Global Output Regulation and Disturbance Attenuation with Global Stability via Measurement Feedback for a Class of Nonlinear Systems

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Abstract—**In this paper, we consider the problem of global stabilization via output feedback for a class of nonlinear systems which have been recently considered by many authors and are characterized by having nonlinear terms depending only on the output y. Our result incorporates many recent results. When static output feedback is considered, it is shown that the existence of an output control Lyapunov function, satisfying a suitable continuity property, is sufficient for constructing a continuous output feedback law $u = k(y)$ which globally (or semiglobally) stabilizes the above class of systems. When dynamic output feedback is allowed, it is shown that the stabilization problem can be split into two independent stabilization subproblems: one is the corresponding problem via state feedback, and the other is the problem via output injection. From solving the two subproblems, one obtains two Lyapunov functions which, combined, give a candidate Lyapunov function for solving the output feedback stabilization problem. The proofs of our results give systematic procedures for constructing output feedback controllers, once two such Lyapunov functions are known. One can also consider the problem of output regulation and disturbance attenuation with global stability via measurement feedback and show that a similar "separation" condition holds.**

I. Basic Definitions and Problem Statement

We will consider nonlinear systems of the form (or globally diffeomorphic to)

$$\begin{aligned}\dot{x}_1 &= A_1(y)x + B_1(y)u\\ \dot{z} &= A_2(y)x + B_2(y)u\\ y &= x_1\end{aligned} \tag{1}$$

where $x = \binom{x_1}{z}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, $y \in \mathbb{R}^p$, $A_j(y)$, and $B_j(y)$ are matrices with smooth entries. In (1), u defines the input variables and y are the measured variables. System (1) has been considered first in [1] and recently in [2]–[9].

Moreover, we will consider the class of continuous feedback laws

$$\begin{aligned}u &= \eta(\sigma, y),\ \sigma \in \mathbb{R}^q\\ \dot{\sigma} &= \varrho(\sigma, y)\end{aligned} \tag{2}$$

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with $\eta(0, 0) = 0$ and $\varrho(0, 0) = 0$. In what follows, we will refer to these feedback laws as (static or dynamic) output (or measurement) feedback laws. In the case that $y = x$, we will say (static or dynamic) state feedback laws.

Let us begin with formulating the problem of globally asymptotically stabilizing (1).

Global stabilization problem via measurement feedback: Given (1), find a control law (2) such that the closed-loop system (1) and (2), with $w(t) = 0$, is globally asymptotically stable in $(x, \sigma) = (0, 0)$.

Semiglobal stabilization problem via measurement feedback: Given (1) and compact $\Omega^e \subset \mathbb{R}^{n+q}$, find a control law (2) such that the closed-loop system (1) and (2), with $w(t) = 0$, is locally asymptotically stable in $(x, \sigma) = (0, 0)$ with basin of attraction containing Ω^e.

Let $V(x)$ be a smooth function. Moreover, let

$$A(y) = \begin{pmatrix}A_1(y)\\ A_2(y)\end{pmatrix},\ B(y) = \begin{pmatrix}B_1(y)\\ B_2(y)\end{pmatrix}$$

and

$$\begin{aligned}a(x) &= \frac{\partial V}{\partial x}(x)A(y)x\\ b(x) &= \frac{\partial V}{\partial x}(x)B(y).\end{aligned}$$

We will say that a proper and positive definite function $V(x)$ is a Lyapunov function for (1) if $a(x) < 0$ for all $x \neq 0$.

According to [10], we say that a proper and positive definite function $V(x)$ is a control Lyapunov function for (1) if for each $x \neq 0$, such that $b(x) = 0$, we have $a(x) < 0$ ($\|\cdot\|$ is the euclidean norm of $\mathbb{R}^n$), or, equivalently, for each $x \neq 0$ there exists a $m \times 1$ vector k_x such that $a(x) + b(x)k_x < 0$. We will say that the control Lyapunov function $V(x)$ satisfies the small control property if for each $\epsilon > 0$ there is $\delta_\epsilon > 0$ such that, for each $x \neq 0$ satisfying $\|x\| < \delta_\epsilon$, we have $a(x) + b(x)k_x < 0$ for some k_x with $\|k_x\| < \epsilon$. An elegant proof is given in [10], showing that there exists a control Lyapunov function $V(x)$, which satisfies the small control property, if and only if it is possible to find a feedback law $u = k(x)$ which globally asymptotically stabilizes (1) in $x = 0$. A former proof, based on unity partition, was given in [11]. The feedback law

$$u = k(x) = \begin{cases}0 & \text{if } b(x) = 0,\\ -\frac{a(x) + \sqrt{a^2(x) + \|b(x)\|^4}}{\|b(x)\|^2} b^T(x) & \text{else}\end{cases}$$

has the claimed properties.

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According to [6] (see also [7]), we say that a proper and positive definite function $V(x)$ is an output control Lyapunov function for (1) if for each $y \neq 0$ there exists a $m \times 1$ vector k_y such that $a(x) + b(x)k_y < 0$ for all z and $a(0, z) < 0$ for all $z \neq 0$. If $V(x)$ satisfies the above properties in a neighborhood of $x = 0$, we will say that $V(x)$ is a local output control Lyapunov function. We will say that the output control Lyapunov function $V(x)$ satisfies the small control property if there exists a positive real function $L(y)$, defined in a neighborhood of $y = 0$, such that $L(y) \to 0$ as $y \to 0$ and $\|k_y\| < L(y)$, where for each $y \neq 0 k_y$ satisfies $a(x) + b(x)k_y < 0$ for all z. It is shown in [7] that there exists a local output control Lyapunov function $V(x)$ which satisfies the small control property if and only if it is possible to find a feedback law $u = k(y)$ which locally asymptotically stabilizes (1) in $x = 0$. Implicitly, this is shown on arbitrary compact sets, but the compactness is actually an intrinsic limitation of the procedure proposed. The proposed control law can be explicitly constructed once an output control Lyapunov function and a unity partition are available. Further results are given in [7] in the direction of finding expressions of a stabilizing control law once an output control Lyapunov function is available without passing through the construction of a unity partition.

It is possible to extend the above definitions to the case of dynamic output feedback. Let us introduce the additional dynamics $\dot{\sigma} = v$, $v \in \mathbb{R}^q$. Let $V(x, \sigma)$ be a smooth, proper, and positive definite function. Correspondingly, let

$$a(x, \sigma) = \frac{\partial V}{\partial x}(x, \sigma)A(y)x$$
$$b(x, \sigma) = \left(\frac{\partial V}{\partial x}(x, \sigma)B(y) \quad \frac{\partial V}{\partial \sigma}(x, \sigma) \right).$$

We will say that a proper and positive definite $V(x, \sigma)$ is a dynamic output control Lyapunov function for (1) if for each $(y, \sigma) \neq (0, 0)$ there exists a $n + q$ vector $k_{y,\sigma}$ such that $a(x, \sigma) + b(x, \sigma)k_{y,\sigma} < 0$ for all z and $a(0, z, 0) < 0$ for all $z \neq 0$. We will say that the dynamic output control Lyapunov $V(x, \sigma)$ satisfies the small control property if there exists a positive real function $L(y, \sigma)$, defined in a neighborhood of $(y, \sigma) = (0, 0)$, such that $L(y, \sigma) \to 0$ as $\|(y, \sigma)\| \to 0$ and $\|k_{y,\sigma}\| < L(y, \sigma)$, where for each $(y, \sigma) \neq (0, 0) k_{y,\sigma}$ satisfies $a(x, \sigma) + b(x, \sigma)k_{y,\sigma} < 0$ for all z.

Along the same lines, one can give the definition of dynamic control Lyapunov function for (1).

The systems of (1) have been considered first in the seminal paper [4] for the case of $A(y)$ constant. In general, (1) is not feedback linearizable, nonminimum phase, and, since $B(y)$ may have a singularity in $y = 0$, may even not have vector relative degree at $x = 0$. A first important step towards the stabilization (1) is given in [2]–[5], where for the case $m = p = 1$, $A(y)x = Ax + \psi(y)$ with A in Brunovsky form, and $B(y) = B$ a Hurwitz vector and relative degree at $x = 0$, it is shown that global stabilization can be achieved via dynamic output feedback.

In the case of single-input–single-output feedback linearizable systems, the stabilization problem via dynamic output feedback is solved in [13] under the assumption that a certain map is globally invertible. This assumption guarantees that the system is globally "observable" so that a global observer can be constructed.

Another important step is marked in [8], where the problem of asymptotically stabilizing (1) via dynamic output feedback is considered under the two following main assumptions: a) there exists a stabilizing controller $u = k(x)$, and b) there exists a constant matrix Q satisfying a certain inequality ("weak observability" property). While the first assumption is common to ours and is quite natural in solving a dynamic output feedback stabilization problem, the second one is somewhat "coupled" with the first one.

A similar solution to the problem of stabilizing (1) via dynamic output feedback has been given in [9] a) under the assumption that there exists a stabilizing controller $u = k(x)$, b) under some "growth" conditions, and c) under a certain unboundedness observability. These conditions are stronger than those proposed in [8] due to the difficulty of separating the problem of output feedback stabilization into the subproblems via full information and via output injection, respectively.

Finally, using some seminal ideas contained in [8] and [14], the problem of semiglobally stabilizing via dynamic output feedback for the class of nonlinear systems, which are minimum phase and in normal form, is solved under a suitable "small-gain" assumption in [15]. Following a similar approach, the problem of stabilizing a general nonlinear system via dynamic output feedback is considered in the subsequent [16], and a nice generalization to nonlinear systems of the so-called "separation" principle for linear systems is given. It is shown that state-feedback stabilizability plus complete uniform observability imply semiglobal stabilizability via dynamic output feedback. The property of complete uniform observability involves the solution of a system of first-order differential equations [i.e., the solution of (1)] and is not sufficient for global stabilization via dynamic output feedback, since, as shown in the recent paper [17], the "unboundedness unobservability" phenomenon may arise.

Another problem which will be discussed in this paper is the one of achieving output regulation for (1), where the state vector z may include reference signals as well as disturbances. In particular, the z-equation of (1) includes the case in which the reference signals and/or the disturbances are modeled by a linear first-order differential equation $\dot{w} = Sw$. In general, if this is the case, one can try to solve a disturbance attenuation problem (see later discussion). Let us formulate the output regulation problem according to [8].

Global output regulation problem via measurement feedback: Given (1), find a control law (2) such that for each initial condition (x_0, σ_0) the trajectories of the closed-loop system (1) and (2) are bounded and $\|y(t)\| \to 0$ as $t \to \infty$.

The local problem for a general nonlinear system has been given an elegant solution in [18] which closely follows the corresponding linear solution. In this case, the problem of output regulation is split into two separate subproblems which are the natural extension of the corresponding counterparts in the linear case. A different approach, as much as valuable, is pursued in [19]. On the other hand, an important contribution to the solution of the global output regulation problem via output feedback has been given in [8], followed immediately after by [2]–[5] (see discussion on the contributions to the stabilization problem).

