# Nonlinear Robust Disturbance Rejection Controllers for Rotating Stall and Surge in Axial Flow Compressors

Wassim M. Haddad, Alexander Leonessa, Vijaya-Sekhar Chellaboina, and Jerry L. Fausz

***Abstract*—In this paper we develop globally stabilizing robust/disturbance rejection controllers for rotating stall and surge in axial flow compressors with uncertain system dynamics and exogenous disturbances. Specifically, using the nonlinear-nonquadratic disturbance rejection optimal control framework for systems with bounded energy (square-integrable) $L_2$ disturbances developed in [15] and the nonlinear-nonquadratic robust optimal control framework for systems with nonlinear structured parametric uncertainty developed in [16], a *family* of globally robustly stabilizing controllers for jet engine compression systems is developed. The proposed controllers are compared with the locally stabilizing bifurcation-based controllers of [17]–[20] and the recursive backstepping controllers of [21].**

***Index Terms*— Compression systems, disturbance rejection, globally stabilizing control, parametric uncertainty, robust control, rotating stall, surge.**

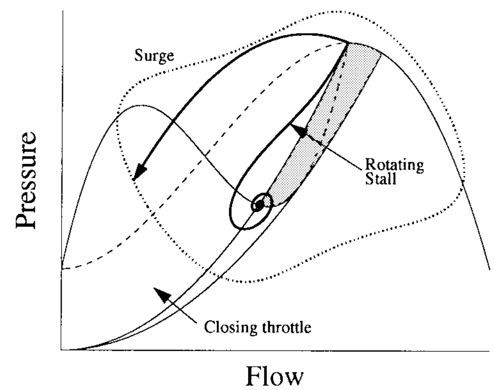


Fig. 1. Schematic of compressor characteristic map for a typical compression system (— stable equilbria, - - - unstable equilibria).

## I. Introduction

THE desire for developing an integrated control system-design methodology for advanced propulsion systems has led to significant activity in modeling and control of flow compression systems in recent years (see, for example, [1]–[14] and the numerous references therein). However, unavoidable discrepancies between compression system models and real-world compression systems can result in degradation of control-system performance including instability. In particular, jet engine compression systems with poorly modeled dynamics and exogenous disturbances can severely limit jet engine compression system performance by inducing the compressor aerodynamic instabilities of rotating stall and surge. Rotating stall is an inherently two-dimensional local compression system oscillation which is characterized by regions of flow that rotate at a fraction of the compressor rotor speed while surge is a one-dimensional axisymmetric global compression system oscillation which involves axial flow oscillations and in some cases even axial flow reversal which can damage engine components and cause flameout to occur.

Rotating stall and surge arise due to perturbations in stable system operating conditions involving steady, axisymmetric flow and can severely limit compressor performance. The transition from stable compressor operating conditions to rotating stall and surge is shown in Fig. 1 representing a schematic of a compressor characteristic map where the abscissa corresponds to the circumferentially averaged mass flow through the compressor and the ordinate corresponds to the normalized total-to-static pressure rise in the compressor. For maximum compressor performance, operating conditions require that the pressure rise in the compressor correspond to the maximum pressure operating point on the stable axisymmetric branch for a given throttle opening. In practice, however, compression system uncertainties and compression system disturbances can perturb the operating point into an unstable region driving the system to a stalled stable equilibrium, a stable limit cycle (surge), or both. In the case of rotating stall, an attempt to recover to a high pressure operating point by increasing the flow through the throttle traps the system within a flow range corresponding to two stable operating conditions involving steady axisymmetric flow and rotating stall resulting in severe hysteresis.

To avoid rotating stall and surge, traditionally system designers allow for a safety margin (rotating stall or surge margin) in compression system operation. However, to account for compression system uncertainties such as system modeling errors, in-service changes due to aging, etc., and compression system disturbances such as compressor speed fluctuations, combustion noise, etc., operating at or below the rotating stall/surge margin significantly reduces the efficiency of the compression system. In contrast, active control can enhance stable compression system operation to achieve peak compressor performance. However, compression system uncertainty and compression system disturbances are often significant and the need for robust disturbance rejection control is severe.

Manuscript received July 8, 1997; revised February 1, 1998. Recommended by Associate Editor, K. Wise. This work was supported in part by the Army Research Office under Grant DAAH04-96-1-0008, the Air Force Office of Scientific Research under Grant F49620-96-1-0125, and the National Science Foundation under Grant ECS-9496249.

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Publisher Item Identifier S 1063-6536(99)03277-7.

Disturbances having the largest destabilizing effect in compression systems include system transients, circumferential distortion, planar turbulence, and combustion [14]. Transient disturbances can be characterized by system initial conditions. Circumferential distortion involves nonaxisymmetric flow disturbances generated by boundary layer separation caused by high angles of attack at the jet engine inlet. Planar turbulence involves axisymmetric disturbances in the flow field caused by wake ingestion. Finally, combustion disturbances involve unsteady back-pressure perturbations to the compressor from the combustor. Alternatively, uncertainties having the largest destabilizing effect in compression systems include system modeling errors, in-service changes due to aging, manufacturing quality variations, and unmodeled system dynamics. System modeling errors such as uncertainty in the compressor performance pressure-flow characteristic map, in-service changes due to aging, and manufacturing quality variations can be captured as structured parametric uncertainty. Parametric uncertainty refers to system errors that are modeled as real (possibly nonlinear) parameter uncertainty. Unmodeled system dynamics such as truncated system modes, system time delays, and unmodeled actuator-sensor dynamics can be captured as unstructured nonparametric uncertainty. Nonparametric uncertainty refers to uncertain operators that may be modeled as nonlinear arbitrarily time-varying operators.

In this paper we apply the optimality-based disturbance rejection control framework developed in [15] and the optimality-based robust nonlinear control framework developed in [16] to control rotating stall and surge in axial flow compressors with system parametric uncertainty subjected to transient disturbances and bounded energy (square-integrable) $L_2$ disturbances. Specifically, disturbance rejection controllers are constructed such that the time derivative of the control Lyapunov function is negative along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing solutions to the Hamilton–Jacobi–Isaacs equation and guaranteeing that the closed-loop output system energy is less than the closed-loop net weighted input system energy at any time $T$. Thus, our results provide a family of globally stabilizing disturbance rejection controllers guaranteeing that the closed-loop nonlinear input-output map from $L_2$ disturbances to performance variables is nonexpansive (gain bounded). Furthermore, robust nonlinear controllers are constructed such that the robust control Lyapunov function provides sufficient conditions for the existence of asymptotically *robustly* stabilizing solutions to the Hamilton–Jacobi–Bellman equation. Even though the frameworks for robustly stabilizing nonlinear control and nonlinear disturbance rejection control developed in [16] and [15], respectively, can be combined to develop a *unified* robust disturbance rejection control framework with parametric robustness and disturbance rejection guarantees, for simplicity of exposition we consider the robustness and disturbance rejection problems separately.

