. 4. (*Continued.*) Numerical simulations for the considered example system with nonzero process noise. The diagrams show the estimation error component $\zeta_{2,n}$ as a function of the discrete time n for several cases: (c) large initial error.

TABLE I
INITIAL VALUES AND NOISE WEIGHTING MATRICES FOR THE NUMERICAL SIMULATIONS

	Small initial error and small noise		Large noise		Large initial error	
$\hat{z}_0$	$[0.5 \quad 0.5]^T$		$[0.5 \quad 0.5]^T$		$[1.5 \quad 1]^T$	
G_n	0	$\sqrt{10^{-5}}I$	0	$\sqrt{10^{-3}}I$	0	$\sqrt{10^{-5}}I$
D_n	$\sqrt{10}$		$\sqrt{10^3}$	$\sqrt{10}$	$\sqrt{10}$	
Figure	1a, 2a	3a, 4a	1b, 2b	3b, 4b	1c, 2c	3c, 4c

given by [according to (100), (101)] $\kappa_\varphi = 0.06$ and $\kappa_\chi = 0$, respectively. Conditions (28)–(30) are verified by numerical simulations yielding $\bar{a} = 1$, $\bar{c} = 1$, $\underline{p} = 0.6$ and $\bar{p} = 1.3$, respectively. We consider the following cases: small initial error and small process noise, large process noise as well as large initial error. The simulation results are depicted in Figs. 3 and 4, where sample paths for the unknown state $z_{2,n}$ and the estimated state $\hat{z}_{2,n}$ as well as for the estimation error $\zeta_{2,n}$ are plotted versus the discrete time n. It can be seen in Figs. 3(a) and 4(a) that for small initial error and small process noise [cf. conditions (35)–(37)] the estimation error remains bounded. However, if the process noise or the initial error is large (i.e., conditions (35)–(37) are violated), then the estimation error is no longer bounded and diverges, as shown in Fig. 3(b), 3(c) and 4(b), 4(c).

In the numerical simulations the estimation error is bounded if $\|\zeta_0\| \leq 0.4$ and $G_n G_n^T \leq 10^{-5} I$, $D_n D_n^T \leq 10$ is fulfilled. However, estimating ϵ and δ via (78) and (81) yields much smaller values for the bounds. For a considered compact subset $\mathcal{K} \subset R^q$ with $\mathcal{K} = B_0(10) = \{b \in R^2 \mid \|b\| \leq 10\}$ we get $\epsilon \leq 5 \cdot 10^{-3}$ and $\delta \leq 10^{-10}$, respectively. This shows that the estimates for ϵ, δ are very conservative, and therefore only of theoretical interest. To quantify the error bound according to (18) we make use of (35) and $\underline{v} = 1/\bar{p}$, $\bar{v} = 1/\underline{p}$, $\mu = \kappa_{\text{noise}}\delta$ with (71) and (81). With the theoretical values $\epsilon \leq 5 \cdot 10^{-3}$ and $\delta \leq 10^{-10}$ we get $E\{\|\zeta_n\|^2\} \leq 10^{-4}$. For the values $\epsilon = 0.4$ and $\delta = 10^{-5}$ resulting from the numerical simulations we obtain $E\{\|\zeta_n\|^2\} \leq 0.4$, in agreement with the values of $\|\zeta_n\|$ in the simulations [cf. Figs. 2(a) and 4(a)].

VI. Conclusion

We have analyzed the error behavior of the extended Kalman filter, if it is applied to general estimation problems for nonlinear stochastic discrete-time systems. In Section III we have shown that under certain conditions the estimation error is bounded in mean square and bounded with probability one. These conditions include the requirements that the initial estimation error as well as the disturbing noise terms are small enough, the nonlinearities are not discontinuous, and the solution of the Riccati difference equation (which is the error covariance in the linear case) remains positive definite and bounded. As shown in Section IV, for autonomous systems this condition on the solution of the Riccati difference equation can be reduced to a nonlinear observability rank condition, which can be checked in advance. The numerical simulations in Section V verify the error boundedness if the initial estimation error and the disturbing noise terms are small. Moreover, they indicate that the estimation error is divergent if either the initial error or the noise terms are large. The results presented in this paper are limited to the discrete-time case; the continuous-time case is treated in the companion paper [41].

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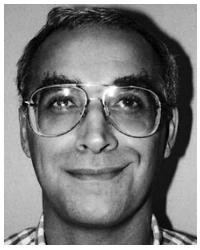

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