When either the disturbances cannot be modeled through an exosystem or the asymptotic tracking cannot be performed, it can be useful to formulate a disturbance attenuation problem. In this case, the class of systems considered is

$$\begin{aligned}\dot{x}_1 &= A_{11}(y)x + B_{111}(y)w + B_{112}(y)u\\ \dot{z} &= A_{12}(y)x + B_{121}(y)w + B_{122}(y)u\\ \mathcal{I} &= A_2(y)x + B_{21}(y)w + B_{22}(y)u\\ y &= x_1\end{aligned}$$

where w are exogenous disturbances, and $\mathcal{I}$ defines a penalty variable which may include a tracking error or a cost on the control. For simplicity of calculations, we will assume $B_{21}(y) = 0$, $B_{22}^T(y)B_{22}(y) = I$, and $B_{22}^T(y)A_2(y) = 0$ (see the simplifying assumptions in [20] and [21]). The purpose of the control (2) is, on one hand, to achieve global internal stability when the effect of the disturbances is not present (i.e., $w(t) = 0$) and, on the other hand, to attenuate the influence of the disturbances on the penalty variables. Although there exist several ways of characterizing the requirement of disturbance attenuation, we will follow the one pursued in [22].

Disturbance attenuation problem with global stability via measurement feedback: Given (1) and a real number $0 < \gamma < 1$, find a controller (2) and a smooth and nonnegative function $V^e(x, \sigma)$ such that

$$\dot{V}^e(x, \sigma, w) + \|\mathcal{I}\|^2 - \gamma^2\|w\|^2 \le 0$$

and the closed-loop system (1) and (2) is globally asymptotically stable in $(x, \sigma) = (0, 0)$ for $w(t) = 0$.

In this formulation, γ represents a bound for the $\mathcal{L}_2$-gain $(w - \mathcal{I})$ of (1) (see the seminal papers [23]). Sufficient conditions for disturbance attenuation with local stability have been given in [21] for affine-in-the-input nonlinear systems, and necessary conditions for disturbance attenuation have been given in [22, Theorem 4.2] for (2) being linear in y. In particular, in [22] it has been shown that the solution of the disturbance attenuation problem amounts to the solution of two "uncoupled" Hamilton–Jacobi (not strict) inequalities plus a "coupling" condition between these two solutions, giving a nice generalization of the corresponding conditions for linear systems.

II. A Short Discussion of the Main Results

Let us now briefly discuss the main results of this paper. Our first result is the following and concerns semiglobal stabilization of (1) via static output feedback.

Theorem 1: Assume $m = 1$ and let $\Omega \subset \mathbb{R}^n$ be a compact set. Let $V(x) = \frac{1}{2}z^TPz + z^T\zeta(y) + \xi(y)$ be an output control Lyapunov function for (1), satisfying the small control property. Then, it is possible to find a static output feedback law (2) such that the closed-loop system (1) and (2) is locally asymptotically stable in $x = 0$ with the basin of attraction containing Ω. □

Theorem 1 proves the existence of a stabilizing output feedback law through a constructive proof, once an output control Lyapunov function is available. Moreover, it considers for simplicity single-input systems, but more complex arguments can lead to prove a similar result for multiple-input systems (see Section IV for an illustrative example). The sufficient condition of Theorem 1 is not necessary, since an output control Lyapunov function of the required form does not necessarily exist.

A simple consequence of the arguments contained in the proof of Theorem 1 is the possibility of obtaining global stabilization, as long as one assumes that the rank of a certain matrix is constant. This improves the procedure proposed in [6], in the sense that the compactness of the region of attraction is no longer an intrinsic limitation of the procedure itself. Let

$$\alpha(y) = \frac{\partial \zeta}{\partial y}(y)[A(y)]_y^z + P[A(y)]_z^z$$

where $[A(y)]_z^y$ and $[A(y)]_z^z$ are the matrices obtained from $A(y)$ by selecting the first p rows and the last $n-p$ columns and, respectively, the last $n-p$ rows and $n-p$ columns. The matrix $\alpha(y)$ is the Hessian (with respect to z) of the function $a(x) + b(x)u$, with $a(x) = \frac{\partial V}{\partial x}(x)A(y)x$ and $b(x) = \frac{\partial V}{\partial x}(x)B(y)$.

Theorem 2: Assume that $m = 1$. If for (1) there exists an output control Lyapunov function $V(x) = (1/2)z^TPz + z^T\zeta(y) + \xi(y)$ which satisfies the small control property, and, in addition, the matrix $\alpha(y)$ has constant rank, then it is possible to find an output feedback law $u = k(y)$ which solves the global stabilization problem for (1). □

The stabilizing controllers, which can be constructed through the proof of Theorem 1, may be sometimes complex to construct, except when the additional assumption on $\alpha(y)$ is stated. If this is the case, the expression of the controller is straightforward once an output control Lyapunov function is available. By this reason, we will prove Theorem 1 first, and, finally, we will remove the rank assumption on $\alpha(y)$, giving the procedure only for the two-dimensional case. In Section IV, Example 1 shows the procedure for a higher-dimensional case and two inputs.

Theorem 1 can also be proven, with far more complicated notations, by simply requiring that $V(x)$ is an output control Lyapunov function for (1) on $\{x \in \mathbb{R}^n : V(x) \le c\} \supset \Omega$.

When dynamic controllers are also allowed, we will show that the global stabilization problem via measurement feedback can be split up into two subproblems: the problem via state feedback and the problem via output injection. As is well known, the first subproblem quantifies the possibility of globally stabilizing (1) via state feedback. To clarify the significance of the second subproblem and for simplicity, let us consider the case of linear system (1) and define by

$$\begin{aligned}\dot{x} &= Ax + u,\ u \in \mathbb{R}^n\\ y &= Cx = x_1\end{aligned}$$

the system associated to (1) with u being a "fictitious" input vector. In this case, the control vector u has nothing to do with the control vector u in (1) and has dimension n. The problem of globally stabilizing the associated system via output feedback is commonly known as stabilization via output injection, and it is dual to the stabilization problem via the state feedback of (1). A necessary and sufficient condition for the associated system of (1) to be stabilizable via output feedback is that the pair (C, A) is detectable. For nonlinear systems

$$\begin{aligned} \dot{x} &= f(x) + g(y)u \\ y &= h(x) \end{aligned} \tag{3}$$

this kind of detectability condition does not make sense. The stabilization problem via output injection of (3) still can be characterized through the existence of an output control Lyapunov function for the associated system

$$\begin{aligned} \dot{x} &= f(x) + u \\ y &= h(x). \end{aligned}$$

It is easy to realize that (1), being asymptotically stabilizable via output injection, is the natural condition needed to prevent the unboundedness unobservability phenomenon [17] and to guarantee detectability. In this paper, we will prove the following "separation" condition.

Theorem 3: Let $V(x) = \frac{1}{2}z^T Pz + z^T\zeta(y) + \xi(y)$ be a control Lyapunov function for (1), satisfying the small control property. Also, let $\overline{V}(x) = \frac{1}{2}z^T\overline{P}z + z^T\overline{\zeta}(y) + \overline{\xi}(y)$ be an output control Lyapunov function for its associated system. Then it is possible to find an output feedback law (2), with $q = n$, which solves the global stabilization problem via measurement feedback. □

Theorem 3 gives a nonlinear analogous to the fact that if (1) is both stabilizable via state feedback and via output injection, it can be asymptotically stabilized via dynamic output feedback. The result of Theorem 3 has already been proven in [24] as far as the existence of a stabilizing output feedback is concerned. Indeed, the proof is not constructive, since it is based on unity partition. Here, we give an explicit expression of a stabilizing controller once a control Lyapunov function for (1) and an output control Lyapunov function for its associated system are available (see Recipe 2).

As is well known in the literature, if a linear system is both controllable and observable (actually, stabilizable and detectable is sufficient), a simple expression of a stabilizing dynamic output feedback law is

$$\begin{aligned} u &= K\sigma \\ \dot{\sigma} &= (A + BK)\sigma + H(-C\sigma + y) \end{aligned}$$

where K and H are such that $A+BK$ and $A-HC$ are Hurwitz matrices. Note that the dynamics of σ incorporates a copy of the asymptotically stable system $\dot{x} = (A + BK)x$. Moreover, this design is based on the construction of an observer which gives an estimate σ of the state x of the plant. The structure of the dynamic feedback law we propose here is not based on imposing to it the structure of an observer and provides an alternative expression of a stabilizing controller even in the linear case.

We want to remark that if we know the expression of a stabilizing controller $u = k(x)$ and that of a Lyapunov function $V(x)$ for (1) with $u = k(x)$, clearly $V(x)$ can be taken as a control Lyapunov function for (1) (a similar remark holds for an output control Lyapunov function).

Let us spend few more words on the structure we require for the Lyapunov functions. Although this structure may seem restrictive even for the class of nonlinear systems (1), many recent results on both (robust) global and semiglobal stabilization via output feedback can be recovered through the solution of a state feedback and, respectively, an output injection stabilization problem with Lyapunov functions having indeed this structure (see [25]–[27]). Further applications of the ideas contained in the proof of Theorems 3 and 6 are given in [28].

By relaxing the requirement on the structure of the Lyapunov functions Theorem 3, we obtain also necessary conditions.

Theorem 4: Let $V(x, \sigma_1) = \frac{1}{2}z^T Pz + z^T\zeta(y, \sigma_1) + \xi(y, \sigma_1)$, $\sigma_1 \in \mathbb{R}^{s_1}$, be a dynamic control Lyapunov function for (1), satisfying the small control property. Also, let $\overline{V}(x, \sigma_2) = \frac{1}{2}z^T\overline{P}z + z^T\overline{\zeta}(y, \sigma_2) + \overline{\xi}(y, \sigma_2)$, $\sigma_2 \in \mathbb{R}^{s_2}$, be a dynamic output control Lyapunov function for its associated system. Then it is possible to find (2), with $q = n + \max\{s_1, s_2\}$, which solves the global stabilization problem via measurement feedback.

Conversely, assume that there exists (2) which solves the global stabilization problem via measurement feedback for (1). Then, there exists a dynamic control Lyapunov function $V(x, \sigma_1)$ for (1), satisfying the small control property, and a dynamic output control Lyapunov function $\overline{V}(x, \sigma_2)$ for its associated system. □

Theorem 4 arises from the fact that for nonlinear systems, stabilization via dynamic feedback is no more equivalent to stabilization via static feedback. Moreover, the sufficient part of Theorem 3 (and 4) can be proven with a far more complicated notation by simply requiring that $V(x, \sigma_1)$ and $\overline{V}(x, \sigma_2)$ are a dynamic control Lyapunov function and a dynamic output control Lyapunov function, respectively, for (1) and its associated system on $\{(x, \sigma) \in \mathbb{R}^{n+q_1} : V(x, \sigma_1) \leq c_1\} \supset \Omega_1$ and $\{(x, \sigma) \in \mathbb{R}^{n+q_2} : \overline{V}(x, \sigma_2) \leq c_2\} \supset \Omega_2$, with given compact sets $\Omega_1 \subset \mathbb{R}^{n+q_1}$ and $\Omega_2 \subset \mathbb{R}^{n+q_2}$. The resulting dynamic output feedback law semiglobally asymptotically stabilizes (1).

