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Global set point control via link position measurement for flexible joint robots☆

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Abstract

In this paper it is shown that a linear controller solves the global set point control problem for a flexible joint robot via link position measurements. Moreover, the controller is shown to be robust with respect to uncertainties in the gravity term.

Keywords: Output feedback; Global set point control; Elastic joint robots

1. Introduction

In the literature, there exist essentially two models for the flexible joint robot, depending on how the inertia of the actuators is modeled (see [7] for a survey). In what follows, we will refer to these two models as *simplified* and *full* models. The global set point control problem via full *state*-feedback is solvable in both cases since exact feedback linearization can be achieved by means of *static* state-feedback, for the simplified model, or by *dynamic* state-feedback [1], for the full model.

Some recent results solve the global set point control problem by using *partial state*-feedback. In [3, 10] a *static* controller is proposed for the full model, based only on the motor position and velocity measurements plus a suitable feedforward term. In [4, 5, 9] observer-based *dynamic* controllers are introduced, achieving set point control for any initial condition in an arbitrary compact set and under the additional hypothesis that the link and motor velocities are bounded. Therein, when the full model is considered, either the link and actuator positions or the link positions and the actuator velocities are taken as measured variables; on the other hand, for the simplified model, the link positions are the only measured variables. A different approach is followed in [6], where a global solution is obtained under a dynamic output feedback controller, using the measurements of link position and velocity.

In this paper, we assume only the link position as available for measurement and consider the simplified model. We show that a *linear* output feedback controller solves the *global* set point control problem (Theorem 1). This controller has dimension equal to twice the number of links and is not observer based. Although the model considered is fully linearizable via state-feedback, the recent techniques for output

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feedback stabilization [2, 8, 11] either do not apply or achieve stabilization on arbitrary compact sets through a high-gain observer. Our approach gives a very simple controller, which not only achieves global set point control, but also avoids the use of high-gain observers. See also [12] for the motor position measurement case. Moreover, we prove that the proposed controller is robust with respect to uncertainties in the gravitational term (Lemma 1).

2. Flexible joint robot model and properties

The simplified dynamic model of an elastic joint robot is given by [7]

$$B_1(q_l)\ddot{q}_l + C(q_l, \dot{q}_l)\dot{q}_l + K(q_l - q_m) + h(q_l) = 0, \tag{1a}$$

$$B_2\ddot{q}_m - K(q_l - q_m) = u \tag{1b}$$

with q_l and q_m being, respectively, the $(n \times 1)$ vectors of the links and rotors relative displacements, $K = \mathrm{diag}\{k_i\}$ with k_i the elastic constant of the ith joint and $h(q_l)$ representing the gravity forces acting on the links. We assume that no damping is present. The positive-definite inertia matrix is

$$B(q_l) = \begin{pmatrix} B_1(q_l) & 0 \\ 0 & B_2 \end{pmatrix},$$

while $C(q_l, \dot{q}_l)\dot{q}_l$ represents the centrifugal and Coriolis forces. Note that while $C(q_l, \dot{q}_l)\dot{q}_l$ is unique, $C(q_l, \dot{q}_l)$ is not.

Property 1. There exists a suitable choice of the matrix $C(q_l, \dot{q}_l)$ such that the matrix, along the motion trajectories,

$$\dot{B}_1(q_l) - 2C(q_l, \dot{q}_l)$$

is skew symmetric.

We want to point out that, in general, the vector $C(q_l, \dot{q}_l)\dot{q}_l$ contains terms quadratic in $\dot{q}_l$.

Since the gravity term $h(q_l)$ is formed by trigonometric functions of the link variables q_l, we can also state the following property.

