# Minimum Sensitivity Controllers With Application to VTOL Aircraft

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**Abstract**

**A technique is described for the design of a fixed-gain feedback controller for a vertical takeoff and landing (VTOL) aircraft, which minimizes the effect of arbitrary variations in the aircraft dynamics on aircraft performance over varying flight conditions. The design method involves the assignment of the eigenvalues of the aircraft model to prescribed locations in the complex plane and the minimization of their sensitivities to model parameter variations. This controller is shown to possess better tracking and regulating capabilities than another fixed-gain controller designed merely for assignment of eigenvalues using nominal parameter values, without any consideration of their sensitivity to plant parameter variations.**

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## I. Introduction

The design of controllers for vertical takeoff and landing (VTOL) aircraft has engaged the attention of many researchers in recent years. Various methods have been proposed to take into account the fact that the dynamics of the aircraft change over varying flight conditions. One such controller is an adaptive controller [1, 2] which requires on-board computers which are not at present available in many VTOL aircraft. Also it is quite sophisticated and can be expensive. Another controller which has been recently proposed [3] utilizes a sensitivity-reduction method to improve the performance of a fixed-gain feedback controller designed on the basis of nominal values of the aircraft model parameters. Although an on-board computer is not needed, this method calls for a feed-forward input to be stored and fed in at proper instants as a varying function of time. Another drawback of this method is that the nature of the parameter variation must be assumed or known in order to compute the feed-forward input required to be stored. A different method of designing a fixed-gain feedback controller which minimizes the effect of arbitrary parameter variations on aircraft performance is presented in this paper. This design procedure involves the assignment of the eigenvalues of the aircraft model matrix to prescribed locations in the complex plane in such a manner that the sensitivity of the eigenvalues to model parameter variations is a minimum. Being a constant-gain feedback controller, it is relatively easy to implement at a reasonable cost. By computer simulation studies it is shown that this controller has satisfactory tracking and regulating properties under varying flight conditions. The design procedure is described in Section II and in the Appendix. Sections III, IV, and V illustrate the application of this procedure to a particular helicopter model. The conclusions are given in Section VI.

## II. Design Procedure

Consider a linear multivariable multi-input dynamic system described by

$$\dot{x} = Ax + Bu$$

$$y = Cx \tag{1}$$

where $x$ is the $n$-dimensional state vector, $u$ is the $m$-dimensional input vector, and $y$ is the $p$-dimensional output vector. $A$, $B$, and $C$ are respectivey $n \times n$, $n \times m$, and $p \times n$ matrices.

It is desired to design a fixed-gain feedback controller such that some of the state variables satisfy certain specified requirements, which, in the case of the VTOL aircraft are that the transient responses of the controlled variables do not deviate excessively from acceptable values and these variables attain in the steady state certain predetermined values. The former requirement can be met by assigning the eigenvalues of the closed-loop system matrix to prescribed locations in the complex plane and making them *least* sensitive to parameter variations. In order to ensure that the steady-state error will be zero, the procedure suggested in [4] is followed and we introduce a new state vector $z$, governed by the differential equation

$$\dot{z} = -Cx + Ix_r \tag{2}$$

where $z$ is a $p$-dimensional vector, $x_r$ represents the $p$ variables to be controlled, and $I$ is a $p \times p$ identity matrix. $C$ is the same as in (1). Equations (1) and (2) are then combined to form the augmented system

$$\dot{\hat{x}} = A^* x + B^* u + I^* x_r \tag{3}$$

where

$$A^* = \left[\begin{array}{c|c} A & 0 \\ \hline -C & 0 \end{array}\right] ; \quad B^* = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] ; \quad I^* = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right]. \tag{4}$$

The effect of this augmentation is to make the steady-state error in the states to be controlled ($x_r$) equal to zero.

In order to assign the eigenvalues of the closed-loop system matrix to prescribed locations in the complex plane and also make them least sensitive to parameter variations, a procedure described in detail in the Appendix is followed. However, in order to do so, it is necessary that the system matrix $A^*$ be cyclic ($A^*$ will be cyclic if there exists an $n$-vector $w$ such that $[w\, A^* w \ldots A^{*n-1} w]$ is of full rank). Unfortunately, $A^*$ will never be cyclic since it has multiple eigenvectors corresponding to its zero eigenvalue. The first step therefore is to remove the noncyclicity of $A^*$. This is achieved by a method suggested in [4]. This step will be referred to as the first stage.

*First Stage:* Let

$$u = u^1 + u^2 + R \tag{5}$$

where $u^1$ and $u^2$ are, respectively, the first and second stage controls, and $R$ is the reference input.

We define the first stage control as

$$u^1 = -K_{T1} \; [x \mid z]^{\mathrm{T}} \tag{6}$$

where

$$K_{T1} = [0 \mid K_I]. \tag{7}$$

Superscript T means matrix or vector transpose. In general, $K_{T1}$ can be chosen arbitrarily subject to the requirement that $K_I$ is of rank $p$. However, when $p = m$, as will be the case in the sequel, choice of $K_I = I$, a $p \times p$ identity matrix will be sufficient to remove the noncyclicity of the system matrix.