In the framework of global output regulation via measurement feedback, we have the following result which can be proven along the lines of Theorem 1 and by using LaSalle's invariance principle [29]. Extensions of this result in the spirit of Theorem 4 are still possible.

Theorem 5: Let $V(x) = \frac{1}{2}z^T Pz + z^T\zeta(y) + \xi(y)$ be a smooth, proper, and positive definite function such that its derivative with respect to time along the trajectories of (1), for some continuous $u = k(x)$, $k(0) = 0$, is less than or equal to zero for all x and equal to zero only if $y = 0$. Also, let $\overline{V}(x) = \frac{1}{2}z^T\overline{P}z + z^T\overline{\zeta}(y) + \overline{\xi}(y)$ be an output control Lyapunov function for (1). Then it is possible to find an output feedback law (2), with $q = n$, which solves the global output regulation problem via measurement feedback. □

The result of Theorem 5 generalizes the local [18] and global [2] results with respect to the class of systems considered. Moreover, the problem of global output regulation via measurement feedback is split up into two "separate" subproblems which are the natural extension of the corresponding counterparts in the linear case.

The second assumption of Theorem 5 is not very natural even in the linear case (see [18]). By arguments similar to those contained in the proof of Theorem 5, we can show that Theorem 5 still holds if we replace the condition on $\overline{V}(x)$ by the more natural one which gives (even locally) a weaker result than the one contained in [18].

2a): There exists a continuous function $l(y)$, $l(0) = 0$ and a smooth, proper, and positive definite function $\overline{V}(x) = \frac{1}{2}z^T\overline{P}z + z^T\zeta(y) + \overline{\xi}(y)$ such that its derivative with respect to time along the trajectories of (1), with $u = l(y)$, is less than or equal to zero.

The sufficient conditions of Theorem 5 [with the second condition replaced by 2a)] can be further weakened in the spirit of Theorem 4.

We want to remark that for the purpose of constructing an output feedback controller which achieves global output regulation, it is not necessary to know the expression of $l(y)$ (this will be clear from the proof of Theorem 5). A similar remark holds for the problem of output feedback stabilization.

As already stated, when either the disturbances cannot be modeled through an exosystem or the asymptotic tracking cannot be performed, it can be useful to look at a disturbance attenuation problem with stability. In our paper, the relevance of the disturbance attenuation problem with stability is with respect to the robust stabilization problem via output feedback (see [25]), since as is well known, a robust stabilization problem can be formulated as a suitable disturbance attenuation problem with internal stability.

Theorem 6: Assume that the system

$$\begin{aligned}\dot{x} &= A_1(y)x\\ \mathcal{I}_0 &= A_2(y)x\end{aligned}\tag{3}$$

with output $\mathcal{I}_0$, is detectable in the sense of [2]. Moreover, assume that there exist smooth, positive definite, and proper functions $V(x) = \frac{1}{2}z^TPz + z^T\zeta(y) + \xi(y)$ and $\overline{V}(x) = \frac{1}{2}z^T\overline{P}z + z^T\overline{\zeta}(y) + \overline{\xi}(y)$ such that

1) There exists a continuous function $u = k(x)$, $k(0) = 0$, such that

$$\begin{aligned}&\frac{\partial V}{\partial x}(x)(A_1(y)x + B_{12}(y)k(x))\\ &\quad + \frac{1}{4\gamma^2}\frac{\partial V}{\partial x}(x)B_{11}(y)B_{11}^T(y)\frac{\partial V^T}{\partial x}(x)\\ &\quad + \|A_2(y)x + B_{22}(y)k(x)\|^2 \le 0.\end{aligned}$$

2) There exists a continuous function $l(y)$, $l(0) = 0$, such that

$$\begin{aligned}&\frac{\partial \overline{V}}{\partial x}(x)(A_1(y)x + l(y))\\ &\quad + \frac{1}{4}\frac{\partial \overline{V}}{\partial x}(x)B_{11}(y)B_{11}^T(y)\frac{\partial \overline{V}^T}{\partial x}(x)\\ &\quad + \frac{1}{\gamma^2}\|A_2(y)x\|^2 \le 0\end{aligned}$$

and the Hessian (with respect to z) of

$$\begin{aligned}&\frac{\partial \overline{V}}{\partial x}(x)(A_1(y)x + l(y))\\ &\quad + \frac{1}{4}\frac{\partial \overline{V}}{\partial x}(x)B_{11}(y)B_{11}^T(y)\frac{\partial \overline{V}^T}{\partial x}(x)\\ &\quad + \frac{1}{\gamma^2}\|A_2(y)x\|^2\end{aligned}$$

is definite negative for $y = 0$.

3) $\gamma^2\overline{P} - P > 0$.

Then, the disturbance attenuation problem with global stability via measurement feedback is solvable. □

Theorem 6 shows that if the two "uncoupled" Hamilton–Jacobi inequalities 1) and 2) have a solution, and a "coupling" condition between the two corresponding solutions is satisfied, then the disturbance attenuation problem with global stability via measurement feedback is solvable, proving again that the problem can be split up into two subproblems: one is the corresponding problem via full information and the other is the dual problem via output injection. This "separation" principle has been proven to be a necessary condition in [22] for the first time, in the case that the class of controllers (2) is linear in y. Our result gives an alternative proof even in the linear case, and our procedures give systematic tools for designing *ad hoc* controllers.

In the linear case, $V(x)$ and $\overline{V}(x)$ are quadratic in x. In this case, 1) and 2) are the classical Riccati inequalities, dual to each other.

Before ending the section, we remark that by using the arguments contained in the proof of Theorem 6 and since, as is well known, a robust stabilization problem can be formulated as a suitable disturbance attenuation problem, one can recover many recent results on global robust stabilization via output feedback ([25]–[28]).

III. Proofs of the Main Results

For a given $n \times n$ matrix M by $[M]_z$ ($[M]_y$), we will denote the last $n-p$ (the first p) rows of M. Dually, by $[M]^z$ ($[M]^y$) we will denote the last $n-p$ (the first p) columns of M.

Proof: Proof of Theorem 1: We first assume that $\alpha(y)$ has constant rank. Finally, we will remove this assumption and show the constructive procedure for the two-dimensional case (see comments in Section II).

After direct computations along the trajectories of (1), one has

$$\begin{aligned}\dot{V}(x,u) = a(x) + b(x)u &= z^T\alpha(y)z\\ &\quad + z^T(\beta(y) + \delta(y)u) + \gamma(y) + \epsilon(y)u\end{aligned}\tag{4}$$

with $a(x) = \frac{\partial V}{\partial x}(x)A(y)x$, $b(x) = \frac{\partial V}{\partial x}(x)B(y)$ and for some $\alpha(y)$, $\beta(y)$, $\gamma(y)$, $\delta(y)$ and $\epsilon(y)$ (see Recipe 1).

From the structure of (4), one can assume that $\alpha(y)$ is symmetric. Moreover, since $\alpha(y)$ has constant rank and $\frac{\partial \xi}{\partial y}(0) = 0$, $\zeta(0) = 0$, $\psi(0) = 0$, $\beta(0) = 0$, and $\gamma(0) = 0$, from the definition of output control Lyapunov function we conclude that $\alpha(y)$ is negative definite for all y.

Since $\alpha(y)$ is nonsingular for all y and $V(x)$ is an output control Lyapunov function, it follows that for each $y \neq 0$ the function $\varphi_{y,u}(z) = z^T\alpha(y)z + z^T(\beta(y)+\delta(y)u)+\gamma(y)+\epsilon(y)u$ has a global maximum at $z^*_{y,u} = \frac{1}{2}\alpha^{-1}(y)(\beta(y)+\delta(y)u)$. We have $\varphi_{y,u}(z^*_{y,u}) = -\frac{1}{4}(\beta(y)+\delta(y)u)^T\alpha^{-1}(y)(\beta(y)+\delta(y)u)+\gamma(y)+\epsilon(y)u$. Since for each $y \neq 0$ there exists u such that $\varphi_{y,u}(z^*_{y,u}) < 0$, we obtain for each $y \neq 0$

$$\dot{V}(x, u) \leq u^2\Phi(y) + u\Phi(y) + H(y) < 0 \tag{5}$$

for some u, $\Phi(y)$, $\Psi(y)$, and $H(y)$, with $\Phi(y) \geq 0$ for each y.

Let $\Phi(y) = \Psi^2(y)\Phi(y)$, $\Psi(y) = \Psi^2(y)$ and $\overline{H}(y) = H(y)$. By construction, $\overline{\Phi}(y) \geq 0$, $\overline{\Psi}(y) \geq 0$, $\Phi(0) = \Psi(0) = \overline{H}(0) = 0$. Moreover, let

$$\overline{k}(y) = \begin{cases} 0 \\ \quad \text{if } \overline{\Psi}(y) = 0 \\ -\dfrac{\overline{H}(y)+\sqrt{\overline{H}^2(y)+\frac{16}{27}\overline{\Psi}^4(y)\varrho(\overline{\Psi}(y)+\overline{H}(y))}}{2\overline{\Psi}(y)} \\ \quad \text{if } \overline{\Phi}(y) = 0 \text{ and } \overline{\Psi}(y) \neq 0 \\ \dfrac{2\sqrt{\overline{\Psi}^2(y)-3\overline{\Phi}\overline{H}(y)}\cos\left(\frac{1}{3}\operatorname{arcos}\left\{\frac{R(y)}{\sqrt{-Q^3(y)}}\right\}+\frac{4}{3}\pi\right)-\overline{\Psi}(y)}{3\overline{\Phi}(y)} \\ \quad \text{if } \overline{\Phi}(y) \neq 0 \end{cases} \tag{6}$$

where $0 < \operatorname{arcos}\left\{\frac{R(y)}{\sqrt{-Q^3(y)}}\right\} < \pi$

$$\begin{aligned} R(y) &= -\frac{1}{2}[2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y) \\ &\quad + 27G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))\overline{\Phi}^2(y)] \\ Q(y) &= -\overline{\Psi}^2(y) + 3\overline{H}(y)\overline{\Phi}(y) \\ G(\overline{\Phi}(y)&, \overline{\Psi}(y), \overline{H}(y)) \\ &= -\frac{1}{27}[(2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)) \\ &\quad + 2\sqrt{(\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y))^3}] \\ &\quad \cdot \varrho(\overline{\Psi}(y) + \overline{H}(y))e^{-\overline{\Phi}^2}(y) \end{aligned}$$

with $\varrho(\cdot)$ any smooth function such that $0 \leq \varrho(\theta) \leq 1$ for all θ, $\varrho(\theta) > 0$ for all $\theta > 0$, and $\varrho(\theta) = 0$ for $\theta \leq 0$.