The globally stabilizing robust/disturbance rejection controllers presented are predicated on the three-state nonlinear Moore–Greitzer model [3] involving a one-mode expansion of the disturbance velocity potential in the compression system with a nonlinear (cubic) characteristic pressure-flow performance map. This simple model, modified to include uncertain exogenous disturbances and parametric system uncertainty, captures the nonlinear features of rotating stall and surge [3]. Finally, we compare our globally stabilizing robust/disturbance rejection controllers to the locally stabilizing bifurcation-based controllers of Liaw and Abed [17], [18] and Badmus *et al.* [19], [20] along with the recursive backstepping controller of Krstić *et al.* [21].

## II. Disturbance Rejection Control for Rotating Stall and Surge

In this section we apply the optimal disturbance rejection backstepping control framework developed in [15] to the control of rotating stall and surge in jet engine compression systems with $L_2$ input disturbances. To capture post-stall transients in axial flow compression systems we use the one-mode Galerkin approximation model for the nonlinear partial differential equation characterizing the disturbance velocity potential at the compressor inlet proposed by Moore and Greitzer [3]. Specifically, invoking a momentum balance across the compression system, conservation of mass in the plenum, and using a Galerkin projection based on a one-mode circumferential spatial harmonic approximation for the nonaxisymmetric flow disturbances yields [3]

$$\dot{A}(t) = \frac{\sigma}{3\pi}\int_0^{2\pi} \Psi_C(\Phi(t) + A(t)\sin\theta)\sin\theta\, d\theta, \quad A(0) = A_0, \quad t \geq 0 \tag{1}$$

$$\dot{\Phi}(t) = -\Psi(t) + \frac{1}{2\pi}\int_0^{2\pi} \Psi_C(\Phi(t) + A(t)\sin\theta)\, d\theta, \quad \Phi(0) = \Phi_0 \tag{2}$$

$$\dot{\Psi}(t) = \frac{1}{\beta^2}(\Phi(t) - \Phi_T(t)), \quad \Psi(0) = \Psi_0 \tag{3}$$

where $\Phi$ is the circumferentially averaged axial mass flow in the compressor, $\Psi$ is the total-to-static pressure rise, $A$ is the normalized stall cell amplitude of angular variation capturing a measure of nonuniformity in the flow, $\Phi_T$ is the mass flow through the throttle, $\sigma$, $\beta$ are positive constant parameters, and $\Psi_C(\cdot)$ is a given compressor pressure-flow map. The compliance coefficient $\beta$ is a function of the compressor rotor speed and plenum size. For large values of $\beta$ a surge limit cycle can occur while rotating stall can occur for any value of $\beta$. Now, assuming that the compressor pressure-flow map $\Psi_C(\cdot)$ is analytic, the integral terms in (1)–(3) can be expressed in terms of an infinite Taylor expansion about the circumferentially averaged flow to give

$$\dot{A}(t) = \frac{2\sigma}{3}\sum_{k=1}^{\infty}\frac{1}{k!(k-1)!}\left.\frac{d^{2k-1}\Psi_C(\xi)}{d\xi^{2k-1}}\right|_{\xi=\Phi(t)}\left(\frac{A(t)}{2}\right)^{2k-1}, \quad A(0) = A_0, \quad t \geq 0 \tag{4}$$

$$\dot{\Phi}(t) = -\Psi(t) + \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left.\frac{d^{2k}\Psi_C(\xi)}{d\xi^{2k}}\right|_{\xi=\Phi(t)}\left(\frac{A(t)}{2}\right)^{2k}, \quad \Phi(0) = \Phi_0 \tag{5}$$

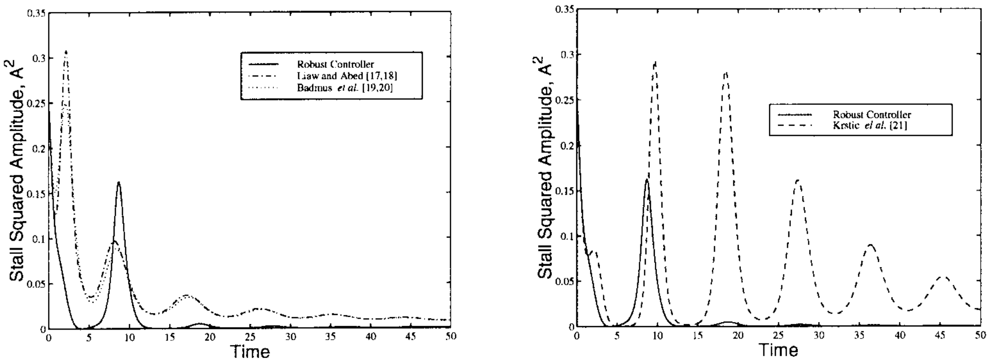



Fig. 2. Squared stall cell amplitude versus time.

$$\dot{\Psi}(t) = \frac{1}{\beta^2}(\Phi(t) - \Phi_T(t)), \quad \Psi(0) = \Psi_0. \tag{6}$$

The specific compressor characteristic $\Psi_C$ which was considered in [3] is

$$\Psi_C(\Phi) = \Psi_{C_0} + 1 + \frac{3}{2}\Phi - \frac{1}{2}\Phi^3 \tag{7}$$

where $\Psi_{C_0}$ is a constant parameter. In this case, (4)–(6) become

$$\dot{A}(t) = \frac{\sigma}{2}A(t)\left[1 - \Phi^2(t) - \frac{1}{4}A^2(t)\right] + \beta_1 w_1(t), \quad A(0) = A_0, \quad t \geq 0, \tag{8}$$

$$\dot{\Phi}(t) = -\Psi(t) + \Psi_C(\Phi(t)) - \frac{3}{4}\Phi(t)A^2(t) + \beta_2 w_2(t), \quad \Phi(0) = \Phi_0, \tag{9}$$

$$\dot{\Psi}(t) = \frac{1}{\beta^2}(\Phi(t) - \Phi_T(t)) + \beta_3 w_3(t), \quad \Psi(0) = \Psi_0 \tag{10}$$

to which we have added the $L_2$ external disturbance signals $w_1(t)$, $w_2(t)$, and $w_3(t)$, $t \geq 0$, with scaling factors $\beta_1$, $\beta_2$, $\beta_3 \in R$. The proposed additive disturbance model can be used to capture combustion noise and compressor speed fluctuations. Specifically, $w_3(t)$, $t \geq 0$, might reflect back-pressure disturbances to the compressor from the combustor as well as turbine speed fluctuation disturbances.