Combining (3) - (7) we get

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & -BK_I \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u^2 + \begin{bmatrix} B \\ 0 \end{bmatrix} R + \begin{bmatrix} 0 \\ I \end{bmatrix} x_r \tag{8}$$

or

$$\dot{\hat{x}} = \hat{A}^* \hat{x} + B^* u^2 + B^* R + I^* x_r$$

where the definition of $\hat{A}^*$ is obvious and $B^*$ and $I^*$ were previously defined. The augmented state vector $[x \;\; z]^{\mathrm{T}}$ has dimension $n + p$. $A^*$ will be treated as the new system matrix for eigenvalue assignment in the second stage of the design.

*Second Stage*: The procedure described in the Appendix is now used to determine the unity-rank (dyadic) feedback gain matrix $K_{T2}$ which will assign poles of the closed-loop system to prescribed locations in the complex plane and at the same time minimize their sensitivity to parameter variations. It should be noted that $p = m$ in this discussion.

Let

$$u^2 = -qu^0 \tag{9}$$

where $u^0$ is a scalar control and $q$ is an $m \times 1$ (or $p \times 1$) vector of the form

$$q = [1 \;\; q_1 \;\; q_2 \;\; \ldots \;\; q_{m-1}]^{\mathrm{T}}. \tag{10}$$

Let

$$u^0 = k_{t2} \; [x \;\; z]^{\mathrm{T}} \tag{11}$$

where $k_{t2}$ is a row vector with $n + p$ elements. We write

$$k_{t2} = [k_d \mid k_p] \tag{12}$$

where $k_d$ represents the first $n$ elements and $k_p$ represents the remaining $p = m$ elements of $k_{t2}$. Combining (9) and (11) we get

$$u^2 = -qk_{t2} \; [x \mid z]^{\mathrm{T}} = -K_{T2} \; [x \mid z]^{\mathrm{T}} \tag{13}$$

where

$$K_{T2} = qk_{t2} = [K_D \mid K_P]. \tag{14}$$

Substituting (13) into (8) and making use of (14), we get

$$\begin{bmatrix} \dot{x} \\ \hline \dot{z} \end{bmatrix} = \left[\begin{array}{c|c} A - BK_D & -BK_I - BK_P \\ \hline -C & 0 \end{array}\right] \begin{bmatrix} x \\ \hline z \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} R + \begin{bmatrix} 0 \\ \hline I \end{bmatrix} x_r$$

or

$$\dot{\hat{x}} = \hat{A}\hat{x} + B^*R + I^*x_r \tag{15}$$

where the closed-loop system matrix is given by

$$\hat{A} = \left[\begin{array}{c|c} A - BK_D & -B(K_I + K_P) \\ \hline -C & 0 \end{array}\right]. \tag{16}$$

A schematic diagram of the closed-loop system represented by (15) is shown in Fig. 1. The control $u$ in Fig. 1 can be written as

$$u = -K_{T1}[x \mid z]^{\mathrm{T}} - K_{T2}[x \mid z]^{\mathrm{T}} + R$$

$$u = -[K_{T1} + K_{T2}][x \mid z]^{\mathrm{T}} + R$$

or

$$u = -K_T[x \mid z]^{\mathrm{T}} + R \tag{17}$$

where

$$K_T = K_{T1} + K_{T2} = [K_D \mid K_P + K_I]. \tag{18}$$

## III. Minimum Sensitivity Controller For a VTOL Aircraft

In this section the design procedure described in Section II is applied to a specific VTOL aircraft (helicopter).

### Model Dynamics

The model used is that used by Narendra and Tripathi [1]. As in most other techniques used for VTOL controller design, this model is obtained by linearization of the system dynamics around a nominal air speed.

The linearized model of the VTOL aircraft in the vertical plane is described by

$$\dot{x} = Ax + Bu \tag{19}$$

where $A$ is the $(4 \times 4)$ system matrix, $B$ is the $(4 \times 2)$ control matrix, $x$ is the 4-dimensional state vector, and $u$ is the 2-dimensional control vector. The state variables are

- $x_1$ horizontal velocity
- $x_2$ vertical velocity
- $x_3$ pitch rate
- $x_4$ pitch angle.

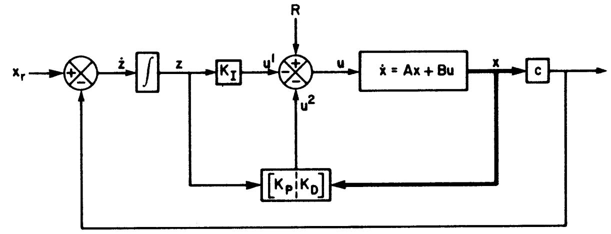



Fig. 1. Block diagram representation of (15).