We will show that the function $k(y) = \overline{k}(y)\Psi(y)$ solves the global stabilization problem for (1). We will prove subsequently the following facts: a) (6) is the pointwise solution of a couple of algebraic equations [see (7a) and (7b)], b) the coefficients of these algebraic equations are a smooth function of y, c) use the implicit function theorem in the spirit of [10] to conclude that (6) is a smooth function of y, and d) (6) is continuous in $y = 0$ and, finally, $u = k(y)$ solves (5) for all $(y, z) \neq (0, 0)$.

1) Let us consider the set of equations

$$3\overline{\Phi}(y)\overline{k}^2(y) + 2\overline{\Psi}(y)\overline{k}(y) + \overline{H}(y) < 0 \tag{7a}$$

$$\begin{aligned} &\overline{\Phi}(y)\overline{k}^3(y) + \overline{\Psi}(y)\overline{k}^2(y) + \overline{H}(y)\overline{k}(y) \\ &\quad + G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y)) = 0 \end{aligned} \tag{7b}$$

with $G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))$ as above. Using (5) and by straightforward computations, one shows that for each nonzero y inequality, (7a) is satisfied by some real number $\overline{k}(y)$.

The left-hand part of (7b) is, for each nonzero y such that $\overline{\Phi}(y) \neq 0$, a polynomial in $\overline{k}(y)$ of degree 3. By standard arguments of algebra, it can be shown that this polynomial has three distinct real roots if

$$\begin{aligned} &-\frac{1}{27}\left[\frac{\overline{\Psi}^2(y)-3\overline{H}(y)\overline{\Phi}(y)}{3}\right]^3 \\ &+\frac{1}{4}\left[\frac{2\overline{\Psi}^3(y)-9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)+27G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))\overline{\Phi}^2(y)}{27}\right]^2 < 0 \\ &\overline{\Phi}(y) \neq 0. \end{aligned} \tag{8}$$

Since for $\overline{\Phi}(y) \neq 0$

$$\begin{aligned} &-4[\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)]^3 \\ &\quad + [2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)]^2 \\ &= -27(\overline{\Psi}^2(y) - 4\overline{H}(y)\overline{\Phi}(y))\overline{H}^2(y)\overline{\Phi}^2(y) \\ &\leq 0 \end{aligned} \tag{9}$$

it follows that

$$\begin{aligned} &-\frac{[2\overline{\Psi}^3(y)-9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)]+2\sqrt{[\overline{\Psi}^2(y)-3\overline{H}(y)\overline{\Phi}(y)]^3}}{27\overline{\Phi}^2(y)} \leq 0 \leq \\ &< -\frac{[2\overline{\Psi}^3(y)-9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)]-2\sqrt{[\overline{\Psi}^2(y)-3\overline{H}(y)\overline{\Phi}(y)]^3}}{27\overline{\Phi}^2(y)}. \end{aligned} \tag{10}$$

From (10), since $\frac{1}{\overline{\Phi}^2(y)} > \varrho(\overline{\Psi}(y) + \overline{H}(y))e^{-\overline{\Phi}^2(y)} > 0$ when $\overline{H}(y) = 0$ and $\overline{\Phi}(y) \neq 0$ (remember that $\overline{\Psi}(y) = 0 \Rightarrow \overline{H}(y) < 0$), and $-4[\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)]^3 + [2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)]^2 = 0$ for $\overline{\Phi}(y) \neq 0$ if and only if $\overline{H}(y) = 0$, we have for $\overline{\Phi}(y) \neq 0$ (X) shown at the bottom of the next page. We conclude that (11) is always satisfied with $G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))$ defined as above. Under the assumption $\overline{\Phi}(y) \neq 0$, it is well known from elementary algebra that the three distinct roots of (7b) are given by (Y) shown at the bottom of the next page. It is easy to see that, under our assumptions, it is $\overline{k}_1(y) > \overline{k}_3(y) > \overline{k}_2(y)$ and, since $\overline{\Phi}(y) \geq 0$, the derivative of (7b) with respect to $\overline{k}(y)$, i.e., the left-hand part of (7a), evaluated in $\overline{k}(y) = \overline{k}_3(y)$, is strictly negative as long as $\overline{\Phi}(y) \neq 0$.

On the other hand, for each $y \neq 0$ such that $\overline{\Phi}(y) = 0$ and $\overline{\Psi}(y) \neq 0$, the left-hand part of (7b) boils down to a polynomial in $\overline{k}(y)$ of degree 2. It is easy to see that this polynomial has two distinct real roots if

$$\begin{aligned} &\overline{H}^2(y) - 4\overline{\Psi}(y)G(0, \overline{\Psi}(y), \overline{H}(y)) < 0, \\ &\overline{\Psi}(y) \neq 0, \overline{\Phi}(y) = 0. \end{aligned}$$

Since $\varrho(\cdot) > 0$ when $\overline{H}(y) = 0$ and $G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y)) = -\frac{4}{27}\overline{\Psi}^4(y)\varrho(\overline{\Psi}(y) + \overline{H}(y))$, the above inequality is always satisfied with $G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))$ defined as above. The two distinct roots are given by

$$\overline{k}_1(y) = \frac{\overline{H}(y) - \sqrt{\overline{H}^2(y) + \frac{16}{27}\overline{\Psi}^4(y)\sigma(\overline{\Psi}(y) + \overline{H}(y))}}{2\overline{\Psi}(y)}$$

$$\overline{k}_2(y) = -\frac{\overline{H}(y) + \sqrt{\overline{H}^2(y) + \frac{16}{27}\overline{\Psi}^4(y)\sigma(\overline{\Psi}(y) + \overline{H}(y))}}{2\overline{\Psi}(y)}.$$

It is easy to see that under our assumptions, $\overline{k}_1(y) > \overline{k}_2(y)$ and, since $\overline{\Psi}(y) \geq 0$, the derivative of (7b) with respect to $\overline{k}(y)$, i.e., the left-hand part of (7a) evaluated in $\overline{k}(y) = \overline{k}_2(y)$, is negative as long as $\overline{\Psi}(y) \neq 0$ and $\overline{\Phi}(y) = 0$.

2) Note that $G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))$, defined as above, is smooth at each $y \neq 0$, since $\overline{\Phi}(y)$, $\overline{\Psi}(y)$, and $\overline{H}(y)$ are themselves smooth, and for each $\overline{y} \neq 0$ such that $\overline{\Psi}^2(\overline{y}) - 3\overline{H}(\overline{y})\overline{\Phi}(\overline{y}) = 0$ we have $\varrho(\overline{\Psi}(y) + \overline{H}(y)) = 0$ for all y in a sufficiently small neighborhood of $\overline{y}$. Thus, each coefficient of the polynomials (7a) and (7b) are smooth functions of y.

3) Since the polynomial (7b) is equal to zero and its derivative with respect to $\overline{k}(y)$, i.e., the left-hand part of (7a) is strictly negative invoking the implicit function theorem in the spirit of [10], it follows that $\overline{k}(y)$ satisfies (7a) for all $(y, z) \neq (0, 0)$, and it is smooth at each $y \neq 0$.

4) Since the proof is quite technical and consists of lengthy algebraic computations, we will omit it (a complete proof may be obtained via electronic mail at stefbatt@riscdis.uniroma1.ing.it).

Now, we will remove the assumption that $\alpha(y)$ has constant rank. For the reader's convenience, we will prove the theorem in the two-dimensional case with $m = p = 1$. The general procedure goes in the same way.

Since $a(y, z) + b(y, z)u < 0$ for all z and for some u and, in addition, $\alpha(0) < 0$, it follows that $\alpha(y)$ is negative semidefinite for all y. Now, define the sets

$$S_1 = \mathbb{R}$$
$$S_0 = \{y \in \mathbb{R} : \alpha(y) = 0\}.$$

Since $\alpha(0) < 0$, it follows that $0 \notin S_0$. By easy arguments, one can conclude that the set S_0 is closed in $\mathbb{R}$.

Let $\varphi_{y,u}(z) = z^T\alpha(y)z + z^T(\beta(y) + \delta(y)u) + \gamma(y) + \epsilon(y)u$. The idea of the proof consists of constructing continuous feedback laws $u = k_j(y)$, $j = 1, 2$, such that $\varphi_{y,k_j(y)}(z) < 0$ for all z and $y \in S_1/S_0$ or $y \in S_0$, respectively, such that $(y, z) \in \Omega/\{(0, 0)\}$ and finally putting all these feedback laws together.

More explicitly, at each point $y \in S_1/S_0$, $\alpha(y)$ is negative and nonzero. In this case, one proceeds as in the case that $\alpha(y)$ has constant rank to obtain a continuous feedback law $k_1(y)$ such that $\varphi_{y,k_1(y)}(z) < 0$ for all $y \in S_1/S_0$ and z but $(y, z) = (0, 0)$.

On the other hand, at the points $y \in S_0$ one can proceed as follows. At each point y such that $\alpha(y) = 0$, since $V(x)$ is an output control Lyapunov function, necessarily $\beta(y) + \delta(y)u = 0$. Let

$$T_0 = \{y \in \mathbb{R} : \delta^2(y) \neq 0\}.$$

Moreover, let

$$\mu(y) = \frac{\gamma(y) + \sqrt{\gamma^2(y) + \epsilon^4(y)}}{\epsilon(y)}. \tag{11}$$

Since, by definition of output control Lyapunov function, at each $y \neq 0$ if $\epsilon(y) = 0$ then $\gamma(y) < 0$, using the small control property of $V(x)$ and the same arguments contained in [10], one can prove that the function (11) is continuous at each $y \in S_0 \cap (\mathbb{R}/T_0)$. Moreover, since at each $y \in S_0 \cap (\mathbb{R}/T_0)$, we have $\delta(y) = \beta(y) = \alpha(y) = 0$, the function (11) is such that $\varphi_{y,\mu(y)}(z) < 0$ for all $y \in S_0 \cap (\mathbb{R}/T_0)$ and z.

Finally, let

$$\lambda(y) = -\frac{\beta(y)}{\delta(y)}. \tag{12}$$

At each point $y \in T_0$, the function (12) is continuous and is such that $\beta(y) + \delta(y)\lambda(y) = 0$.