Property 2. There exists a positive constant α such that

$$\left\| \frac{\partial h(q_l)}{\partial q_l} \right\| \leqslant \alpha \quad \forall q_l \in \mathbb{R}^n.$$

By the mean value theorem, Property 2 implies that

$$\| h(q_l) - h(q_l^*) \| \leqslant \alpha \| q_l - q_l^* \| \quad \forall q_l, q_l^* \in \mathbb{R}^n.$$

Choosing as state variables $x^{\mathrm{T}} = (q_l^{\mathrm{T}}\ \dot{q}_l^{\mathrm{T}}\ q_m^{\mathrm{T}}\ \dot{q}_m^{\mathrm{T}})$, the second order model (1) can be rewritten in state space form as

$$\dot{x}_1 = x_2, \tag{2a}$$

$$\dot{x}_2 = -B_1^{-1}(x_1)[C(x_1, x_2)x_2 + K(x_1 - x_3) + h(x_1)], \tag{2b}$$

$$\dot{x}_3 = x_4, \tag{2c}$$

$$\dot{x}_4 = B_2^{-1}[K(x_1 - x_3) + u]. \tag{2d}$$

3. Controller design

Suppose the output of interest is $y = q_l = x_1$. Our control objective is to ensure, through the control input u, that y tends to a desired position x_{1d} asymptotically, while maintaining internal stability. Accordingly, let us consider the point

$$x_d = \begin{pmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \\ x_{4d} \end{pmatrix} = \begin{pmatrix} y_d \\ 0 \\ y_d + K^{-1}h(y_d) \\ 0 \end{pmatrix}.$$

We can state the two following control problems.

- *State feedback global set point control problem.* Find a controller

$$u = k(x, x_d) \tag{3}$$

such that the closed loop system (2)–(3) is globally asymptotically stable in $x = x_d$.

- *Output feedback global set point control problem.* Find a dynamic controller

$$\begin{aligned} \dot{\hat{x}} &= k(y, \hat{x}, x_d), \quad \hat{x} \in \mathbb{R}^q, \\ u &= \eta(y, \hat{x}, x_d) \end{aligned} \tag{4}$$

such that the closed loop system (2)–(4) is globally asymptotically stable in $(x, \hat{x}) = (x_d, 0)$.

Before stating the main result, we briefly recall some results about global state feedback stabilization of flexible joint robots. It has been shown [3, 10] that the feedback law

$$u_{\text{STAB}} = -K_p(x_3 - x_{3d}) - K_d x_4 + h(x_{1d}) \tag{5}$$

with K_p and K_d positive-definite symmetric matrices, solves the state feedback global set point control problem.

In order to prove the global stability of the closed loop system (2)–(5), defining $\Delta x = x - x_d$, the following Lyapunov function was used:

$$V_{\text{STAB}}(\Delta x) = \frac{1}{2}(x_2^{\text{T}} \; x_4^{\text{T}}) \begin{pmatrix} B_1(x_1) & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + U(\Delta x_1, \Delta x_3)$$

with

$$U(\Delta x_1, \Delta x_3) = \frac{1}{2}(\Delta x_1^{\text{T}} \; \Delta x_3^{\text{T}}) \begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_3 \end{pmatrix} + F(x_1) - F(x_{1d}) - \Delta x_1^{\text{T}} h(x_{1d})$$

and $F(x_1)$ denoting the potential energy associated with the conservative force $h(x_1)$, i.e., $F(x_1)$ is such that

$$\frac{\partial F(x_1)}{\partial x_1} = h^{\text{T}}(x_1).$$

The following assumption guarantees that $U(\Delta x_1, \Delta x_3)$, and thus $V_{\text{STAB}}(\Delta x)$, is definite positive.

Assumption 1 (*Lanari and Wen* [3] *and Tomei* [10]). The spring at the joint has to be sufficiently stiff so that, for sufficiently large K_p, $U(\Delta x_1, \Delta x_3)$ is positive definite in $(\Delta x_1, \Delta x_3)$.

We briefly explain the meaning of "sufficiently stiff" and "sufficiently large". Note that the gradient of $U(\Delta x_1, \Delta x_3)$ w.r.t. $(\Delta x_1, \Delta x_3)$

$$\nabla_{(\Delta x_1, \Delta x_3)} U(\Delta x_1, \Delta x_3) = \begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_3 \end{pmatrix} + \begin{pmatrix} h(x_1) - h(x_{1d}) \\ 0 \end{pmatrix}$$

is zero only in $(\Delta x_1, \Delta x_3) = (0,0)$. On the other hand, its Hessian is given by

$$H(\Delta x_1, \Delta x_3) = \nabla^2_{(\Delta x_1, \Delta x_3)} U(\Delta x_1, \Delta x_3) = \begin{pmatrix} K + \partial h(\Delta x_1 + x_{1d})/\partial \Delta x_1 & -K \\ -K & K + K_p \end{pmatrix}.$$