The control inputs are

- $u_1$ collective
- $u_2$ longitudinal cyclic

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The control $u_1$ is located on the collective pitch lever at the pilot's side. Its main use is the control over the vertical velocity of the VTOL aircraft by the selection of a desired flight angle. The flight path or angle is changed merely by the pilot applying collective control pressure in an up or down motion. This control also has some effect on the horizontal velocity. The control $u_2$ is one of the controls located on the cyclic control stick immediately in front of the pilot. Its main use is to control the horizontal velocity of the VTOL aircraft. The up and down movement of the control (the one we are considering) changes the forward horizontal velocity of the helicopter while a left or right movement of the control will change the heading reference to that of the flight indicator.

$u_1$ and $u_2$ are not to be confused with $u^1$ and $u^2$ defined earlier as the first and second stage controls.

Following [1], the nominal airspeed in assumed to be 135 knots. At this nominal airspeed the system matrix $A$ and control matrix $B$ are

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.42 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0.0 & 0.0 \end{bmatrix}.$$

As the aircraft deviates from its nominal air speed, all the elements in the first three rows of both matrices change. Since the most significant changes occur in the elements $a_{32}$ and $a_{34}$, the rest of the elements can be assumed to remain constant without serious loss of accuracy.

## Design Objectives

Stated in general terms, the objectives of the design are: 1) to enable the VTOL aircraft to go from one steady state to another steady state without excessive transient fluctuations in the controlled variables (this will determine the tracking capability of the controller); and 2) to minimize the transient fluctuations in the controlled variables when the VTOL aircraft is subject to disturbances (this will determine the regulating capability of the controller).

By proper choice of the controlled variables, a suitable controller can be designed for achieving various practical objectives. In this paper, the following two specific objectives are chosen: 1) It should be possible to take the VTOL aircraft from one horizontal velocity to another horizontal velocity in a smooth fashion with minimum fluctuations in the vertical velocity which is zero to begin with. 2) When cruising horizontally at a constant speed the effect of wind gusts and similar disturbances on both the horizontal velocity and the zero vertical velocity should be minimized.

The horizontal velocity $x_1$ and the vertical velocity $x_2$ of the VTOL aircraft are therefore chosen as the output variables to be controlled. This means that the $C$ matrix in (1) is given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that $p = m = 2$ in this example; $n = 4$. If we define the vector $z$ in (2) as

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}$$

then (2) becomes

$$\dot{x}_5 = -x_1 + x_{r1}$$

$$\dot{x}_6 = -x_2 + x_{r2}$$

where $x_{r1}$ and $x_{r2}$ are the reference values for the horizontal and vertical velocities, respectively.

*First Stage:* Since $p = m$, we choose the matrix $K_{T1}$ in (6) as

$$K_{T1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \overbrace{1 & 0}^{K_I} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix $\hat{A}^*$ at the nominal air speed of 135 knots is given by (see (8))

$$\hat{A}^* = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 & -0.4422 & -0.1761 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 & -3.5446 & 7.5922 \\ 0.1002 & 0.3681 & -0.707 & 1.42 & 5.52 & -4.49 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

We now consider $\hat{A}^*$ as the open-loop system matrix for the second stage of the design.

*Second Stage:* It is now desired to obtain the dyadic feedback gain matrix $K_{T2}$ to assign the closed-loop poles of the augmented helicopter system to the following locations in the complex frequency plane: $s_1 = -4.0$, $s_2 = -3.0$, $s_3 = -2.0$, $s_4 = -1.5$, $s_5 = -1.0 + j1.0$, and $s_6 = -1.0 - j1.0$.

Since only the parameters $a_{32}$ and $a_{34}$ of $A$ are varying with changes in the horizontal air speed, the minimum eigenvalue sensitivity functional is given by

$$J = \sum_{i=1}^{n+p} (|S_{32}^i|)^2 + (|S_{34}^i|)^2$$

where the definition of the sensitivity functions $S$ is given in the Appendix. The functional $J$ is a function of $q$. The minimum value of $J$ occurs when $q = [1 \quad 0.562]^T$. This results in the minimum eigenvalue sensitivity matrix $K_{T2}$ of

$$K_{T2} = \begin{bmatrix} \underbrace{\begin{matrix} 8.8193 & 0.6028 & -2.1379 & -10.1264 \\ 4.9553 & 0.3381 & -1.2012 & -5.6897 \end{matrix}}_{K_D} & \underbrace{\begin{matrix} -9.6836 & 8.3666 \\ -5.4408 & 4.7009 \end{matrix}}_{K_P} \end{bmatrix}$$

and the total minimum eigenvalue sensitivity feedback matrix $K_T$ of

$$K_T = \left[\underbrace{\begin{matrix} 8.8193 & 0.6028 & -2.1379 & -10.1264 \\ 4.9553 & 0.3381 & -1.2012 & -5.6897 \end{matrix}}_{K_D} \quad \underbrace{\begin{matrix} -8.6836 & 8.3666 \\ -5.4408 & 5.7009 \end{matrix}}_{K_P + K_I}\right].$$