Now, given $r \in \mathbb{R}^+$, let $\varrho(\theta, r)$ be any smooth function such that $0 \leq \varrho(\theta, r) \leq 1$ for all θ, $\varrho(\theta, r) = 1$ for $\theta \geq r$ and $\varrho(\theta, r) = 0$ for $\theta \leq \frac{r}{2}$. By compactness arguments, it can be shown that there always exists $\tau > 0$ such that $\varphi_{y,\mu(y)}(z) < 0$ for all z and $y \in S_0 \cap (\mathbb{R}/T_0)$ such that both $\delta^2(y) < \tau$ and $(y, z) \in \Omega$.

Correspondingly, choose

$$k_0(y) = (1 - \varrho(\delta^2(y), \tau)\lambda(y) + \varrho(\delta^2(y), \tau)\mu(y).$$

$$\begin{aligned} &-\frac{[2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)] + 2\sqrt{[\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)]^3}}{27\overline{\Phi}^2(y)} < G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y)) \\ &< -\frac{[2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)] - 2\sqrt{[\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)]^3}}{27\overline{\Phi}^2(y)} \end{aligned} \tag{X}$$

$$\begin{aligned} \overline{k}_1(y) &= \frac{2\sqrt{\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)}\cos\left(\frac{1}{3}\arccos\left\{\frac{R(y)}{\sqrt{-Q^3(y)}}\right\}\right) - \overline{\Psi}(y)}{3\overline{\Phi}(y)} \\ \overline{k}_2(y) &= \frac{2\sqrt{\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)}\cos\left(\frac{1}{3}\arccos\left\{\frac{R(y)}{\sqrt{-Q^3(y)}}\right\} + \frac{2}{3}\pi\right) - \overline{\Psi}(y)}{3\overline{\Phi}(y)} \\ \overline{k}_3(y) &= \frac{2\sqrt{\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y)}\cos\left(\frac{1}{3}\arccos\left\{\frac{R(y)}{\sqrt{-Q^3(y)}}\right\} + \frac{4}{3}\pi\right) - \overline{\Psi}(y)}{3\overline{\Phi}(y)} \end{aligned} \tag{Y}$$

It is easy to see that the function $k_0(y)$ is continuous at each S_0 and is such that $\varphi_{y,k_0(y)}(z) < 0$ for all z and $y \in S_0$ such that $(y, z) \in \Omega$.

By compactness arguments, it can be shown that there always exists $\sigma > 0$ such that $\varphi_{y,\mu(y)}(z) < 0$ for all z and $y \in S_0$ such that $-\alpha(y) < \sigma$ and $(y, z) \in \Omega$. If necessary, since $0 \notin S_0$, σ can be taken smaller so that $0 \notin \{y \in S_0 \colon -\alpha(y) < r\}$.

It is easy to show that the output feedback law

$$\begin{aligned} k(y) =& (1 - \varrho(-\alpha(y), \sigma))k_0(y) \\ &+ \varrho(-\alpha(y), \sigma)k_1(y) \end{aligned} \tag{13}$$

is continuous and such that $\varphi_{y,k(y)}(z) < 0$ for all $(y, z) \in \Omega/\{(0, 0)\}$, i.e., solves the semiglobal stabilization problem for (1). Indeed, we have

$$\varphi_{y,k(y)}(z) = (1 - \varrho)\varphi_{y,k_0(y)}(z) + \varrho\varphi_{y,k_1(y)}(z)$$

where for simplicity we have omitted the arguments of the function ϱ. Since $0 \leq \varrho(\cdot, \cdot) \leq 1$ and since $\varphi_{y,k_0(y)}(z) < 0$ for all z and $y \in S_0$ such that $-\alpha(y) < \sigma$ and $(y, z) \in \Omega$ and $\varphi_{y,k_1(y)}(z) < 0$ for all z and $y \in S_1/S_0$ such that $(y, z) \in \Omega/\{(0, 0)\}$, it follows that $\varphi_{y,k(y)}(z) < 0$ for all $(y, z) \in \Omega/\{(0, 0)\}$. This completes the proof of the theorem. □

At this point, we can give a systematic procedure to construct stabilizing output feedback controllers, once an output control Lyapunov function $V(x)$ is available. For the sake of simplicity, we assume that $\alpha(y)$ has constant rank. A general, but far more complex procedure, can be worked out in the case that this assumption is violated.

IV. Recipe 1

- Let $V(x) = \frac{1}{2}z^T Pz + z^T\zeta(y) + \xi(y)$ be an output control Lyapunov function for (1) satisfying the small control property and, correspondingly, define

$$\begin{aligned} \alpha(y) &= \frac{\partial\zeta}{\partial y}(y)[A(y)]_y^z + P[A(y)]_z^z \\ \gamma(y) &= \frac{\partial\xi}{\partial y}(y)[A(y)]_y^y y + \zeta^T(y)[A(y)]_z^y y \\ \beta(y) &= \frac{\partial\zeta}{\partial y}(y)[A(y)]_y^y y + ([A(y)]_y^z)^T\frac{\partial\xi^T}{\partial y}(y) \\ &\quad + P[A(y)]_z^y y + ([A(y)]_z^z)^T\zeta(y) \\ \delta(y) &= \frac{\partial\zeta}{\partial y}(y)[B(y)]_y + P[B(y)]_z \\ \epsilon(y) &= \frac{\partial\xi}{\partial y}(y)[B(y)]_y + \zeta^T(y)[B(y)]_z. \end{aligned}$$

Symmetrize $\alpha(y)$ if necessary. Moreover, let

$$\begin{aligned} \Phi(y) &= -\frac{1}{4}\delta^T(y)\alpha^{-1}(y)\delta(y) \\ \Psi(y) &= \epsilon(y) - \frac{1}{2}\delta^T(y)\alpha^{-1}(y)\beta(y) \\ H(y) &= \gamma(y) - \frac{1}{4}\beta^T(y)\alpha^{-1}(y)\beta(y) \\ \overline{\Phi}(y) &= \Psi^2(y)\Phi(y),\ \overline{\Psi}(y) = \Psi^2(y),\ \overline{H}(y) = H(y) \end{aligned}$$

$$\begin{aligned} R(y) &= -\frac{1}{2}[2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y) \\ &\quad + 27G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y))\overline{\Phi}^2(y)] \\ Q(y) &= -\overline{\Psi}^2(y) + 3\overline{H}(y)\overline{\Phi}(y) \end{aligned}$$

$$\begin{aligned} &G(\overline{\Phi}(y), \overline{\Psi}(y), \overline{H}(y)) \\ &\quad = -\frac{1}{27}[(2\overline{\Psi}^3(y) - 9\overline{\Psi}(y)\overline{H}(y)\overline{\Phi}(y)) \\ &\quad\quad + 2\sqrt{(\overline{\Psi}^2(y) - 3\overline{H}(y)\overline{\Phi}(y))^3}] \\ &\quad\quad \cdot \varrho(\overline{\Psi}(y) + \overline{H}(y))e^{-\overline{\Phi}^2(y)} \end{aligned}$$

with $\varrho(\cdot)$ any smooth function such that $0 \leq \varrho(\theta) \leq 1$ for all θ, $\varrho(\theta) > 0$ for all $\theta > 0$, and $\varrho(\theta) = 0$ for $\theta \leq 0$.

- The output feedback law

$$u = \begin{cases} 0 \\ \quad \text{if } \Psi(y) = 0 \\ -\frac{\overline{H}(y) + \sqrt{\overline{H}^2(y) + \frac{16}{27}\overline{\Psi}^4(y)\varrho(\overline{\Psi}(y) + \overline{H}(y))}}{2\overline{\Psi}(y)}\Psi(y) \\ \quad \text{if } \Phi(y) = 0 \text{ and } \Psi(y) \neq 0 \\ \frac{2\sqrt{\overline{\Psi}^2(y) - 3\overline{\Phi}\overline{H}(y)}\cos\left(\frac{1}{3}\operatorname{arcos}\left\{\frac{R(y)}{\sqrt{-Q^3(y)}}\right\} + \frac{4}{3}\pi\right) - \overline{\Psi}(y)}{3\overline{\Phi}(y)}\Psi(y) \\ \quad \text{if } \Phi(y) \neq 0 \text{ and } \Psi(y) \neq 0 \end{cases}$$

solves the global stabilization problem for (1).

Proof: Proof of Theorem 3: Let us consider the extended system

$$\begin{aligned} \dot{x} &= A(y)x + B(y)u(y, \sigma) \\ \dot{\sigma} &= v(y, \sigma),\ \sigma \in \mathbb{R}^n \\ y &= x_1. \end{aligned} \tag{14}$$

To prove the theorem, we will construct a feedback law (2) which globally asymptotically stabilizes (14) in $(x, \sigma) = (0, 0)$.

Let

$$k(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ -\frac{a(x) + \sqrt{a^2(x) + \|b(x)\|^4}}{\|b(x)\|^2}b^T(x), & \text{else} \end{cases}$$

with $a(x) = \frac{\partial V}{\partial x}(x)A(y)x$ and $b(x) = \frac{\partial V}{\partial x}(x)B(y)$. Moreover, let $u = k(y, [\sigma]_z)$ and define

$$V^e(x, \sigma) = V_m(x, \sigma) + \lambda V(x) \tag{15}$$

with

$$\begin{aligned} V_m(x, \sigma) &= \frac{1}{2}(z - [\sigma]_z)^T P_m(z - [\sigma]_z) \\ &\quad + (z - [\sigma]_z)^T\zeta_m(y, [\sigma]_y) + \xi_m(y, [\sigma]_y) \\ P_m &= \overline{P} - \lambda P \\ \zeta_m(y, [\sigma]_y) &= -\lambda\zeta(y) + \overline{\zeta}(y) + \lambda\zeta([\sigma]_y) - \overline{\zeta}([\sigma]_y) \\ \xi_m(y, [\sigma]_y) &= \frac{1}{2}\|y - [\sigma]_y\|^2 \\ &\quad + \frac{1}{2}\zeta_m^T(y, [\sigma]_y)P_m^{-1}\zeta_m(y, [\sigma]_y) \end{aligned}$$

and $\lambda > 0$ such that P_m is positive definite.

We will show first that $V^e(x, \sigma)$ is positive definite. Note that since P_m is positive definite and symmetric and since

$\frac{1}{2}(z - [\sigma]_z)^T P_m (z - [\sigma]_z) + (z - [\sigma]_z)^T \zeta_m(y, [\sigma]_y) \geq -\frac{1}{2}\zeta_m^T(y, [\sigma]_y) P_m^{-1} \zeta_m(y, [\sigma]_y)$ for all (x, σ), we have $V_m(x, \sigma) \geq \frac{1}{2}\|y - [\sigma]_y\|^2 \geq 0$. Moreover, since $V_m(x, \sigma) \geq \frac{1}{2}\|y - [\sigma]_y\|^2$ and the equality holds if and only if $z - [\sigma]_z = -\frac{1}{2}P_m^{-1}\zeta_m(y, [\sigma]_y)$, we conclude that $V_m(x, \sigma) = 0$ if and only if $y = [\sigma]_y$ and $z - [\sigma]_z = -P_m^{-T}\zeta_m(y, [\sigma]_y)$ or, equivalently, since $\zeta_m([\sigma]_y, [\sigma]_y) = 0$, if and only if $x = \sigma$. Since $\lambda V(x)$ is positive definite, it follows that $V^e(x, \sigma) > 0$ for all $(x, \sigma) \neq (0, 0)$ and $V^e(0, 0) = 0$.