If $H(\Delta x_1, \Delta x_3)$ is positive definite in $(\Delta x_1, \Delta x_3) = (0,0)$ for any x_{1d}, then $U(\Delta x_1, \Delta x_3)$ is positive definite. Note that

$$\frac{\partial h(\Delta x_1 + x_{1d})}{\partial \Delta x_1} = \frac{\partial h(x_1)}{\partial x_1}.$$

Moreover since $\partial h(x_1)/\partial x_1$ is also the Hessian of the gravitational potential energy $F(x_1)$, $H(\Delta x_1, \Delta x_3)$ is a symmetric matrix. Choosing

$$T = \begin{pmatrix} I & (K + [\partial h(x_1)/\partial x_1]|_{x_{1d}})^{-1} K \\ 0 & I \end{pmatrix},$$

$H(0,0)$ is congruent to

$$T^{\mathrm{T}} H(0,0) T = \begin{pmatrix} K + [\partial h(x_1)/\partial x_1]|_{x_{1d}} & 0 \\ 0 & K_p + K - K(K + [\partial h(x_1)/\partial x_1]|_{x_{1d}})^{-1} K \end{pmatrix}.$$

Therefore, if

(A1) $$K + \left.\frac{\partial h(x_1)}{\partial x_1}\right|_{x_{1d}} > 0, \quad \forall x_{1d},$$

(A2) $$K_p + K - K\left(K + \left.\frac{\partial h(x_1)}{\partial x_1}\right|_{x_{1d}}\right)^{-1} K > 0, \quad \forall x_{1d}$$

$U(\Delta x_1, \Delta x_3)$ is positive definite. Note that assuming the linear spring at the joint sufficiently stiff, in the sense of (A1), has the following simple physical interpretation. To move the link arm to a desired position, the rotor winds up so is to counterbalance the gravity effect on the link side. Therefore the spring has to be sufficiently stiff so that no multiple rotor windings are necessary.

We can now state the main theorem.

Theorem 1. *Under Assumption* 1, *the dynamic output feedback controller*

$$\dot{\hat{x}}_3 = [(K - \lambda(K + K_p))^{-1} \lambda K_p + I] \hat{x}_4, \tag{6a}$$

$$\dot{\hat{x}}_4 = -\frac{\lambda}{1-\lambda} B_2^{-1} K_d \hat{x}_4 + B_2^{-1} [K(\Delta x_1 - \hat{x}_3) - K_p \hat{x}_3 - K_d \hat{x}_4], \tag{6b}$$

$$u = -K_p \hat{x}_3 - K_d \hat{x}_4 + h(x_{1d}) \tag{6c}$$

solves the output feedback global set point control problem.

Proof. Choose a sufficiently small strictly positive scalar $\lambda < 1$ such that the matrix

$$P = \begin{pmatrix} K - \lambda(K + K_p) & 0 \\ 0 & (1-\lambda) B_2 \end{pmatrix}$$

is positive definite (since K, K_p and B_2 are all positive definite, this is always possible). Choose a candidate output feedback controller of the form

$$\begin{aligned} \dot{\hat{x}}_3 &= v_3(\hat{x}, y, y_d), \\ \dot{\hat{x}}_4 &= v_4(\hat{x}, y, y_d) \end{aligned} \tag{7}$$

with

$$\hat{x} = \begin{pmatrix} \hat{x}_3 \\ \hat{x}_4 \end{pmatrix}$$

and v_3 and v_4 yet to be determined. Let us define

$$V^e(\Delta x, \hat{x}) = \lambda V_{\mathrm{STAB}}(\Delta x) + \frac{1}{2}((\Delta x_3 - \hat{x}_3)^{\mathrm{T}} \ (x_4 - \hat{x}_4)^{\mathrm{T}}) P \begin{pmatrix} \Delta x_3 - \hat{x}_3 \\ x_4 - \hat{x}_4 \end{pmatrix}.$$

Under Assumption 1, the first term $\lambda V_{\mathrm{STAB}}(\Delta x)$ is zero only for $\Delta x = 0$, while the second term is zero only for $\hat{x}_3 = \Delta x_3$ and $\hat{x}_4 = x_4$. $V^e(\Delta x, \hat{x})$ is thus positive definite.