Our objective is to stabilize the equilibrium ($A(t) \equiv 0$, $\Phi(t) \equiv 1$, $\Psi(t) \equiv \Psi_{C_0} + 2$), corresponding to maximum pressure operation, by controlling the throttle mass flow $\Phi_T$ which is related to the throttle opening, $\gamma_{\rm throt}$, by $\Phi_T = \gamma_{\rm throt}\sqrt{\Psi}$ [3]. To translate the desired equilibrium to the origin we apply the linear transformations $\Phi_s \triangleq \Phi - 1$ and $\Psi_s \triangleq \Psi - \Psi_{C_0} - 2$. Furthermore, we define the control variable to be

$$u \triangleq \frac{1}{\beta^2}(\Phi_T - 1 - \Phi_s) \tag{11}$$

yielding the transformed nonlinear disturbed system

$$\dot{A}(t) = -\frac{\sigma}{2}A(t)\left[\frac{1}{4}A^2(t) + 2\Phi_s(t) + \Phi_s^2(t)\right] + \beta_1 w_1(t), \quad A(0) = A_0, \quad t \geq 0 \tag{12}$$

$$\dot{\Phi}_s(t) = -\frac{3}{2}\Phi_s^2(t) - \frac{1}{2}\Phi_s^3(t) - \frac{3}{4}A^2(t)[1 + \Phi_s(t)] - \Psi_s(t) + \beta_2 w_2(t), \quad \Phi_s(0) = \Phi_{s_0} \tag{13}$$

$$\dot{\Psi}_s(t) = -u(t) + \beta_3 w_3(t), \quad \Psi_s(0) = \Psi_{s_0}. \tag{14}$$

Note that (12)–(14) can be written as

$$\dot{x}(t) = f(x(t)) + g(x(t))\hat{x}(t) + J_1(x(t))w(t), \quad x(0) = x_0, \quad w(\cdot) \in L_2, \quad t \geq 0 \tag{15}$$

$$\dot{\hat{x}}(t) = u(t) + J_3(\hat{x}(t))w(t), \quad \hat{x}(0) = \hat{x}_0 \tag{16}$$

where $x \triangleq [A \;\; \Phi_s]^T$, $\hat{x} \triangleq \Psi_s$, and

$$f(A, \Phi_s) \triangleq \begin{bmatrix} -\frac{\sigma}{2}A(\frac{1}{4}A^2 + 2\Phi_s + \Phi_s^2) \\ -\Phi_s(\frac{3}{2}\Phi_s + \frac{1}{2}\Phi_s^2 + \frac{3}{4}A^2) - \frac{3}{4}A^2 \end{bmatrix}$$

$$g(A, \Phi_s) \triangleq \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$J_1(A, \Phi_s) \triangleq \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{bmatrix}, \quad J_3(\hat{x}(t)) = [0 \quad 0 \quad \beta_3].$$

To apply Theorem 4.1 of [15] we require a stabilizing feedback for the subsystem (12), (13) and a corresponding Lyapunov function $V_{\rm sub}(A, \Phi_s)$ such that $V_{\rm sub}(0, 0) = 0$ and $V_{\rm sub}(A, \Phi_s) > 0$, $(A, \Phi_s) \neq (0, 0)$. For the nonlinear subsystem (12), (13) we define the Lyapunov function candidate

$$V_{\rm sub}(A, \Phi_s) \triangleq \varepsilon A^4 + \Phi_s^2 \tag{17}$$

where $\varepsilon > 0$, and the stabilizing feedback control

$$\alpha(A, \Phi_s) \triangleq c_\alpha \Phi_s - \frac{3}{2}\Phi_s^2 - \frac{3}{4}A^2 - 2\varepsilon\sigma A^4 + \frac{\beta_2^2}{2\gamma^2}\Phi_s, \tag{18}$$

where $c_\alpha \geq 0$ and $\gamma > \beta_1\sqrt{\frac{8\varepsilon}{\sigma}}$. Now, it is straightforward to show that (17) and (18) satisfy

$$V'_{\rm sub}(x)[f(x) + g(x)\alpha(x)] + \frac{1}{4\gamma^2}V'_{\rm sub}(x)J_1(x)J_1^T(x)V'^T_{\rm sub}(x) < 0, \quad x \in R^2, \quad x \neq 0. \tag{19}$$

Finally, define the weighted performance variable $z \triangleq E(\Psi_s - \alpha(A, \Phi_s))$, where $E \in R$ is a weighting constant.

Applying Theorem 4.1 of [15] to the system (12)–(14) yields the family of control laws

$$u = -\tilde{\phi}(A, \Phi_s, \Psi_s) = \frac{1}{2}R_2^{-1}[2\hat{P}(\Psi_s - \alpha(A, \Phi_s)) + \tilde{L}_2(A, \Phi_s, \Psi_s)] \tag{20}$$

where $R_2 > 0$, $\hat{P} > 0$, and $\tilde{L}_2(A, \Phi_s, \Psi_s)$ provide flexibility in choosing the control law (20) and minimize a nonlinear-nonquadratic performance functional. For details

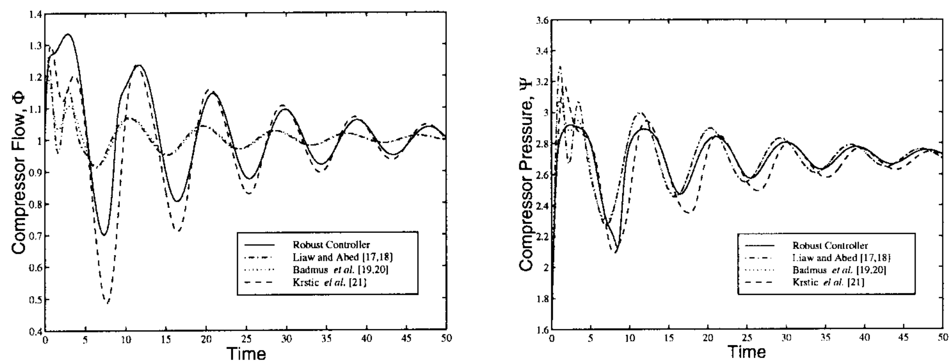



Fig. 3. Compressor flow and pressure rise versus time.

on constructing (20) see [15]. Next, consider the control Lyapunov function candidate

$$V(A,\Phi_s,\Psi_s)=\varepsilon A^4+\Phi_s^2+\hat{P}(\Psi_s-\alpha(A,\Phi_s))^2. \tag{21}$$

Now $\tilde{L}_2(A,\Phi_s,\Psi_s)$ must be chosen to satisfy $\dot{V}(A,\Phi_s,\Psi_s)<0$, $(A,\Phi_s,\Psi_s)\neq(0,0,0)$, which, after some algebraic manipulation, implies $\tilde{L}_2(A,\Phi_s,\Psi_s)$ must be chosen such that

$$\begin{aligned}&(\Psi_s-\alpha(A,\Phi_s))\\&\Big\{E^2(\Psi_s-\alpha(A,\Phi_s))-2\Phi_s\\&\quad-2\hat{P}\alpha'(A,\Phi_s)(f(A,\Phi_s)+g(A,\Phi_s)\Psi_s)\\&-\hat{P}R_2^{-1}(2\hat{P}(\Psi_s-\alpha(A,\Phi_s))+\tilde{L}_2(A,\Phi_s,\Psi_s))\\&-\frac{\hat{P}}{\gamma^2}\{\alpha'(A,\Phi_s)J_1(A,\Phi_s)J_1^T(A,\Phi_s)V_{\rm sub}'^T(A,\Phi_s)\\&-\big[\alpha'(A,\Phi_s)J_1(A,\Phi_s)J_1^T(A,\Phi_s)\alpha'^T(A,\Phi_s)\\&+J_3(\Psi_s)J_3^T(\Psi_s)\big]\hat{P}(\Psi_s-\alpha(A,\Phi_s))\}\Big\}<0.\end{aligned} \tag{22}$$