Moreover, $V^e(x, \sigma)$ is smooth and proper. Indeed, $V(x)$ is proper, $V^e(x, \sigma) \geq \frac{1}{2}\|y - [\sigma]_y\|^2 \geq 0$, and for each $(y, [\sigma]_y)$ the function $V^e(x, \sigma)$ is quadratic in $z - [\sigma]_z$.

After straightforward computations, along the trajectories of (14) one obtains

$$\dot{V}(x, \sigma, u, v) = z^T\alpha^e(y)z + z^T[\beta^e(y, \sigma) + \delta^e(y, \sigma)v] + \gamma^e(y, \sigma) + \epsilon^e(y, \sigma)v \quad (16)$$

for some $\alpha^e(y)$, $\beta^e(y, \sigma)$, $\gamma^e(y, \sigma)$, $\delta^e(y, \sigma)$ and $\epsilon^e(y, \sigma)$. Moreover, $\alpha^e(y) = \frac{\partial\zeta}{\partial y}[A(y)]_y^z + \overline{P}[A(y)]_z^z$.

To prove the theorem, it is sufficient to show that there exists a continuous function $v(y, \sigma)$ such that (16) is negative for all nonzero (x, σ). Since for each (y, σ), (16) is a quadratic function of z, we can rearrange the entries of $\alpha^e(y)$ in such a way that $\alpha^e(y)$ is a symmetric matrix.

By our assumptions on $\overline{V}(x)$, as in the proof of Theorem 1, we can prove that $\alpha^e(y)$ is negative semidefinite for all y and negative definite for $y = 0$.

Now, redefine v as $T^{-1}(y, \sigma)v$, where

$$T(y, \sigma) = \begin{pmatrix} 0_{p\times(n-p)} & I_{p\times p} \\ -P_m^{-1} & P_m^{-1}\frac{\partial\zeta_m}{\partial[\sigma]_y}(y, [\sigma]_y) \end{pmatrix}.$$

With $T(y, \sigma)$ as above, we have a partition $\delta_1^e(y, \sigma)$, $\delta_2^e(y, \sigma)$ of $\delta^e(y, \sigma)T(y, \sigma)$ and $\epsilon_1^e(y, \sigma)$, $\epsilon_2^e(y, \sigma)$ of $\epsilon^e(y, \sigma)T(y, \sigma)$, respectively, such that

$$\begin{aligned} \delta_1^e(y, \sigma) &= I_{(n-p)\times(n-p)} \\ \delta_2^e(y, \sigma) &= 0_{(n-p)\times p} \\ \epsilon_1^e(y, \sigma) &= -[\sigma]_z^T + \zeta_m^T(y, [\sigma]_y)P_m^{-1} \end{aligned}$$

$$\epsilon_2^e(y, \sigma) = \frac{\partial\xi_m}{\partial[\sigma]_y}(y, [\sigma]_y) - \zeta_m^T(y, [\sigma]_y)P_m^{-1}\frac{\partial\zeta_m}{\partial[\sigma]_y}(y, [\sigma]_y) = -y^T + [\sigma]_y^T$$

and (16) can be rewritten as

$$z^T\alpha^e(y)z + z^T[\beta^e(y, \sigma) + v_1(y, \sigma)] + \gamma^e(y, \sigma) + \epsilon_1^e(y, \sigma)v_1(y, \sigma) + \epsilon_2^e(y, \sigma)v_2(y, \sigma) < 0$$

with $\binom{v_1(y, \sigma)}{v_2(y, \sigma)}$ a partition of $T^{-1}(y, \sigma)v$ corresponding to that of $\delta^e(y, \sigma)T(y, \sigma)$ and $\epsilon^e(y, \sigma)T(y, \sigma)$. Choose $v_1(y, \sigma)$ in such a way that

$$2\alpha^e(y)[\sigma]_z + \beta^e(y, \sigma) + v_1(y, \sigma) = 0. \quad (17)$$

We will show that it is always possible to choose $v_2(y, \sigma)$ in such a way that

$$z^T\alpha^e(y)z + z^T[-2\alpha^e(y)[\sigma]_z] + \gamma^e(y, \sigma) - \epsilon_1^e(y, \sigma) \cdot [2\alpha^e(y)[\sigma]_z + \beta^e(y, \sigma)] + \epsilon_2^e(y, \sigma)v_2(y, \sigma) < 0$$

for all nonzero (x, σ), i.e., (16) is negative for all nonzero (x, σ). Indeed, since $\alpha^e(y)$ is negative semidefinite for all y, we have

$$\begin{aligned} &z^T\alpha^e(y)z + z^T[-2\alpha^e(y)[\sigma]_z] + \gamma^e(y, \sigma) \\ &\quad - \epsilon_1^e(y, \sigma)[2\alpha^e(y)[\sigma]_z + \beta^e(y, \sigma)] \\ &\quad + \epsilon_2^e(y, \sigma)v_2(y, \sigma) \\ &= [\sigma]_z^T\alpha^e(y)[\sigma]_z - [\sigma]_z^T\alpha^e(y)[\sigma]_z \\ &\quad + z^T\alpha^e(y)z + z^T[-2\alpha^e(y)[\sigma]_z] \\ &\quad + \gamma^e(y, \sigma) - \epsilon_1^e(y, \sigma)[2\alpha^e(y)[\sigma]_z \\ &\quad + \beta^e(y, \sigma)] + \epsilon_2^e(y, \sigma)v_2(y, \sigma) \\ &= ([\sigma]_z - z)^T\alpha^e(y)([\sigma]_z - z) - [\sigma]_z\alpha^e(y)[\sigma]_z \\ &\quad - \epsilon_1^e(y, \sigma)[2\alpha^e(y)[\sigma]_z + \beta^e(y, \sigma)] \\ &\quad + \gamma^e(y, \sigma) + \epsilon_2^e(y, \sigma)v_2(y, \sigma) \end{aligned}$$

with $\epsilon_1^e(y, \sigma) = -[\sigma]_z^T + \zeta_m^T(y, [\sigma]_y)P_m^{-1}$ and $\epsilon_2^e(y, \sigma) = -y^T + [\sigma]_y^T$. By straightforward computations, since $\zeta_m(y, [\sigma]_y)$ and $\frac{\partial\xi_m}{\partial y}(y, [\sigma]_y)$ are smooth (vector) functions of their arguments and $\zeta_m(y, [\sigma]_y) = 0$ and $\frac{\partial\xi_m}{\partial y}(y, [\sigma]_y) = 0$ when $y = [\sigma]_y$, we obtain

$$\begin{aligned} &-[\sigma]_z\alpha^e(y)[\sigma]_z - (-[\sigma]_z^T + \zeta_m^T(y, [\sigma]_y)P_m^{-1}) \\ &\quad \cdot [2\alpha(y)[\sigma]_z + \beta^e(y, \sigma)] + \gamma^e(y, \sigma) \\ &= \lambda M(y, [\sigma]_z) + (y - [\sigma]_y)^T Q(y, \sigma) \end{aligned} \quad (18)$$

with

$$M(x) = \frac{\partial V}{\partial x}(x)A(y)x + B(y)k(x))$$

and for some continuous (vector) function $Q(y, \sigma)$, vanishing at the origin.

From (18), if we choose

$$v_2(y, \sigma) = Q(y, \sigma) + (y - [\sigma]_y) \quad (19)$$

and since $\epsilon_2^e(y, \sigma) = -y^T + [\sigma]_y^T$, we obtain

$$\begin{aligned} &z^T\alpha^e(y)z + z^T[\beta^e(y, \sigma) + v_1(y, \sigma)] + \gamma^e(y, \sigma) \\ &\quad + \epsilon_1^e(y, \sigma)v_1(y, \sigma) + \epsilon_2^e(y, \sigma)v_2(y, \sigma) \\ &= \lambda M(y, [\sigma]_z) + (z - [\sigma]_z)^T\alpha^e(y)(z - [\sigma]_z) \\ &\quad - (y - [\sigma]_y)^T(y - [\sigma]_y). \end{aligned} \quad (20)$$

Since $\alpha^e(y)$ is semidefinite negative for all y and $\alpha^e(0)$ is definite negative, it follows that $(z-[\sigma]_z)^T\alpha^e(0)(z-[\sigma]_z) = 0$ if and only if $z = [\sigma]_z$. Since $\lambda M(y, [\sigma]_z) < 0$ for all nonzero $(y, [\sigma]_z)$, it follows that (20) is nonpositive for all (x, σ). Moreover, using the above facts, we conclude that (20) is zero only when $y = [\sigma]_y = 0$ and $z = [\sigma]_z = 0$, i.e., $x = \sigma = 0$. This, by standard arguments on Lyapunov functions [21], completes the proof of the theorem. □

At this point, we are ready to give a systematic procedure for designing output feedback stabilizing controllers, once the conditions of Theorem 3 are satisfied.

V. Recipe 2

• Let $V(x) = \frac{1}{2}z^T Pz + z^T\zeta(y) + \xi(y)$ be a control Lyapunov function for (1), satisfying the small control property, and let $\overline{V}(x) = \frac{1}{2}z^T\overline{P}z + z^T\overline{\zeta}(y) + \overline{\xi}(y)$ be an output control Lyapunov function for its associated system. Let

$$\begin{aligned} P_m &= \overline{P} - \lambda P \\ \zeta_m(y, [\sigma]_y) &= -\lambda\zeta(y) + \overline{\zeta}(y) + \lambda\zeta([\sigma]_y) - \overline{\zeta}([\sigma]_y) \\ \xi_m(y, [\sigma]_y) &= \frac{1}{2}\|y - [\sigma]_y\|^2 + \frac{1}{2}\zeta_m^T(y, [\sigma]_y)P_m^{-1}\zeta_m(y, [\sigma]_y) \end{aligned}$$

with $\lambda \in \mathbb{R}^+$ such that P_m is positive definite.