The matrix $\hat{A}$ in (16) becomes

$$\hat{A} = \begin{bmatrix} -4.8091 & -0.2986 & 1.1757 & 5.0244 & 4.7980 & -4.7036 \\ 6.4084 & -0.5760 & -1.5394 & -11.3237 & -10.5281 & 13.6260 \\ 26.5339 & 2.1718 & -7.1149 & -28.9313 & -23.5041 & 20.5868 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

It can be verified that the eigenvalues of $\hat{A}$ are at the desired locations in the complex plane. As mentioned earlier the above design was carried out at the nominal air speed of 135 knots. It has been shown in [1] that the parameters $a_{32}$ and $a_{34}$ of the matrix $A$ vary nonlinearly with air speed. In range of 60 knots to 170 knots, the bounds for these two elements are $0.06635 < a_{32}(t) < 0.5047$ and $0.1198 < a_{34}(t) < 2.526$. The sensitivity of the eigenvalue locations for variations in the elements $a_{32}$ and $a_{34}$ when the air speed changed from 135 to 60 knots and from 135 to 170 knots are given in Table I. The table also includes corresponding figures obtained with another fixed-gain controller, the details of which follow immediately.

Another controller was designed to assign the poles of the closed-loop system to the same locations as before. No parameter variations were considered. The controller design was carried out using the nominal values of the system matrix. Choice of $q = [1 \;\; 3]^{\mathrm{T}}$ resulted in the dyadic feedback gain matrix $K_{T2}$ as

$$K_{T2} = \begin{bmatrix} 13.3175 & -1.2647 & -3.3333 & -7.1020 & -15.3930 & 3.4701 \\ 39.9526 & -3.7940 & -10.0000 & -21.3059 & -46.1789 & 10.4103 \end{bmatrix}$$

using the same $K_{T1}$ as in the minimum sensitivity design, the complete feedback controller turned out to be

$$K_T = \begin{bmatrix} 13.3175 & -1.2647 & -3.3333 & -7.1020 & -14.3930 & 3.4701 \\ 39.9526 & -3.7940 & -10.0000 & -21.3059 & -46.1789 & 11.4103 \end{bmatrix}.$$

Use of this $K_T$ resulted in the closed-loop system matrix of

$$\hat{A} = \begin{bmatrix} -12.9613 & 1.2544 & 3.2538 & 6.4370 & 14.4967 & -3.5438 \\ 256.1712 & -25.3318 & -64.1044 & -140.6057 & -299.5822 & 74.3289 \\ -105.7743 & 10.4231 & 25.7931 & 57.8806 & 127.8941 & -32.0772 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

which also has the desired pole locations as can be verified. For purposes of convenience this second controller is referred to as the arbitrary controller.

A comparison of the above results reveals that the variations in the poles of the closed-loop system, due to changes in the system plant parameters, are appreciably less when the minimum sensitivity design approach is used to obtain the feedback controller.

**TABLE I**
Comparison of Eigenvalue Sensitivities with Minimum Sensitivity and Arbitrary Controllers

| Change in Air Speed | Percent Change in Elements $a_{32}$ and $a_{34}$ | | Percent Change in Locations of Eigenvalues of $\hat{A}^*$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\Delta a_{32}$ | $\Delta_{34}$ | | $\Delta s_1$ | $\Delta s_2$ | $\Delta s_3$ | $\Delta s_4$ | $\Delta s_5$ | $\Delta s_6$ |
| 135-60 knots | 82 | 92 | minimum sensitivity control | 38 | 62 | 3.2 | 49 | 48 | 48 |
| | | | arbitrary control | 133 | 72 | 84 | 82 | 177 | 177 |
| 135-170 knots | 37 | 78 | minimum sensitivity control | 25.7 | 64 | 84 | 38 | 42 | 42 |
| | | | arbitrary control | 106 | 153 | 3.2 | 70 | 66 | 66 |