A particular admissible choice for $\tilde{L}_2(A,\Phi_s,\Psi_s)$ satisfying (22) is given by

$$\begin{aligned}&\tilde{L}_2(A,\Phi_s,\Psi_s)\\&=R_2\Big\{\hat{P}^{-1}[E^2(\Psi_s-\alpha(A,\Phi_s))-2\Phi_s]\\&\quad-2\alpha'(A,\Phi_s)(f(A,\Phi_s)+g(A,\Phi_s)\Psi_s)\\&\quad-\frac{1}{\gamma^2}\{\big[\alpha'(A,\Phi_s)J_1(A,\Phi_s)J_1^T(A,\Phi_s)\alpha'^T(A,\Phi_s)\\&\quad+J_3(\Psi_s)J_3^T(\Psi_s)\big]\hat{P}(\Psi_s-\alpha(A,\Phi_s))\\&\quad-\alpha'(A,\Phi_s)J_1(A,\Phi_s)J_1^T(A,\Phi_s)V_{\rm sub}'^{\rm T}(A,\Phi_s)\}\Big\}.\end{aligned} \tag{23}$$

For this choice of $\tilde{L}_2(A,\Phi_s,\Psi_s)$ the feedback control (20) becomes

$$\begin{aligned}&\tilde{\phi}(A,\Phi_s,\Psi_s)\\&=\Big\{\Big[R_2^{-1}+\frac{1}{2\gamma^2}\big(\alpha'(A,\Phi_s)J_1(A,\Phi_s)J_1^T(A,\Phi_s)\\&\quad\times\alpha'^T(A,\Phi_s)+J_3(\Psi_s)J_3^T(\Psi_s)\big)\Big]\hat{P}\\&\quad+\frac{1}{2}E^2\hat{P}^{-1}\Big\}(\Psi_s-\alpha(A,\Phi_s))-\hat{P}^{-1}\Phi_s\\&\quad-\alpha'(A,\Phi_s)(f(A,\Phi_s)+g(A,\Phi_s)\Psi_s)\\&\quad-\frac{1}{2\gamma^2}V_{\rm sub}'(A,\Phi_s)J_1(A,\Phi_s)J_1^T(A,\Phi_s)\alpha'^T(A,\Phi_s)\end{aligned} \tag{24}$$

so that (22) satisfies

$$-2R_2^{-1}\hat{P}^2(\Psi_s-\alpha(A,\Phi_s))^2<0,\quad (A,\Phi_s,\Psi_s)\neq(0,0,0). \tag{25}$$

In this case, it follows from Theorem 4.1 of [15] that the closed-loop system (15), (16) satisfies the disturbance rejection constraint

$$\int_0^T z^T(t)z(t)\,dt\le\gamma^2\int_0^T w^T(t)w(t)\,dt+V(x_0,\hat{x}_0),\qquad T\ge0,\quad w(\cdot)\in L_2. \tag{26}$$

As discussed in [15], in the special case where $\varepsilon=0$, $\hat{P}=\frac{1}{2}$, $R_2=\frac{1}{c_2}$, $\beta_1=\beta_2=\beta_3=0$, and $E=0$, (24) specializes to the controller given in [21]. Alternatively, by varying $\varepsilon$, $\hat{P}$, and $c_\alpha$ in the control law (24) we can generate a family of feedback controllers which guarantee global asymptotic stability along with the disturbance rejection constraint (26).

## III. Robust Nonlinear Control for Rotating Stall and Surge

In this section we apply the optimal robust backstepping control framework developed in [16] to the control of rotating stall and surge in jet engine compression systems with uncertain system dynamics. Here, we consider the nominal nonlinear system (4)–(6) and the nominal compressor characteristic map given by

$$\Psi_{C_{\rm nom}}(\Phi)=\Psi_{C_0}+1+\frac{3}{2}\Psi-\frac{1}{2}\Psi^3. \tag{27}$$

In order to account for system modeling uncertainty in our nominal model, we introduce structured parametric uncertainty to the system dynamics (4)–(6). Specifically, we add structured uncertainty of the form $\Delta f_1(A,\Phi)$ and $\Delta f_2(A,\Phi)$ to (4) and

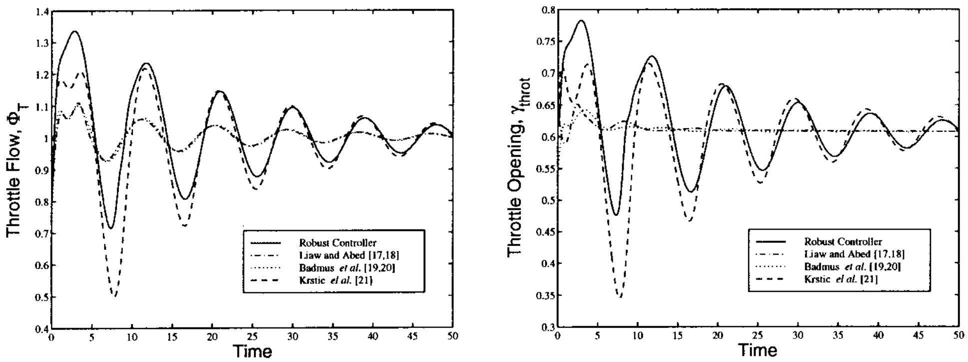



Fig. 4. Control effort versus time.

(5). This, for example, can account for compressor pressure-flow map uncertainty since in actual compressor data [19], [22] the compressor characteristic map exhibits a noncubic morphology that can drive the compression system to deep hysteresis during rotating stall. Hence, in this case, to account for compressor performance map uncertainties one can assume that

$$\Psi_C(\Phi) = \Psi_{C_{\text{nom}}}(\Phi) + \Delta\Psi_C(\Phi) \tag{28}$$

where $\Delta\Psi_C(\Phi)$ is an uncertain perturbation of a given structure of the nominal performance characteristic $\Psi_{C_{\text{nom}}}(\Phi)$. Substituting (27) and (28) in (4)–(6), we obtain

$$\begin{aligned}\dot{A}(t) &= \frac{\sigma}{2}A(t)\left[1 - \Phi^2(t) - \frac{1}{4}A^2(t)\right]\\ &\quad + \Delta f_1(A(t), \Phi(t)), \quad A(0) = A_0, \quad t \geq 0 \end{aligned}\tag{29}$$

$$\begin{aligned}\dot{\Phi}(t) &= -\Psi(t) + \Psi_C(\Phi(t)) - \frac{3}{4}\Phi(t)A^2(t)\\ &\quad + \Delta f_2(A(t), \Phi(t)), \quad \Phi(0) = \Phi_0\end{aligned}\tag{30}$$

$$\dot{\Psi}(t) = \frac{1}{\beta^2}(\Phi(t) - \Phi_T(t)), \quad \Psi(0) = \Psi_0 \tag{31}$$

where

$$\Delta f_1(A, \Phi) \triangleq \frac{2\sigma}{3}\sum_{k=1}^{\infty}\frac{1}{k!(k-1)!}\frac{d^{2k-1}\Delta\Psi_C(\Phi)}{d\Phi^{2k-1}}\left(\frac{A}{2}\right)^{2k-1} \tag{32}$$

$$\Delta f_2(A, \Phi) \triangleq \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\frac{d^{2k}\Delta\Psi_C(\Phi)}{d\Phi^{2k}}\left(\frac{A}{2}\right)^{2k}. \tag{33}$$

In general, $\Delta f_1(A, \Phi)$ and $\Delta f_2(A, \Phi)$ in (29) and (30) can be used to capture parametric uncertainty in the nominal system dynamics (8)–(10) with $\beta_1 = \beta_2 = \beta_3 = 0$.