• Let $u = k(y, [\sigma]_z)$, with

$$k(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ -\frac{a(x)+\sqrt{a^2(x)+\|b(x)\|^4}}{\|b(x)\|^2}b^T(x) & \text{else} \end{cases}$$

with $a(x) = \frac{\partial V}{\partial x}(x)A(y)x$ and $b(x) = \frac{\partial V}{\partial x}(x)B(y)$. (Actually, any continuous stabilizer $k(x)$ is good for our procedure.) Moreover

$$\begin{aligned} \alpha^e(y) &= \frac{\partial \overline{\zeta}}{\partial y}(y)[A(y)]_y^z + \overline{P}[A(y)]_z^z \\ \beta^e(y, \sigma) &= \left[\frac{\partial \zeta_m}{\partial y}(y, [\sigma]_y) + \lambda\frac{\partial \zeta}{\partial y}(y)\right] \\ &\quad \cdot ([A(y)]_y^y y + [B(y)]_y u) \\ &\quad + ([A(y)]_y^z)^T\left[\frac{\partial \xi_m}{\partial y}^T(y, [\sigma]_y) + \lambda\frac{\partial \xi}{\partial y}^T(y)\right. \\ &\quad \left. - \frac{\partial \zeta_m}{\partial y}^T(y, [\sigma]_y)[\sigma]_z\right] \\ &\quad + \overline{P}([A(y)]_z^y y + [B(y)]_z u) \\ &\quad + ([A(y)]_z^z)^T[\zeta_m(y, [\sigma]_y) \\ &\quad + \lambda\zeta(y) - P_m[\sigma]_z]. \end{aligned}$$

• Let $Q(y, \sigma)$ be any continuous (vector) function, vanishing at the origin, such that

$$\begin{aligned} &(y - [\sigma]_y)^T Q(y, \sigma) \\ &= -\zeta_m^T(y, [\sigma]_z)P_m^{-1}[2\alpha^e(y)[\sigma]_z + \beta^e(y, \sigma)] \\ &\quad + \zeta_m^T(y, [\sigma]_y)\left([A(y)]_z\begin{pmatrix} y \\ [\sigma]_z \end{pmatrix}\right. \\ &\quad \left. + [B(y)]_z k(y, [\sigma]_z)\right) \\ &\quad + \frac{\partial \xi_m}{\partial y}(y, [\sigma]_y)\left([A(y)]_y\begin{pmatrix} y \\ [\sigma]_z \end{pmatrix}\right. \\ &\quad \left. + [B(y)]_z k(y, [\sigma]_z)\right). \end{aligned}$$

The dynamic output feedback law

$$\begin{aligned} u &= k(y, [\sigma]_z), \\ \dot{\sigma} &= \begin{pmatrix} 0_{p\times(n-p)} & I_{p\xi p} \\ -P_m^{-1} & P_m^{-1}\frac{\partial \zeta_m}{\partial [\sigma]_y}(y, [\sigma]_y) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} -2\alpha^e(y) - \beta^e(y, \sigma) \\ Q(y, \sigma) + y - [\sigma]_y \end{pmatrix} \end{aligned}$$

solves the global stabilization problem for (1).

Proof: Proof of Theorem 4 (Necessity): If there exists an output feedback law (2) which globally asymptotically stabilizes (1) in $(x, \sigma) = (0, 0)$, necessarily there also exists a dynamic state-feedback law which globally asymptotically stabilizes (1) in $(x, \sigma) = (0, 0)$. Also, the output feedback law

$$\begin{aligned} u &= B(y)\eta(\sigma, y) \\ \dot{\sigma} &= \varrho(\sigma, y) \end{aligned}$$

globally asymptotically stabilizes the associated system (1) in $(x, \sigma) = (0, 0)$. From here it follows the existence of a dynamic control Lyapunov function $V(x, \sigma_1)$ for (1), satisfying the small control property, and a dynamic output control Lyapunov function $\overline{V}(x, \sigma_2)$ for the associated system of (1).

Sufficiency: Assume that $s_1 \le s_2$ (the other case goes in the same way). Let $\begin{pmatrix} \sigma_{21} \\ \sigma_{22} \end{pmatrix}$ be a partition of σ_2 such that σ_{21} has s_1 components. On the other hand, let us rename the state vector σ_1 as σ_{21}. Moreover, define $V^e(x, \sigma_{21}, \sigma_{22}) = \frac{1}{2}z^T Pz + z^T\zeta(y, \sigma_{21}) + \xi(y, \sigma_{21}) + \frac{1}{2}\|\sigma_{22}\|^2$ and the auxiliary system

$$\begin{aligned} \dot{x} &= A(y)x + B(y)u \\ \dot{\sigma}_{21} &= v_1. \end{aligned} \tag{21}$$

Since $V(x, \sigma_{21})$ satisfies the small control property for (21), from [10] we know how to construct a continuous state-feedback law $u = \eta(x, \sigma_{21})$, $v_1 = \varrho(x, \sigma_{21})$ which globally asymptotically stabilizes (21) in $(x, \sigma_{21}) = (0, 0)$. Clearly, $V^e(x, \sigma_{21}, \sigma_{22})$ is a dynamic control Lyapunov function, satisfying the small control property for the system

$$\begin{aligned} \dot{x} &= A(y)x + B(y)u \\ \dot{\sigma}_{21} &= v_1 \\ \dot{\sigma}_{22} &= v_2 \end{aligned} \tag{22}$$

and the feedback law $u = \eta(x, \sigma_{21})$, $v_1 = \varrho(x, \sigma_{21})$, $v_2 = -\sigma_{22}$ is continuous and globally asymptotically stabilizes (22) in $(x, \sigma_{21}, \sigma_{22}) = (0, 0, 0)$. From here we can proceed as in Theorem 1 with $V(x)$ and $\overline{V}(x)$ replaced by $V^e(x, \sigma_{21}, \sigma_{22})$ and $\overline{V}(x, \sigma_{21}, \sigma_{22})$, respectively. □

Proof: Proof of Theorem 5: Let $V^e(x, \sigma)$, $v_1(y, \sigma)$, and $v_2(y, \sigma)$ be defined as in the proof of theorem. We obtain again

$$\begin{aligned} \dot{V}^e(x, \sigma, v) &= \lambda M(y, [\sigma]_z) - (y - [\sigma]_y)^T(y - [\sigma]_y) \\ &\quad + (z - [\sigma]_z)^T\alpha^e(y)(z - [\sigma]_z) \end{aligned} \tag{23}$$

where $\alpha^e(y)$ is negative semidefinite for all y. Since $\lambda M(y, [\sigma]_z) \le 0$ for all nonzero $(y, [\sigma]_z)$ and the equality holds only if $y = 0$, it follows that (23) is nonpositive for all (x, σ) and is equal to zero only if $y = [\sigma]_y = 0$, i.e., output regulation is achieved and the trajectories of the closed-loop system (1) and (2) are bounded (by LaSalle's theorem: see [29]). □

Proof: Proof of Theorem 6: Let us consider the "extended" system

$$\begin{aligned} \dot{x} &= A_1(y)x + B_{11}(y)w + B_{12}(y)u(y, \sigma) \\ \dot{\sigma} &= v(y, \sigma), \ \sigma \in \mathbb{R}^n \\ \mathcal{I} &= A_2(y)x + B_{22}(y)u \\ y &= x_1. \end{aligned} \tag{24}$$

To prove the theorem we will construct an output feedback law (2) solving the disturbance attenuation problem with global internal stability for (24).

Let $u = k(y, [\sigma]_z)$ and define

$$V^e(x, \sigma) = V_m(x, \sigma) + V(x)$$

with

$$\begin{aligned} V_m(x, \sigma) &= \frac{1}{2}(z - [\sigma]_z)^T P_m (z - [\sigma]_z) \\ &\quad + (z - [\sigma]_z)^T \zeta_m(y, [\sigma]_y) + \xi_m(y, [\sigma]_y) \\ P_m &= \gamma^2 \overline{P} - P \\ \zeta_m(y, [\sigma]_y) &= -\zeta(y) + \gamma^2 \overline{\zeta}(y) + \zeta([\sigma]_y) - \gamma^2 \overline{\zeta}([\sigma]_y) \\ \xi_m(y, [\sigma]_y) &= \frac{1}{2}\|y - [\sigma]_y\|^2 \\ &\quad + \frac{1}{2}\zeta_m^T(y, [\sigma]_y) P_m^{-1} \zeta_m(y, [\sigma]_y). \end{aligned}$$

Note that by our assumptions, the matrix $\gamma^2 \overline{P} - P$ is positive definite. Moreover, following the lines of the proof of Theorem 3, $V^e(x, \sigma)$ can be easily shown to be C^1, proper, and positive definite.

Along the trajectories of (24), we have

$$\begin{aligned} &\frac{\partial V^e}{\partial x}(x)(A_1(y)x + B_{12}(y)u) \\ &\quad + \frac{1}{4\gamma^2}\frac{\partial V^e}{\partial x}(x) B_{11}(y) B_{11}^T(y) \frac{\partial V^e}{\partial x}(x) \\ &\quad + \frac{\partial V^e}{\partial \sigma}(x, \sigma) v(y, \sigma) + \|A_2(y)x + B_{22}(y)u\|^2 \\ &= z^T \alpha^e(y) z + z^T [\beta^e(y, \sigma) + \delta^e(y, \sigma) v(y, \sigma)] \\ &\quad \cdot \gamma^e(y, \sigma) + \epsilon^e(y, \sigma) v(y, \sigma) \end{aligned} \tag{25}$$

for some suitable $\alpha^e(y)$, $\beta^e(y, \sigma)$, $\gamma^e(y, \sigma)$, $\delta^e(y, \sigma)$ and $\epsilon^e(y, \sigma)$. By our assumptions and by construction, $\alpha^e(y)$ is negative semidefinite for all y and $\alpha^e(0)$ is negative definite.

As in the proof of Theorem 3, one can prove that for some continuous $v(y, \sigma)$, (25) is negative for all nonzero (x, σ).

Since (25) is less than or equal to zero (with u and v chosen as above) and greater than or equal to

$$\dot{V}^e(x, \sigma, w) + \|\mathcal{I}\|^2 - \gamma^2\|w\|^2 \tag{26}$$

from the simplifying assumptions it follows that:

$$\dot{V}^e(x, \sigma, w) + \|A_2(y)x\|^2 + \|u\|^2 - \gamma^2\|w\|^2 \le 0. \tag{27}$$

By standard arguments on Lyapunov functions, (27) implies that for $w(t) = 0$ the closed-loop system is stable. Global asymptotic stability follows from the detectability assumption on (3) and the fact that $\alpha^e(y) \le 0$ with $\alpha^e(0) < 0$. □

Note that the roles of λ and $\frac{1}{\gamma^2}$ in the stabilization problem and in the disturbance attenuation problem, respectively, are similar, except for the fact that λ is a design parameter while γ is *a priori* given. For robust stabilization, γ becomes a design parameter and it is strictly connected to the gain of the uncertainties.