Let us compute the time derivative of V^e along the closed loop system's (2)–(7) trajectories. One has

$$\dot{V}^e(\Delta x, \hat{x}) = \lambda x_4^{\mathrm{T}}[K_{\mathrm{p}} \Delta x_3 - h(x_{1\mathrm{d}}) - K_{\mathrm{p}}\hat{x}_3 - K_{\mathrm{d}}\hat{x}_4 + h(x_{1\mathrm{d}})]$$
$$+ (\Delta x_3 - \hat{x}_3 \ \ x_4 - \hat{x}_4)^{\mathrm{T}} P \begin{pmatrix} x_4 - v_3 \\ B_2^{-1}[K(x_1 - x_3) - K_{\mathrm{p}}\hat{x}_3 - K_{\mathrm{d}}\hat{x}_4 + h(x_{1\mathrm{d}})] - v_4 \end{pmatrix}$$

and, since $x_3 = \Delta x_3 + x_{3\mathrm{d}}$ with $x_{3\mathrm{d}} = x_{1\mathrm{d}} + K^{-1}h(x_{1\mathrm{d}})$,

$$\dot{V}^e(\Delta x, \hat{x}) = \lambda x_4^{\mathrm{T}}[K_{\mathrm{p}}(\Delta x_3 - \hat{x}_3) + K_{\mathrm{d}}(x_4 - \hat{x}_4) - K_{\mathrm{d}} x_4]$$
$$+ (\Delta x_3 - \hat{x}_3 \ \ x_4 - \hat{x}_4)^{\mathrm{T}} P \begin{pmatrix} x_4 - v_3 \\ B_2^{-1}[K(x_1 - \Delta x_3) - K_{\mathrm{p}}\hat{x}_3 - K_{\mathrm{d}}\hat{x}_4 - K x_{1\mathrm{d}}] - v_4 \end{pmatrix}.$$

Now choose

$$\begin{pmatrix} v_3(\hat{x}, y, y_{\mathrm{d}}) \\ v_4(\hat{x}, y, y_{\mathrm{d}}) \end{pmatrix}$$

as follows:

$$\begin{pmatrix} v_3(\hat{x}, y, y_{\mathrm{d}}) \\ v_4(\hat{x}, y, y_{\mathrm{d}}) \end{pmatrix} = P^{-1} \begin{pmatrix} \lambda K_{\mathrm{p}} \hat{x}_4 \\ -\lambda K_{\mathrm{d}} \hat{x}_4 \end{pmatrix} + \begin{pmatrix} \hat{x}_4 \\ B_2^{-1}[K(\Delta x_1 - \hat{x}_3) - K_{\mathrm{p}}\hat{x}_3 - K_{\mathrm{d}}\hat{x}_4] \end{pmatrix}.$$

With this choice of v_3 and v_4, we obtain

$$\dot{V}^e(\Delta x, \hat{x}) = \lambda x_4^{\mathrm{T}}[K_{\mathrm{p}}(\Delta x_3 - \hat{x}_3) + K_{\mathrm{d}}(x_4 - \hat{x}_4) - K_{\mathrm{d}} x_4]$$
$$+ (\Delta x_3 - \hat{x}_3 \ \ x_4 - \hat{x}_4)^{\mathrm{T}} P \begin{pmatrix} x_4 \\ B_2^{-1}[K(x_1 - \Delta x_3) - K_{\mathrm{p}}\hat{x}_3 - K_{\mathrm{d}}\hat{x}_4 - K x_{1\mathrm{d}}] \end{pmatrix}$$
$$- (\Delta x_3 - \hat{x}_3 \ \ x_4 - \hat{x}_4)^{\mathrm{T}} \begin{pmatrix} \lambda K_{\mathrm{p}} \hat{x}_4 \\ -\lambda K_{\mathrm{d}} \hat{x}_4 \end{pmatrix}$$
$$- (\Delta x_3 - \hat{x}_3 \ \ x_4 - \hat{x}_4)^{\mathrm{T}} P \begin{pmatrix} \hat{x}_4 \\ B_2^{-1}[K(\Delta x_1 - \hat{x}_3) - K_{\mathrm{p}}\hat{x}_3 - K_{\mathrm{d}}\hat{x}_4] \end{pmatrix}$$
$$= -\lambda \hat{x}_4^{\mathrm{T}} K_{\mathrm{d}} \hat{x}_4.$$