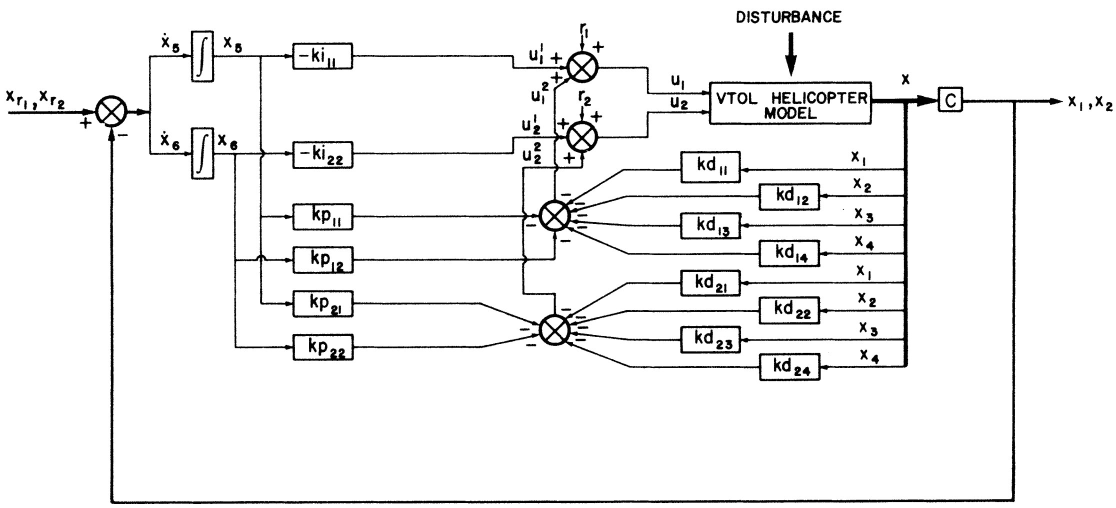



Fig. 2. Schematic diagram of minimum sensitivity controller of VTOL aircraft.

## IV. Tracking Capabilities of Minimum Sensitivity and Arbitrary Controllers

To compare the tracking capabilities of the two controllers designed in the preceding section, the entire system was simulated on a digital computer using continuous system modeling program (CSMP). A schematic diagram of the control system is shown in Fig. 2.

The responses of the system with both controllers were obtained when accelerating from 60 to 135 knots and again when accelerating from 135 to 170 knots. The following steps were involved in the simulations.

1) To begin the simulation at 60 knots,
   a) $R = [r_1 \;\; r_2]^T \; [0.4 \;\; 0.4]^T$ was arbitrarily chosen;
   b) for $x_{r1}$ and $x_{r2}$ equal to 1.0 and 0.0, respectively, the steady-state values were obtained for the other variables as $x_1 = 1.0, x_2 = 0.0, x_3 = 0.0$, and $x_4 = -0.018$.
2) To accelerate from 60 to 135 knots
   a) $R = [1.0 \;\; 1.0]^T$ was chosen;
   b) initial conditions were set as obtained in 1); and
   c) $x_{r1}$ and $x_{r2}$ equaled 3.0 and 0.0, respectively.
3) To accelerate from 135 to 170 knots
   a) $R = [1.25 \;\; 1.25]^T$ was chosen;
   b) initial conditions were set as obtained in 2); and
   c) $x_{r1}$ and $x_{r2}$ equaled 5.0 and 0.0, respectively.

The variable disturbance **w** was omitted during the simulation of tracking capabilities of the two controllers. It was included in regulation studies (see Section V).

The responses, i.e., the horizontal ($x_1$) and vertical ($x_2$) velocities when accelerating from 60 to 135 knots, are shown in Figs. 3 and 4, respectively. In Figs. 5 and 6 $x_1$ and $x_2$ are shown when accelerating from 135 to 170 knots. The responses are shown for both minimum sensitivity and the arbitrary controllers.

## V. Regulating Capabilities of the Two Controllers

In this section we look at the ability of the controllers to maintain a constant output, i.e., zero steady-state error of the chosen outputs, when the system is subject to disturbances. The disturbance vector is

$$w = [w_1 \;\; w_2]^T$$

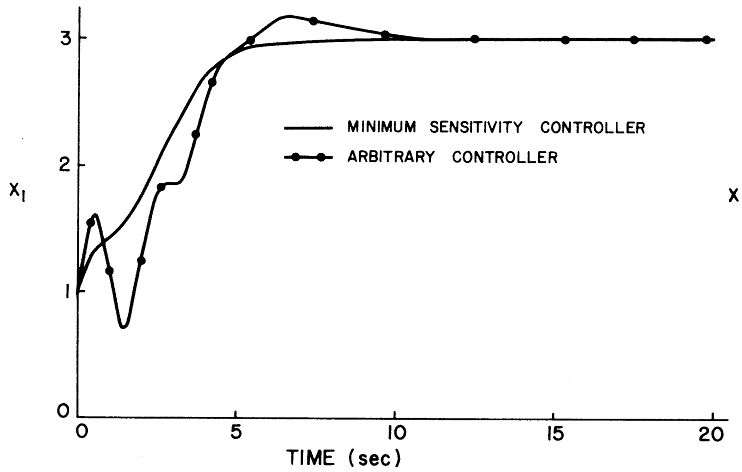
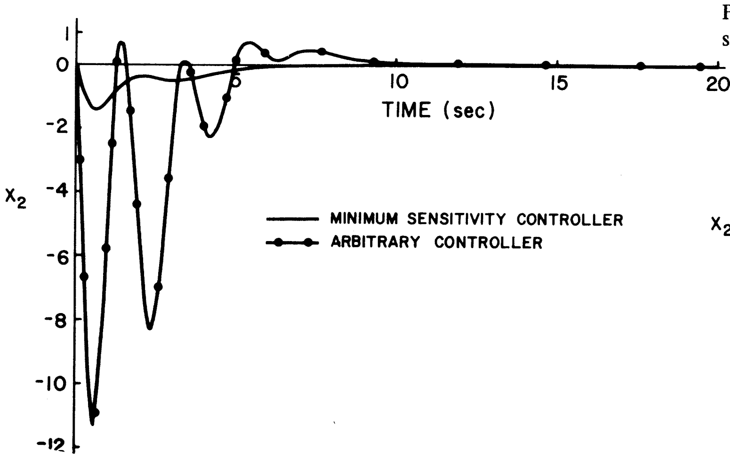



Fig. 3. Transient behavior of horizontal velocity of helicopter when accelerating from 60 to 135 knots.