Before ending the section, we want to remark that all the results contained in this paper can be easily extended to the more general class of systems

$$\begin{aligned} \dot{x}_1 &= A_{11}(y, u)x + B_{111}(y, u)w + B_{112}(y, u) \\ \dot{z} &= A_{12}(y, u)x + B_{121}(y, u)w + B_{122}(y, u) \\ \mathcal{I} &= A_2(y, u)x + B_{21}(y, u)w + B_{22}(y, u) \\ y &= x_1. \end{aligned}$$

VI. Some Examples

Example 1: Let us consider the system

$$\begin{aligned} \dot{y}_1 &= z - y_1 + y_1^2 y_2 + cu_2 \\ \dot{y}_2 &= z + y_2 - 2y_1 y_2^2 - cu_2 \\ \dot{z} &= -z(\|y\|^2 - 1)^2 + u_1. \end{aligned} \tag{28}$$

It is easy to show that $V(x) = (1/2)\|x\|^2$ is an output control Lyapunov function for (28) and satisfies the small control property. In this case, $\alpha(y) = -(\|y\|^2 - 1)^2$ which is equal to zero at all points y lying on the circle with radius one and $\|\delta(y)\|^2 \neq 0$ for all y.

The assumptions of [7] are violated, and it is quite hard to construct a stabilizing output feedback controller following the lines suggested in [6]. We will see how to construct a semiglobally stabilizing output feedback controller, following the lines of the proof of Theorem 1. We have

$$S_0 = \{y \in \mathbb{R}^2 : \|y\|^2 = 1\},\ S_1 = \mathbb{R}^2,\ T_0 = \mathbb{R}^2.$$

Choose

$$\begin{aligned} \lambda_1(y) &= -(y_1 + y_2),\ \mu_2(y) \\ &= -\frac{\gamma(y) + \sqrt{\gamma^2(y) + c^4(y_1 - y_2)^4}}{c(y_1 - y_2)} \end{aligned}$$

with

$$\gamma(y) = (y_1 y_2)\begin{pmatrix} -1 & y_1^2 \\ -2y_2^2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and, correspondingly, $k_0(y) = \begin{pmatrix} \lambda_1(y) \\ \mu_2(y) \end{pmatrix}$. Moreover, choose

$$k_1(y) = \begin{cases} \begin{pmatrix} -\frac{\Psi_1(y)}{2\Phi_1(y)} \\ 0 \end{pmatrix}, & \text{if } y_1 = y_2 \\ \begin{pmatrix} -\frac{\Psi_1(y)}{2\Phi_1(y)} \\ -\frac{\nu(y) + \sqrt{\nu^2(y) + \Psi_2^4(y)}}{\Psi_2(y)} \end{pmatrix}, & \text{else} \end{cases}$$

where

$$\begin{aligned} \Phi_1(y) &= \frac{1}{4(\|y\|^2 - 1)^2} \\ \Psi_1(y) &= \frac{1}{2}(y_1 + y_2),\ \Psi_2(y) = c(y_1 - y_2) \\ H(y) &= \gamma(y) + \frac{1}{4}(y_1 + y_2)^2 \\ \nu(y) &= -\frac{\Psi^2(y)}{4\Phi_1(y)} + H(y). \end{aligned}$$

A stabilizing output feedback law is

$$\begin{aligned} k(y) &= (1 - \varrho((\|y\|^2 - 1)^2, \sigma) k_0(y) \\ &\quad + \varrho((\|y\|^2 - 1)^2, \sigma) k_1(y) \end{aligned}$$

with $\varrho(\cdot, \cdot)$ and σ defined in such a way that, if $u = k_1(y)$, along the trajectories of (28) one has $V(x, u) < 0$ for all z and y such that $(\|y\|^2 - 1)^2 < \sigma$ and $(y, z) \in \Omega$. Since $\min_{\{y\, \|y\|^2=1\}}\{\gamma^2(y) + c^4(y_1 - y_2)^4\} > 0$, we can simply choose σ in such a way that

$$z^2 < \frac{\sqrt{\gamma^2(y) + c^4(y_1 - y_2)^4}}{\sigma}$$

for all $(y, z) \in \Omega$ such that $(\|y\|^2 - 1)^2 < \sigma$.

Example 2: Let us consider the system

$$\begin{aligned}\dot{y} &= z + u\\ \dot{z} &= y^2 - z^3.\end{aligned} \tag{29}$$

Note that (29) is not of the form of (1). It can be globally asymptotically stabilized through a state-feedback law $u = -y - z - yz$ with a Lyapunov function $V(x) = \frac{1}{2}(y^2 + z^2)$. Moreover, it can be globally asymptotically stabilized via output injection through a feedback law $u_1 = -y$ and $u_2 = -y^2 - y$ and with a Lyapunov function $\overline{V}(x) = \frac{1}{2}(y^2 + z^2)$. We claim that it can be globally asymptotically stabilizable also via dynamic output feedback. Consider the extended system

$$\begin{aligned}\dot{y} &= z + u\\ \dot{z} &= y^2 - z^3\\ \dot{\sigma}_1 &= v_1\\ \dot{\sigma}_2 &= v_2\end{aligned} \tag{30}$$

with $\sigma_1, \sigma_2 \in \mathbb{R}$, $\sigma = \binom{\sigma_1}{\sigma_2}$, $x = \binom{y}{z}$, and look for an output-feedback $u = \eta(y, \sigma)$, $\dot{\sigma} = \varrho(y, \sigma)$ which globally asymptotically stabilizes (30). According to the procedure of Theorem 3, let us consider the positive definite function $V^e(x, \sigma) = \frac{1}{2}(1 - \lambda)\|x - \sigma\|^2 + \frac{\lambda}{2}\|x\|^2$, with $0 < \lambda < 1$, and let $u = k(y, \sigma_2) = -y - \sigma_2 - y\sigma_2$. We obtain

$$\begin{aligned}&\dot{V}(y, z, \sigma, v_1, v_2)\\ &\quad= (1 - \lambda)[(y - \sigma_1)(z + k(y, \sigma_2))\\ &\qquad+ (z - \sigma_2)(y^2 - z^3)]\\ &\qquad+ \lambda[y(z + k(y, \sigma_2)) + z(y^2 - z^3)]\\ &\qquad- (1 - \lambda)[(y - \sigma_1)v_1 + (z - \sigma_2)v_2].\end{aligned}$$

Consider the function

$$\begin{aligned}r(x, \sigma, v_2) &= (1 - \lambda)[(y - \sigma_1)z + (z - \sigma_2)(y^2 - z^3)]\\ &\quad- \lambda y[\sigma_2(1 + y) - z(1 + y)] - (1 - \lambda)(z - \sigma_2)v_2.\end{aligned}$$

Choose $v_2 = y - \sigma_1 - \sigma_2^3 + y^2 + \frac{\lambda y}{1-\lambda}(1 + y)$ so that

$$\begin{aligned}r(x, \sigma, v_2) &= -(1 - \lambda)[(z - \sigma_2)^2(z^2 + z\sigma_2 + \sigma_2^2)\\ &\quad- (y - \sigma_1)\sigma_2].\end{aligned}$$

As a consequence, since

$$\begin{aligned}&\lambda[y(z + k(y, \sigma_2)) + z(y^2 - z^3)]\\ &\quad= \lambda\{[y(z + k(y, z)) + z(y^2 - z^3)]\}\\ &\qquad- \lambda y[\sigma_2(1 + y) + yz(1 + y)]\end{aligned}$$

we have $\dot{V}(x, \sigma, v_1, v_2) = \lambda[y(z + k(y, z)) + z(y^2 - z^3)] - (1 - \lambda)[(z - \sigma_2)^2(z^2 + z\sigma_2 + \sigma_2^2) - (y - \sigma_1)\sigma_2 + (y - \sigma_1)(v_1 - k(y, \sigma_2))]$. By choosing $v_1 = k(y, \sigma_2) + \sigma_2 + (y - \sigma_1)$, we obtain $\dot{V}(x, \sigma, v_1, v_2) = \lambda[y(z + k(y, z)) + z(y^2 - z^3)] - (1 - \lambda)[(z - \sigma_2)^2(z^2 + z\sigma_2 + \sigma_2^2) + (y - \sigma_1)^2]$ which is strictly equal to zero for $(x, \sigma) \neq (0, 0)$. This proves our claim.

Note that the resulting stabilizing output dynamic feedback is

$$\begin{aligned}\dot{\sigma}_1 &= \sigma_2 + k(y, \sigma_2) + y - \sigma_1\\ \dot{\sigma}_2 &= -\sigma_2^3 + y^2 + y(1 + y)\frac{\lambda}{1 - \lambda} + y - \sigma_1.\end{aligned}$$

The structure of the above controller differ from that of a classical observer in the term $y(1+y)\frac{\lambda}{1-\lambda}$ (in the $\dot{\sigma}_2$ equation).

Example 3: Assume that (30) is perturbed by some unknown disturbance w and that the penalty variable $\mathcal{I}$ is the output y, i.e.,

$$\begin{aligned}\dot{y} &= z + u + w\\ \dot{z} &= y^2 - z^3\\ \mathcal{I} &= y.\end{aligned}$$

The disturbance attenuation problem can be easily solved in this case following the procedure of Example 2.

Example 4: The simplified dynamic model of an elastic joint robot can be rewritten in state space form as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -B_1^{-1}(x_1)[C(x_1, x_2)x_2 + K(x_1 - x_3) + h(x_1)]\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= B_2^{-1}[K(x_1 - x_3) + u]\end{aligned}$$

where x_1 is the $(n \times 1)$ vector of the link relative displacements [30]. It is well known that under a quite natural assumption on the so-called joint stiffness matrix K, there exists a smooth, proper, and positive definite function $V(x)$ and a smooth feedback law $u = k(x)$ which solve the global stabilization problem. However, the function $V(x)$ is of the form $V(x) = \frac{1}{2}z^T P(y)z + z^T\zeta(y) + \xi(y)$ which does not satisfy the assumptions of Theorem 3. Moreover, if only the link position is available for feedback, i.e., $y = x_1$, the terms $C(x_1, x_2)x_2$ are not linear in the unmeasured variables. However, by using the ideas contained in the proof of Theorem 3, it can be shown that a globally stabilizing linear output feedback controller can be readily implemented [31].

VII. Conclusions

In this paper, we have discussed the solution to several global control problems for a given class of nonlinear systems by means of dynamic output feedback. These systems are characterized by having nonlinear terms depending only on the output y. In a first part, we have shown that, given a compact set $\Omega \subset \mathbb{R}^n$, if there exists an output control Lyapunov function which satisfies the small control property, it is possible to construct an output feedback law $u = k(y)$ such that the resulting closed-loop system is locally asymptotically stable in $x = 0$ with basin of attraction containing Ω. In some cases, the result becomes global.

In a second part, we have allowed dynamic output feedback and shown that for the same class of systems the global output regulation problem (in particular, the global stabilization problem) and the disturbance attenuation problem can be split up into two subproblems: one is the problem via full information and the other is the problem via output injection.

Our result shows the important role of the problem of stabilization via output injection in solving the problem of dynamic output feedback stabilization at least for the class of systems considered. This and the extension of the procedures used in the proofs of our results will be the object of further study.

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