Our objective is to robustly stabilize the equilibrium $(A(t) \equiv 0, \Phi(t) \equiv 1, \Psi(t) \equiv \Psi_{C_0} + 2)$ corresponding to the maximum pressure point for the nominal system over a prescribed uncertainty structure $\Delta f_1(A, \Phi)$ and $\Delta f_2(A, \Phi)$, by controlling the throttle mass flow $\Phi_T$ which, as noted in the previous section, is related to the throttle opening, $\gamma_{\text{throt}}$, by $\Phi_T = \gamma_{\text{throt}}\sqrt{\Psi}$. Once again, we translate the desired equilibrium to the origin using the transformations given in Section II yielding the transformed nonlinear uncertain system

$$\begin{aligned}\dot{A}(t) &= -\frac{\sigma}{2}A(t)\left[\frac{1}{4}A^2(t) + 2\Phi_s(t) + \Phi_s^2(t)\right]\\ &\quad + \Delta f_{s_1}(A(t), \Phi_s(t)), \quad A(0) = A_0, \quad t \geq 0\end{aligned} \tag{34}$$

$$\begin{aligned}\dot{\Phi}_s(t) &= -\frac{3}{2}\Phi_s^2(t) - \frac{1}{2}\Phi_s^3(t) - \frac{3}{4}A^2(t)[1 + \Phi_s(t)]\\ &\quad - \Psi_s(t) + \Delta f_{s_2}(A(t), \Phi_s(t)), \quad \Phi_s(0) = \Phi_{s_0}\end{aligned} \tag{35}$$

$$\dot{\Psi}_s(t) = -u(t), \quad \Psi_s(0) = \Psi_{s_0}. \tag{36}$$

Note that (34)–(36) can be written as

$$\begin{aligned}\dot{x}(t) &= f_0(x(t)) + \Delta f(x(t)) + g_0(x(t))\hat{x}(t),\\ &\qquad x(0) = x_0, \quad t \geq 0\end{aligned} \tag{37}$$

$$\dot{\hat{x}}(t) = u(t), \quad \hat{x}(0) = \hat{x}_0 \tag{38}$$

where $x = [A \;\; \Phi_s]^T$, $\hat{x} = \Psi_s$, and

$$f_0(A, \Phi_s) \triangleq \begin{bmatrix} -\frac{\sigma}{2}A\left(\frac{1}{4}A^2 + 2\Phi_s + \Phi_s^2\right) \\ -\Phi_s\left(\frac{3}{2}\Phi_s + \frac{1}{2}\Phi_s^2 + \frac{3}{4}A^2\right) - \frac{3}{4}A^2 \end{bmatrix}$$

$$g_0(A, \Phi_s) \triangleq \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Delta f(A, \Phi_s) \triangleq \begin{bmatrix} \Delta f_{s_1}(A, \Phi_s) \\ \Delta f_{s_2}(A, \Phi_s) \end{bmatrix}.$$

Here, we assume that the uncertainty $\Delta f$ is characterized by

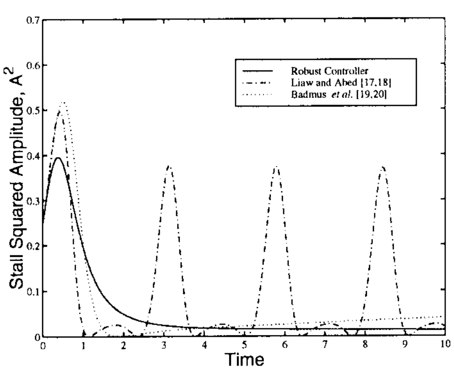
$$\begin{aligned}\Delta f \in \mathcal{F} \triangleq \{&\Delta f : R^2 \to R^2 : \Delta f(x) = g_\delta(x)\delta(h_\delta(x))\\ &x \in R^2, \delta(\cdot) \in \boldsymbol{\Delta}\}\end{aligned} \tag{39}$$

where $h_\delta(A, \Phi_s) \triangleq [A \;\; \Phi_s]^T$ and $g_\delta \triangleq I_2$ capture the structure of the parametric uncertainty, and $\delta(\cdot) \triangleq \Delta f(A, \Phi_s)$ represents the uncertainty in the system dynamics with $\delta(\cdot) \in \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ satisfies

$$\begin{aligned}\boldsymbol{\Delta} = \{&\delta : R^2 \to R^2 : \delta(0) = 0, \delta^T(y)\delta(y) \leq m^T(y)m(y)\\ &y \in R\}\end{aligned} \tag{40}$$

and where $m : R^2 \to R^2$ is a given function such that $m(0) = 0$. To apply Theorem 4.1 of [16] we require a robustly stabilizing feedback for the subsystem (34), (35) and a corresponding Lyapunov function $V_{\text{sub}}(A, \Phi_s)$ such that $V_{\text{sub}}(0, 0) = 0$ and $V_{\text{sub}}(A, \Phi_s) > 0$, $(A, \Phi_s) \neq (0, 0)$. For the nonlinear subsystem (34), (35) we define the Lyapunov function candidate

$$V_{\text{sub}}(A, \Phi_s) \triangleq \varepsilon A^4 + \Phi_s^2 \tag{41}$$



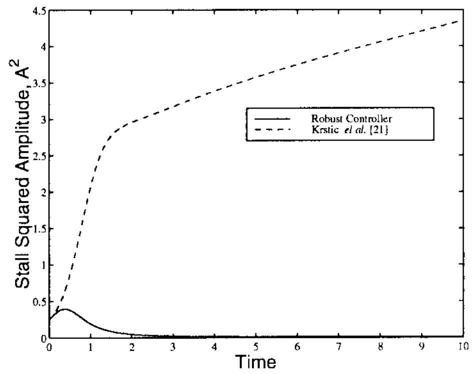




Fig. 5. Squared stall cell amplitude versus time.