Applying La Salle's invariance principle, we can conclude that the closed loop system's (2)–(6) trajectories converge asymptotically to the largest invariant set of $E = \{(\Delta x, \hat{x})\colon \hat{x}_4 = 0\}$. Recall that the closed loop system (2)–(6) is

$$\dot{x}_1 = x_2, \tag{8a}$$

$$\dot{x}_2 = -B_1^{-1}(x_1)[C(x_1, x_2)x_2 + K(x_1 - x_3) + h(x_1)], \tag{8b}$$

$$\dot{x}_3 = x_4, \tag{8c}$$

$$\dot{x}_4 = B_2^{-1}[K(x_1 - x_3) - K_p\hat{x}_3 - K_d\hat{x}_4 + h(x_{1d})], \tag{8d}$$

$$\dot{\hat{x}}_3 = [(K - \lambda(K + K_p))^{-1}\lambda K_p + I]\hat{x}_4, \tag{8e}$$

$$\dot{\hat{x}}_4 = -\frac{\lambda}{1-\lambda}B_2^{-1}K_d\hat{x}_4 + B_2^{-1}[K(\Delta x_1 - \hat{x}_3) - K_p\hat{x}_3 - K_d\hat{x}_4]. \tag{8f}$$

We now show that $E = \{0,0\}$. If $\hat{x}_4 \equiv 0$, from (8e) $\hat{x}_3 \equiv$ const and from (8f) $x_1 \equiv$ const. Therefore, from Eqs. (8a), (8b) and (8c), respectively, we have $x_2 \equiv 0$, $x_3 \equiv$ const and $x_4 \equiv 0$. Moreover, the following set of equalities must hold

$$K(x_1 - x_3) + h(x_1) = 0, \tag{9a}$$

$$K(x_1 - x_3) - K_p\hat{x}_3 + h(x_{1d}) = 0, \tag{9b}$$

$$K(\Delta x_1 - \hat{x}_3) - K_p\hat{x}_3 = 0. \tag{9c}$$

With simple manipulations, since $K(x_{3d} - x_{1d}) = h(x_{1d})$, Eqs. (9) can be rewritten as

$$K(\Delta x_1 - \Delta x_3) + h(x_1) - h(x_{1d}) = 0, \tag{10a}$$

$$K(\Delta x_1 - \Delta x_3) - K_p\hat{x}_3 = 0, \tag{10b}$$

$$K(\Delta x_1 - \hat{x}_3) - K_p\hat{x}_3 = 0. \tag{10c}$$

Subtracting (10b) from (10c) we deduce

$$\Delta x_3 = \hat{x}_3. \tag{11}$$

With (11), Eqs. (10a) and (10c) can be rewritten as

$$\begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix}\begin{pmatrix} \Delta x_1 \\ \Delta x_3 \end{pmatrix} + \begin{pmatrix} h(x_1) - h(x_{1d}) \\ 0 \end{pmatrix} = 0. \tag{12}$$

Note that $(\Delta x_1, \Delta x_3) = (0,0)$ solves Eq. (12). Computing the gradient of (12), we obtain

$$\begin{pmatrix} K + \partial h(\Delta x_1 + x_{d1})/\partial \Delta x_1 & -K \\ -K & K + K_p \end{pmatrix},$$

which, by Assumption 1, is positive definite for any Δx_1 (or, which is the same, for any x_{1d}). Therefore, $(\Delta x_1, \Delta x_3) = (0,0)$ is the unique solution to (10). This also implies $\hat{x}_3 = 0$.

Moreover, $V^e(\Delta x, \hat{x})$ being proper, global asymptotic stability for the closed loop system (8) in $(\Delta x, \hat{x}) = (0,0)$ is achieved. □

Remark 1. Note that, for $\Delta x_3 = \hat{x}_3$ and $x_4 = \hat{x}_4$, the dynamics of the controller (6), on the one hand, and the dynamics of (2c) and (2d), on the other, do not coincide. This shows that the controller (6) is not based on an observer. Moreover, for $\lambda = 0$, we obtain a classical linear (reduced) observer. However, our stabilization result is not true anymore in this case, since $V^e(\Delta x, \hat{x})$ becomes positive semidefinite. This shows also the important role of the parameter λ in the controller design.