Fig. 4. Transient behavior of vertical velocity for the conditions stated in Fig. 3.



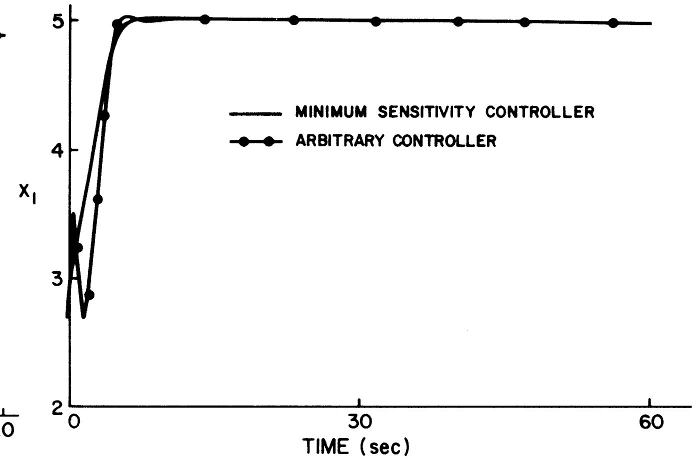
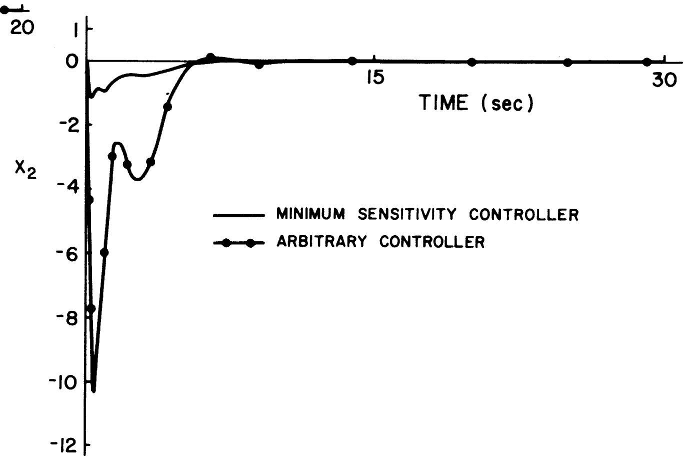




Fig. 5. Transient behavior of horizontal velocity of helicopter when accelerating from 135 to 170 knots.

Fig. 6. Transient behavoir of vertical velocity for the conditions stated in Fig. 5.




For a desired horizontal velocity of 135 knots and zero vertical velocity, the following nominal values were selected:

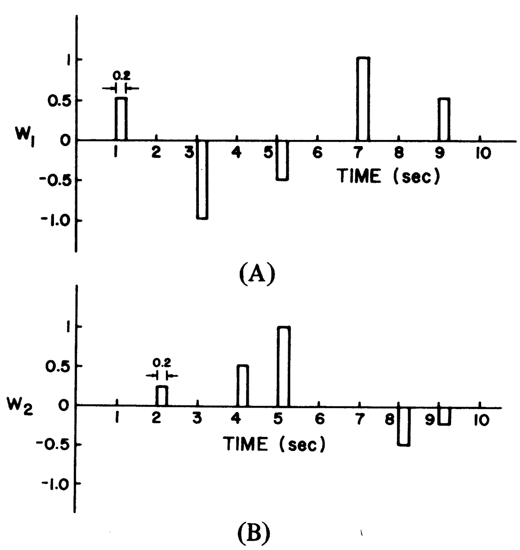
1) pilot inputs:
   $r_1 = 1.0$
   $r_2 = 1.0$
2) reference values:
   $x_{r1} = 3.0$
   $x_{r2} = 0.0.$

The disturbance vector used in the simulations is shown in Fig. 7. The responses (deviations $\delta x_1$ and $\delta x_2$ from the nominal steady-state values) for the two controllers are shown in Figs. 8 and 9. It is seen that the minimum sensitivity controller performs better than the arbitrary controller.

## VI. Conclusions

A method of designing a fixed-gain feedback controller which minimizes the effect of changes in aircraft dynamics on the performance of a VTOL aircraft has been

Fig. 7. Disturbance vector $w = [w_1 \;\; w_2]^T$ used in regulation studies. (A) Component $w_1$. (B) Component $w_2$.