where $\varepsilon > 0$, and the robustly stabilizing feedback control

$$\alpha(A, \Phi_s) \triangleq \left(c_\alpha + \frac{1}{2}\right)\Phi_s - \frac{3}{2}\Phi_s^2 - \frac{3}{4}A^2 - 2\varepsilon\sigma A^4 + \frac{1}{2}\Phi_s \tilde{m}^T(A, \Phi_s)\tilde{m}(A, \Phi_s) \tag{42}$$

where $c_\alpha \geq 0$ and $\tilde{m}(A, \Phi_s)$ is such that $\tilde{m}(A, \Phi_s)\Phi_s = m(A, \Phi_s)$. In this case, it is straightforward to show that (41) and (42) satisfies

$$V'_{\rm sub}(x)[f_0(x) + g_0(x)\alpha(x)] + \frac{1}{4}V'_{\rm sub}(x)g_\delta(x)g_\delta^T(x)V'^T_{\rm sub}(x) + m^T(h_\delta(x))m(h_\delta(x)) < 0, \quad x \in R^n, \quad x \neq 0. \tag{43}$$

Applying Theorem 4.1 of [16] to the system (34)–(36) yields the family of robustly stabilizing control laws

$$\begin{aligned} u &= -\tilde{\phi}(A, \Phi_s, \Psi_s) \\ &= \frac{1}{2}R_2^{-1}[2\hat{P}(\Psi_s - \alpha(A, \Phi_s)) + \tilde{L}_2(A, \Phi_s, \Psi_s)] \end{aligned} \tag{44}$$

where $R_2 > 0$, $\hat{P} > 0$, and $\tilde{L}_2(A, \Phi_s, \Psi_s)$ provide flexibility in choosing the control law (44) and minimize a nonlinear-nonquadratic performance functional. For details on constructing (44) see [16]. Next, consider the robust control Lyapunov function candidate

$$V(A, \Phi_s, \Psi_s) = \varepsilon A^4 + \Phi_s^2 + \hat{P}[\Psi_s - \alpha(A, \Phi_s)]^2. \tag{45}$$

Now $\tilde{L}_2(A, \Phi_s, \Psi_s)$ must be chosen to satisfy $\dot{V}(A, \Phi_s, \Psi_s) < 0$, $(A, \Phi_s, \Psi_s) \neq (0, 0, 0)$, which implies $\tilde{L}_2(A, \Phi_s, \Psi_s)$ must be chosen such that

$$\begin{aligned} &(\Psi_s - \alpha(A, \Phi_s))^T\big\{-2\Phi_s - \hat{P}\big[2\alpha'(A, \Phi_s)(f_0(A, \Phi_s) \\ &\quad + g_0(A, \Phi_s)\Psi_s) + R_2^{-1}(2\hat{P}(\Psi_s - \alpha(A, \Phi_s)) \\ &\quad + \tilde{L}_2^T(A, \Phi_s, \Psi_s)) + \alpha'(A, \Phi_s)g_\delta(A, \Phi_s)g_\delta^T(A, \Phi_s) \\ &\quad \times V'^T_{\rm sub}(A, \Phi_s) - \alpha'(A, \Phi_s)g_\delta(A, \Phi_s)g_\delta^T(A, \Phi_s) \\ &\quad \times \alpha'^T(A, \Phi_s)\hat{P}(\Psi_s - \alpha(A, \Phi_s))\big]\big\} < 0. \end{aligned} \tag{46}$$

A particular admissible choice for $\tilde{L}_2(A, \Phi_s, \Psi_s)$ satisfying (46) is given by

$$\begin{aligned} &\tilde{L}_2(A, \Phi_s, \Psi_s) \\ &\quad = -R_2\big\{2\hat{P}^{-1}\Phi_s + 2\alpha'(A, \Phi_s)(f_0(A, \Phi_s) + g_0(A, \Phi_s)\Psi_s) \\ &\qquad + V'_{\rm sub}(A, \Phi_s)g_\delta(A, \Phi_s)g_\delta^T(A, \Phi_s)\alpha'^T(A, \Phi_s) \\ &\qquad - (\Psi_s - \alpha(A, \Phi_s))\hat{P}\alpha'(A, \Phi_s)g_\delta(A, \Phi_s) \\ &\qquad \times g_\delta^T(A, \Phi_s)\alpha'^T(A, \Phi_s)\big\}. \end{aligned} \tag{47}$$

For this choice of $\tilde{L}_2(A, \Phi_s, \Psi_s)$ the feedback control (44) becomes

$$\begin{aligned} &\tilde{\phi}(A, \Phi_s, \Psi_s) \\ &\quad = -\left[R_2^{-1} + \frac{1}{2}\alpha'(A, \Phi_s)g_\delta(A, \Phi_s)g_\delta^T(A, \Phi_s)\alpha'^T(A, \Phi_s)\right] \\ &\qquad \times \hat{P}(\Psi_s - \alpha(A, \Phi_s)) + \hat{P}^{-1}\Phi_s + \alpha'(A, \Phi_s)(f_0(A, \Phi_s) \\ &\qquad + g_0(A, \Phi_s)\Psi_s) + \frac{1}{2}V'_{\rm sub}(A, \Phi_s)g_\delta(A, \Phi_s)g_\delta^T(A, \Phi_s) \\ &\qquad \times \alpha'^T q(A, \Phi_s) \end{aligned} \tag{48}$$

so that (46) satisfies

$$-2R_2^{-1}\hat{P}^2(\Psi_s - \alpha(A, \Phi_s))^2 < 0, \quad (A, \Phi_s, \Psi_s) \neq (0, 0, 0). \tag{49}$$

In this case, it follows from Theorem 4.1 of [16] the the closed-loop system (37), (38) is robustly stable for all $\Delta f \in \mathcal{F}$. As discussed in [16], in the special case where $g_\delta = [0\ \ 0]^T$, $\varepsilon = 0$, $\hat{P} = 1$, and $R_2 = \frac{1}{c_2}$, (48) specializes to the controller given in [21]. Alternatively, by varying $\varepsilon$, $\hat{P}$, and $c_\alpha$ in the control law (48) we can generate a family of controllers which guarantee global asymptotic stability with respect to nonlinear system parametric uncertainty.

## IV. Active Disturbance Rejection Control of an Axial Flow Compressor Problem

In this section we apply the globally stabilizing disturbance rejection controller developed in Section II to an axial flow compressor problem. Furthermore, we compare our results to the backstepping controller developed by Krstić *et al.* [21] and the bifurcation-based locally stabilizing controllers developed by Liaw and Abed [17], [18] and Badmus *et al.* [19], [20]. Recall that the bifurcation-based Liaw and Abed [17], [18] controller is given by

$$\gamma_{\rm throt}(A) = \gamma_0 + kA^2 \tag{50}$$

where $k > 0$ and $\gamma_0$ corresponds to the throttle opening at the maximum pressure point, while the proposed Badmus *et al.*

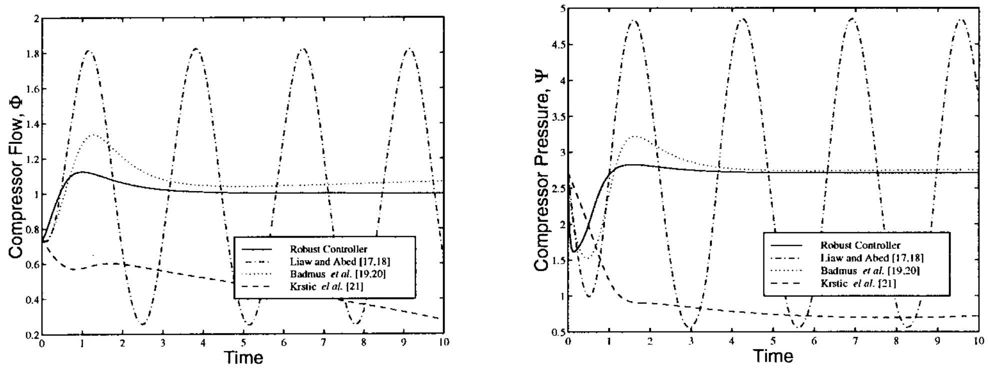



Fig. 6. Compressor flow and pressure versus time.