Remark 2. Note that the controller (6) achieves set point control also for the linear approximation of (2).

It is also interesting to investigate if global set point control can be achieved, under uncertainties on the gravitational parameters. Obviously, the equilibrium point x^*, if it exists, will be in general different from the desired one.

Suppose that the gravity vector $h(x_1)$ is not perfectly known, and let $\bar{h}(x_1)$ be an available estimate. Given x_{1d}, the corresponding available desired joint position will be

$$\bar{x}_{3d} = x_{1d} + K^{-1}\bar{h}(x_{1d}).$$

We can state the following lemma.

Lemma 1. *Under Assumption* 1, *under the dynamic output feedback controller*

$$\dot{\hat{x}}_3 = [(K - \lambda(K + K_p))^{-1}\lambda K_p + I]\hat{x}_4, \tag{13a}$$

$$\dot{\hat{x}}_4 = -\frac{\lambda}{1-\lambda}B_2^{-1}K_d\hat{x}_4 + B_2^{-1}[K(x_1 - \hat{x}_3) - K_p\hat{x}_3 - K_d\hat{x}_4 - Kx_{1d}], \tag{13b}$$

$$u = -K_p\hat{x}_3 - K_d\hat{x}_4 + \bar{h}(x_{1d}), \tag{13c}$$

the flexible joint robot (2) *has a unique equilibrium point* $(x^*, \hat{x}_3^*, 0)$ *which is globally asymptotically stable.*

Proof. Following the lines of [10], we will first show that the closed loop system (2)–(13) has a unique equilibrium point. The candidate equilibrium points are given by the solutions of

$$x_2 = 0, \tag{14a}$$

$$K(x_1 - x_3) + h(x_1) = 0, \tag{14b}$$

$$x_4 = 0, \tag{14c}$$

$$K(x_1 - x_3) - K_p\hat{x}_3 + \bar{h}(x_{1d}) = 0, \tag{14d}$$

$$\hat{x}_4 = 0, \tag{14e}$$

$$K(x_1 - \hat{x}_3) - K_p\hat{x}_3 - Kx_{1d} = 0. \tag{14f}$$

Since $K(\bar{x}_{3d} - x_{1d}) = \bar{h}(x_{1d})$ and $K(x_{3d} - x_{1d}) = h(x_{1d})$, Eqs. (14b), (14d) and (14f) can be rewritten as

$$K((x_1 - x_{1d}) - (x_3 - x_{3d})) = h(x_{1d}) - h(x_1), \tag{15a}$$

$$-K((x_1 - x_{1d}) - (x_3 - x_{3d})) + K_p\hat{x}_3 = \bar{h}(x_{1d}) - h(x_{1d}), \tag{15b}$$

$$K(x_1 - x_{1d}) - (K + K_p)\hat{x}_3 = 0. \tag{15c}$$

From (15b) and (15c) we derive

$$\hat{x}_3 = (x_3 - x_{3d}) + K^{-1}[h(x_{1d}) - \bar{h}(x_{1d})] \tag{16}$$

which, substituted in (15c), gives

$$\begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix}\begin{pmatrix} x_1 - x_{1d} \\ x_3 - x_{3d} \end{pmatrix} = \begin{pmatrix} h(x_{1d}) - h(x_1) \\ -(K + K_p)K^{-1}[h(x_{1d}) - \bar{h}(x_{1d})] \end{pmatrix}. \tag{17}$$

Now consider the function

$$A(x_1,x_3)=\frac{1}{2}((x_1-x_{1d})^{\mathrm{T}}\ (x_3-x_{3d})^{\mathrm{T}})\begin{pmatrix} K & -K \\ -K & K+K_p \end{pmatrix}\begin{pmatrix} x_1-x_{1d} \\ x_3-x_{3d} \end{pmatrix}$$
$$+((x_1-x_{1d})^{\mathrm{T}}\ (x_3-x_{3d})^{\mathrm{T}})\begin{pmatrix} -h(x_{1d}) \\ (K+K_p)K^{-1}[h(x_{1d})-\bar{h}(x_{1d})] \end{pmatrix}+F(x_1)$$
$$=A_1(x_1,x_3)+F(x_1).$$