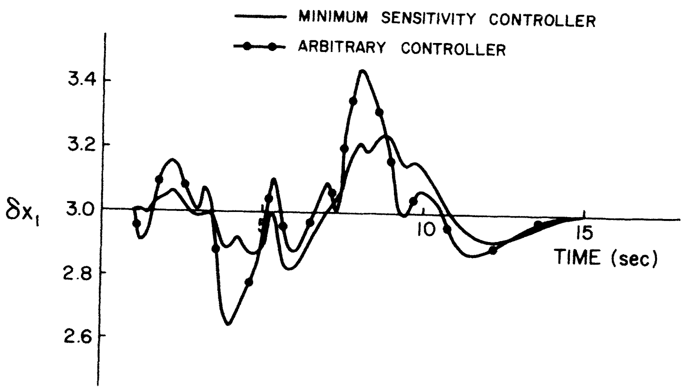
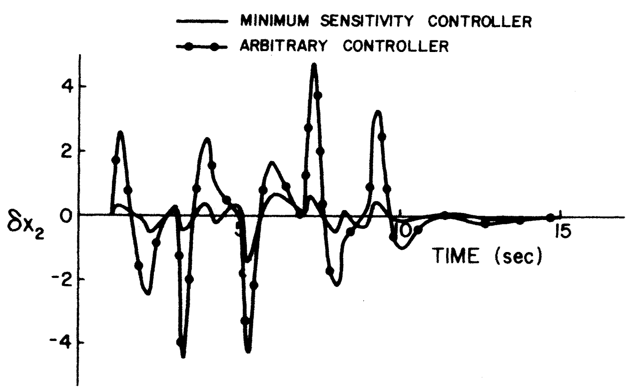



Fig. 8. Deviation from nominal value of horizontal velocity of 135 knots due to wind disturbance w.

Fig. 9. Deviation from nominal value of vertical velocity due to wind disturbance w.




presented. It is shown that this minimum sensitivity controller performs better than another fixed-gain controller designed based on nominal values of the aircraft model without any consideration of parameter variations. Being a fixed-gain controller, the minimum sensitivity controller should be simpler and less expensive than the more expensive (although more sophisticated) adaptive controllers that have been proposed by earlier researchers.

## Appendix

### Pole Assignment with Minimum Eigenvalue Sensitivity to Plant Parameter Variations

In this Appendix a procedure is described for assigning the closed-loop poles of a feedback control system to specified locations in the complex frequency plane in such a way that the poles have minimum sensitivity to plant parameter variations. The discussion follows closely the work of Gourishankar and Ramar [5].

Consider a linear time invariant multivariable system represented by

$$\dot{x} = Ax + Bu \tag{A1}$$

where $x$ is an $n$-dimensional state vector, $u$ is an $r$-dimensional input vector, $A$ is the $n \times n$ system matrix, and $B$ is the $n \times r$ input matrix. This multi-input system is reduced to an equivalent single-input system by defining a new scalar input $u'$ as

$$u = qu' + R \tag{A2}$$

where

$$q = [1, q_1, \ldots, q_m]^T \tag{A3}$$

is an $m$-dimensional column vector, and $R$ is the $m$-dimensional external input control vector. Then using (A2), (A1) becomes

$$\dot{x} = Ax + Bqu' + BR. \tag{A4}$$

Now a feedback control

$$u' = kx \tag{A5}$$

is designed such that the poles (eigenvalues) of the closed-loop system

$$\dot{x} = (A + Bqk)x + BR \tag{A6}$$

are at the prescribed locations $s_1, s_2, \ldots, s_n$ in the complex plane $k$ is an $n$-dimensional row vector. It is a function of the $m$-dimensional vector $q$ in the sense that the values of the elements of $k$ depend on the values of the elements of $q$.

For the sake of convenience we write (A6) as

$$\dot{x} = \hat{A} + BR \tag{A7}$$

where

$$\hat{A} = A + Bqk \quad \text{or} \quad A + b_q k\,. \tag{A8}$$

In general $q$ is chosen arbitrarily. In the present discussion $q$ will be chosen to minimize the sensitivity of the eigenvalues of the closed-loop system to variations in the elements of $A$. Once $q$ is chosen, the vector $k$ is determined for the specified eigenvalue (pole) locations. The $m \times n$ unity rank feedback matrix $K$ for the closed-loop system in (A6) is then obtained as

$$K = qk. \tag{A9}$$

#### Eigenvalue Sensitivity

Following Morgan [6] the sensitivity $S^i_{j\ell}$ of any pole (eigenvalue) $s_i$ of the closed-loop system matrix $\hat{A}$ in equation (A7) with respect to a "small" variation in the element $a_{j\ell}$ of the open-loop system matrix $A$ is defined as

$$S^i_{j\ell} = \partial s_i / \partial a_{j\ell} = [1/g'(s_i)]\,[\mathrm{tr}\, R(s_i)\,(\partial \hat{A} / \partial a_{j\ell})] \tag{A10}$$

where

$g'(s_i)$ is the derivative of the closed-loop system characteristic polynomial with respect to $s$ evaluated at $s = s_i$;
$R(s_i)$ is the adjoint $(s_i I - \hat{A})$, $I$ is the $n \times n$ identity matrix; and
tr denotes the trace of a matrix.