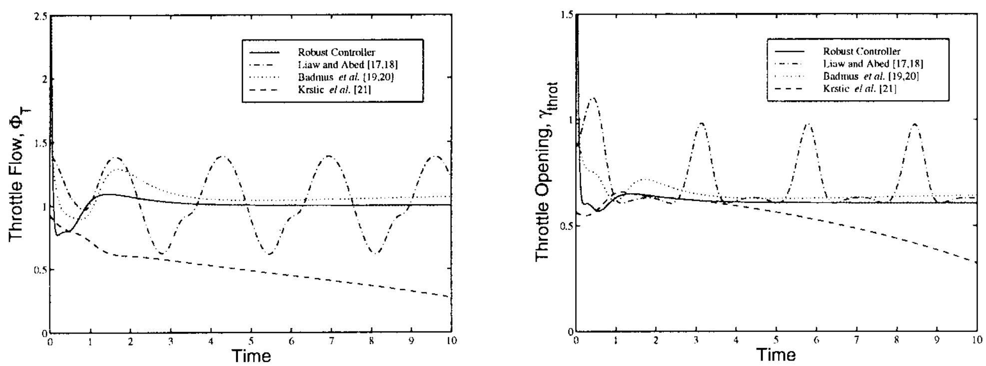



Fig. 7. Control effort versus time.

[19], [20] controller is given by

$$\gamma_{\text{throt}}(A, \dot{\Phi}) = \gamma_0 + k_1 A^2 - k_2 \dot{\Phi} \tag{51}$$

where $k_1,\ k_2 > 0$.

Using the initial conditions $A_0 = 0.5$, $\Phi_{s_0} = 0$, $\Psi_{s_0} = -1.1085$, with data parameters $\Psi_{C_0} = 0.72$, $\sigma = 3.6$, $\beta = 0.356$, and design parameters $\hat{P} = 5$, $R_2 = 1$, $\varepsilon = 2$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $\beta_3 = 1$, $\gamma = 1.1$, and $E = 1$, the optimal disturbance rejection control law (24), the integrator backstepping controller given in [21], the Liaw and Abed control law (50), and the Badmus *et al.* control law (51) were used to compare the closed-loop system response of (8)–(10) with $L_2$ disturbance inputs $w_1(t) = e^{-0.5t}\sin t$, $w_2(t) = e^{-0.05t}\sin 0.7t$, and $w_3(t) = e^{-0.5t}\sin 1.2t$. The squared stall cell amplitude responses for the controllers are compared in Fig. 2, the compressor flow and pressure rise responses are compared in Fig. 3, and the control efforts (throttle flow and throttle opening) are compared in Fig. 4. This comparison illustrates that the present framework provides disturbance rejection control.

## V. Robust Control of an Axial Flow Compressor Problem

In this section we apply the globally stabilizing robust controller developed in Section III to an axial flow compressor problem with the uncertain system dynamics that may arise due to modeling errors in the pressure-flow map, neglected viscous effects in the model, etc. Furthermore, we compare our results to the backstepping controller developed by Krstić *et al.* [21] and the bifurcation-based locally stabilizing controllers developed by Liaw and Abed [17], [18] and Badmus *et al.* [19], [20].

Using the initial conditions $A_0 = 0.5$, $\Phi_{s_0} = -0.25$, $\Psi_{s_0} = 0$, the data parameters $\Psi_{C_0} = 0.72$, $\sigma = 3.6$, $\beta = 0.356$, and design parameters $\hat{P} = 2$, $R_2 = 0.5$, $\varepsilon = 0.44$, and $c_\alpha = 0.5$, the robustly stabilizing controller (48), the integrator backstepping controller given in [21] (with $c_1 = c_2 = 1$), the Liaw and Abed control law (50) (with $k = 1.0$ and $\gamma_0 = 0.6063$), and the Badmus *et al.* control law (51) (with $k_1 = 1$, $k_2 = 0.5$, and $\gamma_0 = 0.6063$) were used to compare the closed-loop system response. Here we model the uncertain system dynamics by

$$\Delta f_{s_1}(A, \Phi_s) = a_1\Phi_s + a_2 A\Phi_s + a_3 A\Phi_s^2 + a_4 A^2\Phi_s^2, \tag{52}$$
$$\Delta f_{s_2}(A, \Phi_s) = a_5\Phi_s + a_6\Phi_s^2 + a_7\Phi_s^3 + a_8 A^2\Phi_s \tag{53}$$

where $a_i \in R$, $i = 1, \cdots, 8$, are the uncertain parameters satisfying $|a_i| \le b_i$, $i = 1, \cdots, 8$, where $b_i > 0$, $i = 1, \cdots, 8$, are given uncertainty bounds. In this case, the bounding function $\tilde{m}(A, \Phi_s)$ is given by

$$\tilde{m}(A, \Phi_s) = \begin{bmatrix} b_1 + b_2 A_{\hat{\epsilon}} + b_3 A_{\hat{\epsilon}}\Phi_{\hat{\epsilon}} + b_4 A^2\Phi_{\hat{\epsilon}} \\ b_5 + b_6\Phi_{\hat{\epsilon}} + b_7\Phi_s^2 + b_8 A^2 \end{bmatrix}$$

where $\hat{\epsilon} > 0$, $A_{\hat{\epsilon}} \triangleq \sqrt{A^2 + \hat{\epsilon}^2}$, and $\Phi_{\hat{\epsilon}} \triangleq \sqrt{\Phi^2 + \hat{\epsilon}^2}$. Here we set $a_i$, $b_i$, $i = 1, \cdots, 8$, to the following values:

$$a = [-0.91 \quad 0 \quad 0 \quad 0 \quad 1.0955 \quad 1.13 \quad 0.92 \quad 0],$$
$$b = [1 \quad 0 \quad 0 \quad 0 \quad 1.1 \quad 1.2 \quad 1 \quad 0]$$

where $a \triangleq [a_1 \ \cdots \ a_8]$ and $b \triangleq [b_1 \ \cdots \ b_8]$.