$A_1(x_1,x_3)$ is clearly a convex function. Since $F(x_1)$ is bounded, $A(x_1,x_3)$ has a global minimum. Therefore the equation

$$\frac{\partial A(x_1,x_3)}{\partial\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}}=0 \tag{18}$$

has at least one solution which will be denoted with $(x_1,x_3)=(x_1^*,x_3^*)$. Since (18) coincides with (17), we can write

$$\begin{pmatrix} K & -K \\ -K & K+K_p \end{pmatrix}\begin{pmatrix} x_1^*-x_{1d} \\ x_3^*-x_{3d} \end{pmatrix}=\begin{pmatrix} h(x_{1d})-h(x_1^*) \\ -(K+K_p)K^{-1}[h(x_{1d})-\bar{h}(x_{1d})] \end{pmatrix}. \tag{19}$$

Subtracting (19) from (17) gives

$$\begin{pmatrix} K & -K \\ -K & K+K_p \end{pmatrix}\begin{pmatrix} x_1-x_1^* \\ x_3-x_3^* \end{pmatrix}=\begin{pmatrix} h(x_1^*)-h(x_1) \\ 0 \end{pmatrix} \tag{20}$$

which, recalling Assumption 1, clearly has the unique solution $(x_1,x_3)=(x_1^*,x_3^*)$. Therefore from (16) we have

$$\hat{x}_3^*=x_3^*-x_{3d}+K^{-1}[h(x_{1d})-\bar{h}(x_{1d})].$$

We have then proved the existence and uniqueness of the equilibrium point $(x^*,\hat{x}_3^*,0)$.

Let $\Delta\bar{x}_3=x_3-\bar{x}_{3d}$ and $\Delta x=x-x_d$, choose as Lyapunov function

$$V^e(\Delta x,\hat{x})=\lambda\bar{V}_{\mathrm{STAB}}(\Delta x)+\frac{1}{2}((\Delta\bar{x}_3-\hat{x}_3)^{\mathrm{T}}\ (x_4-\hat{x}_4)^{\mathrm{T}})\,P\begin{pmatrix} \Delta\bar{x}_3-\hat{x}_3 \\ x_4-\hat{x}_4 \end{pmatrix}+V_c$$

with V_c a constant term such that $V^e(x^*,\hat{x}_3^*,0)=0$ and $\bar{V}_{\mathrm{STAB}}(\Delta x)$ given by [10]:

$$\bar{V}_{\mathrm{STAB}}(\Delta x)=\mathrm{V}_{\mathrm{STAB}}(\Delta x)-\Delta x_3[K_p(\bar{x}_{3d}-x_{3d})+\bar{h}(x_{1d})-h(x_{1d})].$$

It has been shown in [10] that, under Assumption 1, $\bar{V}_{\mathrm{STAB}}(\Delta x)$ is still positive definite in $(x_1-x_1^*,x_3-x_3^*)$. With the controller (13) and following the same calculations as before, we obtain

$$\dot{V}^e(\Delta x,\hat{x})=-\lambda\hat{x}_4^{\mathrm{T}}K_d\hat{x}_4.$$

Again, applying La Salle's invariance principle, the following set of equations has to be satisfied:

$$\begin{aligned} &K(x_1-x_3)+h(x_1)=0,\\ &K(x_1-x_3)-K_p\hat{x}_3+\bar{h}(x_{1d})=0,\\ &K(x_1-\hat{x}_3)-K_p\hat{x}_3-Kx_{1d}=0. \end{aligned} \tag{21}$$

Following the previous discussion on the existence of the equilibrium point, it readily follows that (21) is satisfied only at the equilibrium point. Global asymptotic stability follows. □

4. Conclusions

In this paper, we have considered the simplified model of a flexible joint robot and shown that a linear output feedback controller solves the global set point control problem. The proposed controller has dimension equal to twice the number of links and does not have the structure of an observer. Also, we have proved the robustness of the given controller with respect to uncertainties in the gravitational term.

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