As can be seen from (A7), the closed-loop system matrix $\hat{A}$, is a function of $q$ and $k$; $k$ is a function of $q$ and therefore the sensitivity $s^i_{j\ell}$ is directly a function of $q$. In order to minimize the effect of the variations in the parameter $a_{j\ell}$ on $S^i_{j\ell}$ a sensitivity function is formulated as [6]

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=1}^{n} (|S^i_{j\ell}|)^2. \tag{A11}$$

The objective is to determine $q$ such that $J$ is a minimum. In order to facilitate the minimization procedure, we transform the matrix pair $(\hat{A}, b_q)$ to a phase-variable or canonical form. Following [7] we introduce a transformation matrix $P(q)$ such that

$$x = P(q)^{-1} z$$

or

$$z = P(q)\, x. \tag{A12}$$

Now using the transformation equation (A12) we obtain the following transformed closed-looped system equation:

$$\dot{z} = A_0 z + P(q) BR \tag{A13}$$

where

$$A_0 = P(q)(A + b_q k)\, P(q)^{-1}. \tag{A14}$$

The preselection of the desired poles of the closed-looped system matrix $\hat{A}$ determines the elements of $A_0$, where $A_0$ in canonical form is written as

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 \cdots 0 & \cdots 0 & & 1 \\ -\alpha_1 & -\alpha_2 & & \cdots & -\alpha_n \end{bmatrix} \tag{A15}$$

where $\alpha_1, \alpha_2, \ldots \alpha_n$ are the known coefficients of the closed-loop characteristic equation

$$s^n + \alpha_n s^{n-1} + \alpha_{n-1} s^{n-2} + \ldots + \alpha_2 s + \alpha_1 = 0. \tag{A16}$$

We can rewrite (A10) in terms of the transformed equation as

$$S^i_{j\ell} = [1/g'(s_i)] \operatorname{tr} [L(s_i)(\partial A_0/\partial a_{j\ell})] \tag{A17}$$

where

$L(s_i)$ is the adjoint $(s_i I - A_0)$, noting this is independent of $q$ by the definition of $A_0$ in (A15).
Now substituting (A14) into (A17) we have

$$S^i_{j\ell} = [1/g'(s_i)] \operatorname{tr} \Big[ \Big( L(s_i) \{\partial\, [P(q)(A + b_q k) \cdot P(q)^{-1}]/\partial a_{j\ell}\} \Big)$$

or

$$S^i_{j\ell} = [1/g'(s_i)] \operatorname{tr} \{P(q)^{-1} L(s_i) P(q) \cdot [\partial (A + b_q k)/\partial a_{j\ell}]\}. \tag{A18}$$

Since only the elements of $A$ are varying

$$[\partial (A + b_q k)/\partial a_{j\ell}] = \partial A/\partial a_{j\ell} + [\partial (b_q k)/\partial a_{j\ell}]$$

or

$$[\partial (A + b_q k)/\partial a_{j\ell}] = \partial A/\partial a_{j\ell}. \tag{A19}$$

By using (A19) in (A18), we obtain

$$S^i_{j\ell} = [1/g'(s_i)]\, P_\ell(q)^{-1} L(s_i) P_j(q) \tag{A20}$$

where

$P_\ell(q)^{-1}$ is the $\ell$th row of $P(q)^{-1}$ and
$P_j(q)$ is the $j$th column of $P(q)$.

Using (A20) in (A11) we have the sensitivity functional $J$, strictly as a function of the variable $q$. We can now minimize the functional $J$ with respect to $q$; the required $k$ necessary for the desired pole locations can then be easily determined by use of (A14). Once $k$ is known, the minimum sensitivity controller $K$ is obtained by (A9).

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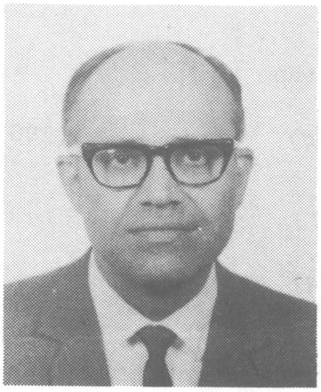

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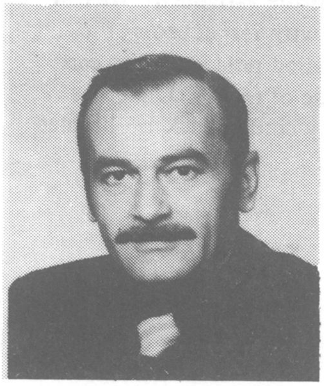

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