The squared stall cell amplitude responses for the controllers are compared in Fig. 5, the compressor flow and pressure rise responses are compared in Fig. 6. Fig. 7 compares the control efforts of all four designs. This comparison illustrates that the robust nonlinear controller globally stabilizes the maximum pressure performance point $(0, 1, 2.72)$ of the nominal system in the face of system uncertainty, while the controller given in [21] and the bifurcation-based Liaw and Abed controller [17], [18] destabilize the system and the bifurcation-based Badmus *et al.* controller [19], [20] drives the uncertain system to a stalled equilibrium.

## VI. Conclusion

The optimality-based nonlinear disturbance rejection controller framework for systems subjected to bounded energy $L_2$ disturbances developed in [15] and the optimality-based robust controller framework for systems with parametric uncertainty developed in [16] were applied to control rotating stall and surge in axial flow compression systems with uncertain exogenous $L_2$ disturbances and uncertain system dynamics, respectively. The proposed nonlinear disturbance rejection controllers were compared with the recursive backstepping controller proposed in [21] and the bifurcation-based controllers developed in [17]–[20]. This comparison illustrates that the proposed nonlinear controller provides disturbance rejection while the backstepping controller is unable to reject the disturbance. Similarly, the proposed robust nonlinear controllers were compared with the recursive backstepping controller proposed in [21] and the bifurcation-based controllers developed in [17]–[20]. This comparison illustrates that the proposed robust controller provides robust global stabilization while the backstepping controller and the locally stabilizing bifurcation-based controllers drive the system to an instability or stalled equilibrium in the presence of uncertainty in the system dynamics.

Finally, we note that there have been several other recursive backstepping controller designs proposed in the literature for the jet engine compression problem based on lean backstepping designs [10], [11] that could outperform the backstepping designs proposed in [21] for the uncertainty and disturbance models considered here. However, none of these approaches *a priori* guarantee robustness or disturbance rejection to parametric system uncertainty or exogenous disturbances. Furthermore, we stress that the objective of this paper is not to establish superiority of a given approach, but rather to demonstrate that compression system uncertainty and compression system disturbances are often significant and the need for robust disturbance rejection control in aeroengines is severe.

## References


[1] E. M. Greitzer, "Surge and rotating stall in axial flow compressors: Part 1 and 2," *J. Eng. Power*, vol. 98, pp. 190–217, 1976.

[2] F. K. Moore, "A theory of rotating stall of multistage axial compressors: Part 1, 2, and 3," *J. Eng. Gas Turbines and Power*, vol. 106, pp. 313–336, 1984.

[3] F. K. Moore and E. M. Greitzer, "A theory of post-stall transients in axial compression systems: Part 1 and 2," *J. Eng. Gas Turbines and Power*, vol. 108, pp. 68–76, 231–239, 1986.

[4] D. C. Liaw and E. H. Abed, "Stability analysis and control of rotating stall," *IFAC Nonlinear Contr. Syst.*, pp. 295–300, 1992.

[5] R. A. Adomaitis and E. H. Abed, "Local nonlinear control of stall inception in axial flow compressors," in *29th Joint Propulsion Conf. Exhibit—Paper AIAA 93-2230*, Monterey, CA, 1993.

[6] D. L. Gysling and E. M. Greitzer, "Dynamic control of rotating stall in axial flow compressors using aeromechanical feedback," in *Proc. ASME Int. Gas Turbines and Aeroengines, Congress and Exposition*, 1994.

[7] J. D. Paduano *et al.*, "Modeling for control of rotating stall," *Automatica*, vol. 30, no. 9, pp. 1357–1373, 1994.

[8] H. O. Wang, R. A. Adomaitis, and E. H. Abed, "Nonlinear analysis and control of rotating stall in axial flow compressors," in *Proc. Amer. Contr. Conf.*, Baltimore, MD, 1994, pp. 2317–2321.

[9] R. L. Behnken, R. D'Andrea, and R. M. Murray, "Control of rotating stall in a low-speed axial flow compressor using pulsed air," in *Proc. IEEE Conf. Dec. Contr.*, New Orleans, LA, 1995, pp. 3056–3061.

[10] M. Krstić, J. M. Protz, J. D. Paduano, and P. V. Kokotović, "Backstepping designs for jet engine stall and surge control," in *Proc. IEEE Conf. Dec. Contr.*, New Orleans, LA, 1995, pp. 3049–3055.

[11] M. Krstić and P. V. Kokotović, "Lean backstepping design for a jet engine compressor model," in *Proc. IEEE Conf. Contr. Appl.*, 1995, pp. 1047–1052.

[12] B. de Jager, "Rotating stall and surge control: A survey," in *Proc. IEEE Conf. Decision Contr.*, New Orleans, LA, 1995, pp. 1857–1862.

[13] O. O. Badmus, S. Chowdhury, and C. N. Nett, "Nonlinear control of surge in axial compression systems," *Automatica*, vol. 32, no. 1, pp. 59–70, 1996.

[14] G. Gu, S. Banda, and A. Sparks, "An overview of rotating stall and surge control for axial flow compressors," in *Proc. IEEE Conf. Decision Contr.*, Kobe, Japan, 1996, pp. 2786–2791.

[15] W. M. Haddad, V. S. Chellaboina, and J. L. Fausz, "Optimal nonlinear disturbance rejection control for nonlinear cascade systems," in *Proc. Amer. Contr. Conf.*, Albuquerque, NM, 1997, pp. 448–452; also in *Int. J. Contr.*, vol. 68, pp. 997–1018, 1997.

[16] W. M. Haddad, V. S. Chellaboina, J. L. Fausz, and A. Leonessa, "Optimal nonlinear robust control for nonlinear uncertain cascade systems," in *Proc. Amer. Contr. Conf.*, Albuquerque, NM, 1997, pp. 403–407.

[17] R. A. Adomaitis and E. H. Abed, "Bifurcation analysis of nonuniform flow patterns in axial-flow gas compressors," in *Proc. 1st World Congr. Nonlinear Analysts*, Tampa, FL, 1992, pp. 1597–1608.

[18] D. C. Liaw and E. H. Abed, "Active control of compressor stall inception: A bifurcation—theoretic approach," *Automatica*, vol. 32, pp. 109–115, 1996.

[19] O. O. Badmus, K. M. Eveker, and C. N. Nett, "Control-oriented high-frequency turbomachinery modeling, Part 1: Theoretical foundations," in *28th Joint Propulsion Conf. Exhibit—Paper AIAA 92-3314*, Nashville, TN, 1992.

[20] O. O. Badmus, S. Chowdhury, K. M. Eveker, and C. N. Nett, "Control-oriented high-frequency turbomachinery modeling: Single-stage compression system 1D model," *J. Turbomachinery*, vol. 117, no. 1, pp. 47–61, 1995.

[21] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, New York: Wiley, 1995.

[22] C. A. Mansoux, D. L. Gysling, J. D. Setiawan, and J. D. Paduano, "Distributed nonlinear modeling and stability analysis of axial compressor stall and surge," in *Proc. Amer. Contr. Conf.*, Baltimore, MD, 1994, pp. 2305–2316.